Using CLNS for FFTs in OFDM Demodulation of UWB Receivers

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***Abstract*— The Complex Logarithmic Number System (CLNS) uses log/polar complex representation. We propose CLNS for the 128-point Fast Fourier Transform (FFT) in Orthogonal Frequency Division Multiplexing (OFDM) demodulation. We show the cost and quality of the FFT depends on the CLNS base b. Memory requirements decrease as b increases. Experimentally, when optimized for Signal-to-Noise Ratio (SNR), $b > 2$ results in memory reduction up to 36% compared to $b = 2$. Dependence of the SNR on the input's standard deviation variations is weaker for the CLNS than for fixed-point and decreases as b increases.**

I. Introduction

The Complex-Logarithmic Number System (CLNS) is an alternative for arithmetic circuits that process complex numbers as log/polar coordinates. CLNS was first described by Arnold et al. [1], and is a generalization of the Logarithmic Number System (LNS) [2]. We propose implementing the 128-point FFT [3] used in Ultra Wide Band (UWB) receivers that adopt the Orthogonal Frequency Division Multiplexing (OFDM) modulation [4].

Most prior FFT implementations (including those implemented with LNS [2], [5], [6]) have used a real pair to represent a complex value. Inherently, the conversion to a finite-precision representation introduces an error due to the quantization performed by the Analog-to-Digital (A/D) converter. For a real-valued LNS, the base b and the integral and fractional bits (k, l) of the $(k + l + 1)$-bit word are the design parameters that impact the error. LNS uses table lookups for addition and subtraction. Paliouras [7] shows there is an optimal b that minimizes such lookup memory, assuming the quality measures for a real-valued LNS are dynamic range and average representational error [8]. However, in DSP applications, the most important measure is SNR. Kwa et al. [9] present analytic formulas for the calculation of SNR performance in a real-valued LNS assuming inputs follow Laplacian or sinusoid distributions. Vouzis et al. [10] propose a different optimal b so that input conversion to a real-valued LNS obtains the maximum achievable SNR. The b proposed in [10] reduces dependence on input's standard deviation variations, and reduces memory requirements for real-valued LNS addition/subtraction.

Since the conversion of an analog signal to a digital representation is followed by arithmetic processing, the selection of design parameters (such as b, k and l) should consider roundoff errors during the FFT. In [6] the optimal (k, l) are evaluated for an FFT, subject to the Bit-Error-Rate (BER) performance of an OFDM receiver using a real-valued $b = 2$ LNS. However, since (k, l) only can be discrete integers, the exploration of the design space in [6] cannot be done with the finest possible resolution. Instead, in this work, SNR is used as the quality measure, and b is also varied. Although this parameter can take any real value $b \neq 1$, it suffices to explore $2 \leq b < 4$, since a (k, l) format word with base b LNS is isomorphic to $(k - 1, l + 1)$ format word with base b^2 LNS.

In the following section CLNS is described briefly. In section III analytic formulas are derived for the calculation of the memory size for CLNS addition. The experimental results for the performance, and the exploration of the design space (including the dependence of the SNR on the input's standard deviation variations) for the 128-point FFT are presented in section IV, before ending with conclusions in section V.

II. Introduction to CLNS

CLNS represents a complex value, X, with a log-polar pair, $X_{\text{CP}} = (X_{\text{CL}}, X_\theta)$, whose components are:

$$\begin{aligned} X_{\text{CL}} &= 0.5 \cdot \log_b(\Re[X]^2 + \Im[X]^2) \\ X_\theta &= \arctan(\Re[X], \Im[X]). \end{aligned} \tag{1}$$

X_{CL} is quantized in the same (k, l) format as the real-valued LNS. The quantization for X_θ is arbitrary, but a sensible approach adopted here is for the quantization step near the unit circle to be the same in both the angular and radial dimensions. This implies $0 \leq X_\theta \leq 2\pi$ is represented with a $(3, l)$ format scaled by 2^{-l}, and the complete CLNS representation occupies $k + 2 \cdot l + 3$ bits.

On output, this log-polar pair is converted back to a rectangular pair that describes the same value:

$$\Re[X] = b^{X_{\text{CL}}} \cos(X_\theta), \quad \Im[X] = b^{X_{\text{CL}}} \sin(X_\theta). \tag{2}$$

CLNS multiply and divide are easy, e.g., to divide X by Y:

$$Z_{\text{CL}} = X_{\text{CL}} + Y_{\text{CL}}, \quad Z_\theta = X_\theta + Y_\theta \mod 2\pi. \tag{3}$$

CLNS addition uses $T = X + Y = Y(Z + 1)$, where $Z = X/Y$. This needs two subtractors, two ROMs, and two adders:

$$\begin{aligned} T_{\mathrm{CL}} &= Y_{\mathrm{CL}} + \Re[S_b(Z_{\mathrm{CP}})] \\ T_\theta &= Y_\theta + \Im[S_b(Z_{\mathrm{CP}})] \mod 2\pi, \end{aligned} \tag{4}$$

where Z_{CP} is (3). $S_b(Z_{\mathrm{CP}})$ implements i) conversion of Z_{CP} via (2); ii) incrementing this; and iii) conversion back via (1). Lewis [11] proposed a 32-bit $b = e$ CLNS ALU based on CORDIC rather than ROM lookup. $b = e$ simplifies Lewis' hardware since the CORDIC algorithm naturally gives the complex logarithm i) and antilogarithm iii) as base e. A different b would require scaling by $\ln(e)$. A higher-speed implementation, such as envisioned here, can use precomputed ROM to hold $S_b(Z_{\mathrm{CP}})$ so b is arbitrary. Then $S_b(Z_{\mathrm{CP}})$ is precomputed in terms of its real and imaginary parts:

$$\begin{aligned} \Re[S_b(Z_{\mathrm{CP}})] &= 0.5 \cdot \log_b(1 + 2b^{Z_{\mathrm{CL}}} \cos(Z_\theta) + b^{2 \cdot Z_{\mathrm{CL}}}) \\ \Im[S_b(Z_{\mathrm{CP}})] &= \arctan(1 + b^{Z_{\mathrm{CL}}} \cos(Z_\theta), b^{Z_{\mathrm{CL}}} \sin(Z_\theta)). \end{aligned}$$

Because of commutativity, we may assume $0 \le Z_\theta \le \pi$.

III. CLNS Memory requirements

A. The real part of $S_b(Z_{CP})$

Due to the finite precision of the CLNS representation, the required interval that the function $S_b(Z_{CP})$ has to be stored is finite as well. The number of words that have to be stored depends on the accuracy and on the value of the base b of the representation. This idea is similar to the effective zero value described in [8], [10]. However, this value has to be recalculated for the case of $S_b(Z_{CP})$. From Figs. 1 and 2 it can be observed that

$$\lim_{Z_{\mathrm{CL}} \to -\infty} \Re[S_b(Z_{\mathrm{CP}})] = 0; \tag{5}$$

thus, if the fractional bits of the two's complement representation are l, then it is sufficient to store only the values that satisfy the inequality $|\Re[S_b(Z_{\mathrm{CP}})]| > 2^{-l}$ with $Z_{\mathrm{CL}} < 0$.

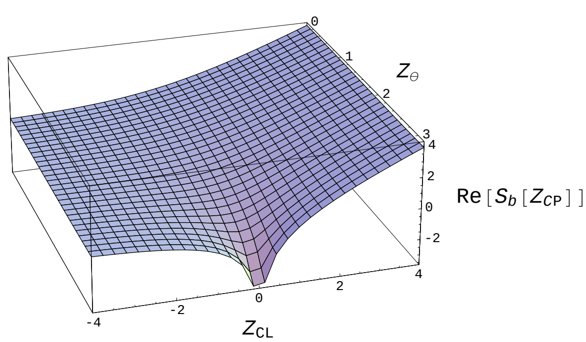

Fig. 1. The real part of $S_b(Z_{\mathrm{CP}})$ for $b = 2$.

The factor $\cos(Z_\theta)$ of the function $\Re[S_b(Z_{\mathrm{CP}})]$ causes a sinusoid fluctuation between a minimum and a maximum value, for a given value of Z_{CL}, as is depicted in Fig. 3. Since, the points that the value of the function has the biggest distance

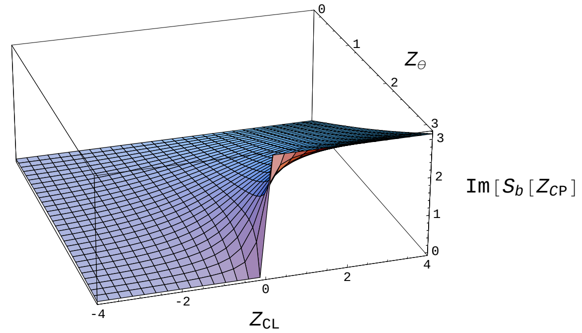

Fig. 2. The imaginary part of $S_b(Z_{\mathrm{CP}})$ for $b = 2$.

from zero are at the two edges of the interval $(0, \pi)$, of Z_θ, the inequalities

$$\begin{matrix} \log_b(1 + b^{Z_{\mathrm{CL}}}) < 2^{-l} & Z_\theta = 0 \\ \log_b(1 - b^{Z_{\mathrm{CL}}}) > -2^{-l} & Z_\theta = \pi \end{matrix} , \tag{6}$$

must be solved. Thus, the points of the effective zero, for the values $Z_\theta = 0$ and $Z_\theta = \pi$, are

$$e_{\Re,\mathrm{a}}(b, l) = \log_b(b^{2^{-l}} - 1) \tag{7}$$

$$e_{\Re,\mathrm{s}}(b, l) = \log_b(1 - b^{-2^{-l}}) , \tag{8}$$

respectively, with $e_{\Re,\mathrm{s}}(b, l) < e_{\Re,\mathrm{a}}(b, l)$, thus $e_{\Re,\mathrm{s}}$ has to be used as the effective zero point for $Z_{\mathrm{CL}} < 0$.

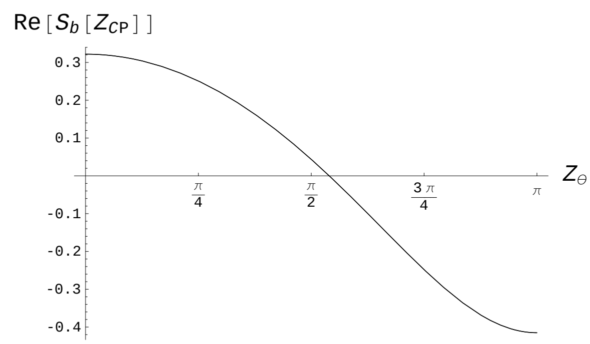

Fig. 3. The fluctuation of $\Re[S_b(Z_{\mathrm{CP}})]$, for $Z_{\mathrm{CL}} = 2$ and $b = 2$, with respect to Z_θ.

Similarly, for $Z_{\mathrm{CL}} > 0$

$$\lim_{Z_{\mathrm{CL}} \to +\infty} \Re[S_b(Z_{\mathrm{CP}})] = Z_{\mathrm{CL}} , \tag{9}$$

Thus, it is sufficient to store the values that give $|\Re[S_b(Z_{\mathrm{CP}})] - Z_{\mathrm{CL}}| > 2^{-l}$, with $Z_{\mathrm{CL}} > 0$. For the rest of the values the function is a tautology, i.e., the result of the function is equal to the input argument Z_{CL}. The process for calculating the effective zero value for $Z_{\mathrm{CL}} > 0$ is the same to the case $Z_{\mathrm{CL}} < 0$. The factor $\cos(Z_\theta)$ causes exactly the same fluctuation to the equation $\Re[S_b(Z_{\mathrm{CP}})] - Z_{\mathrm{CL}}$, and thus the effective zero for $Z_{\mathrm{CL}} > 0$ is equal to $-e_{\mathrm{s}}(b, l)$. Hence, the interval for which the values of $\Re[S_b(Z_{\mathrm{CP}})]$ have to be stored is $[e_{\Re,\mathrm{s}}(b, l), -e_{\Re,\mathrm{s}}(b, l)]$.

Since $0 \leq Z_\theta \leq \pi$ and the scale is 2^{-l}, the number of words, required to be stored for the real part of $S_b(Z_{\text{CP}})$, is equal to

$$w_{\Re}(b,l) = \left\lceil -\frac{2\, e_{\Re,\text{s}}(b,l)\cdot\pi}{2^{-2l}} \right\rceil \quad . \tag{10}$$

B. The imaginary part of $S_b(Z_{CP})$

For the total number of words, the imaginary part of $S_b(Z_{\text{CP}})$ has to be considered as well. Since

$$\lim_{Z_{\text{CL}}\to-\infty} \Im[S_b(Z_{\text{CP}})] = 0^+ \quad , \tag{11}$$

for $Z_\theta \in (0,\pi)$, the effective-zero point for $Z_{\text{CL}} < 0$ is the solution of the inequality

$$\Im[S_b(Z_{\text{CP}})] < 2^{-l} \quad , \tag{12}$$

with regard to Z_{CL}. For a given Z_{CL} the value of $\Im[S_b(Z_{\text{CP}})]$ fluctuates due to the factors $\cos(Z_\theta)$ and $\sin(Z_\theta)$. The point where the function takes its maximum value depends on the value of Z_θ. However, this point is not constant for every value of Z_{CL}, which results in a more complicated calculation of the effective zero value, than the real part of $S_b(Z_{\text{CP}})$.

Initially, the partial derivative of $\Im[S_b(Z_{\text{CP}})]$ with regard to Z_θ is calculated:

$$\frac{\partial\, \Im[S_b(Z_{\text{CP}})]}{\partial\, Z_\theta} = \frac{b^{Z_{\text{CL}}}\left(b^{Z_{\text{CL}}} + \cos(Z_\theta)\right)}{1 + b^{2\,Z_{\text{CL}}} + 2\,b^{Z_{\text{CL}}}\cos(Z_\theta)} \quad . \tag{13}$$

Thus, the point of the maximum value of $\Im[S_b(Z_{\text{CP}})]$, for a given value of $Z_{\text{CL}} < 0$, is the solution, with regard to Z_θ, of the equation

$$\frac{\partial\, \Im[S_b(Z_{\text{CP}})]}{\partial\, Z_\theta} = 0 \quad , \tag{14}$$

which gives

$$r^-_{Z_\theta}(b, Z_{\text{CL}}) = -\arccos(-b^{Z_{\text{CL}}}) \quad . \tag{15}$$

If the factor Z_θ, in $\Im[S_b(Z_{\text{CP}})]$, is substituted by the value $r^-_{Z_\theta}$, the resulting formula gives the maximum value of $\Im[S_b(Z_{\text{CP}})]$ for every $Z_{\text{CL}} < 0$. Thus, the solution of the inequality

$$\Im[S_b(Z_{\text{CP}})] < 2^{-l} \text{ with } Z_\theta = -\arccos(-b^{Z_{\text{CL}}}) \tag{16}$$

gives the effective zero of the imaginary part of $S_b(Z_{\text{CP}})$, for a two's complement representation with l fractional bits:

$$e^-_{\Im}(b,l) = \log_b(\sin(2^{-l})) \quad . \tag{17}$$

Similarly, for $Z_{\text{CL}} > 0$

$$\lim_{Z_{\text{CL}}\to+\infty} \Im[S_b(Z_{\text{CP}})] = Z^+_\theta; \tag{18}$$

thus for values of Z_{CL} greater than the effective zero the function $\Im[S_b(Z_{\text{CP}})]$ becomes a tautology since it produces as a result the value of the quantity Z_θ. However, the value of Z_θ, that $Z_\theta - \Im[S_b(Z_{\text{CP}})]$ achieves its maximum value, depends on the particular value of Z_{CL}, and consequently the process described for $Z_{\text{CL}} < 0$ has to be repeated. The solution of the equation

$$\frac{\partial\,\left(Z_\theta - \Im[S_b(Z_{\text{CP}})]\right)}{\partial\, Z_\theta} = \frac{1 + b^{Z_L}\cos(Z_\theta)}{1 + b^{2\,Z_L} + 2\,b^{Z_L}\cos(Z_\theta)} = 0 \tag{19}$$

with regard to Z_θ, for $Z_{\text{CL}} > 0$, is:

$$r^+_{Z_\theta}(b, Z_{\text{CL}}) = \arccos(-b^{-Z_{\text{CL}}}) \quad . \tag{20}$$

Hence, by substituting in $\Im[S_b(Z_{\text{CP}})]$, Z_θ with $r^+_{Z_\theta}(b, Z_{\text{CL}})$, and by solving the inequality

$$Z_\theta - \Im[S_b(Z_{\text{CP}})] < 2^{-l} \quad , \tag{21}$$

the effective zero for $Z_{\text{CL}} > 0$ can be found:

$$e^+_{\Im}(b,l) = -e^-_{\Im}(b,l) \quad . \tag{22}$$

Thus, the number of words required to be stored, for the imaginary part of $S_b(Z_{\text{CP}})$, is given by

$$w_{\Im}(b,l) = \left\lceil -\frac{2\, e_{\Im}(b,l)\cdot\pi}{2^{-2l}} \right\rceil \quad . \tag{23}$$

Finally, the total number of words required for CLNS addition is equal to

$$w_{CLNS}(b,l) = w_{\Re}(b,l) + w_{\Im}(b,l) \quad . \tag{24}$$

IV. Optimal selection for the base b for the 128-point FFT

The use of the LNS, or the CLNS, representation for the implementation of Digital Signal Processing (DSP) systems is done, so far, without exploring the impact of the base b of the representation. Even though there have been proposed algorithms for the optimal base b selection [7][10], these works consider only the impact of the base on the representation itself, and they do not consider the particular application, in this case, the FFT in OFDM. Also, they consider only a single real value, and not a complex number. The analysis in [10] has shown that the selection of the base has an impact on several characteristics of an LNS representation. However, since the conversion of a signal, to a particular representation, is followed by a digital signal processing, the selection of the base should consider each different case.

A. Experimental study of the 128-point FFT implemented with CLNS

The study of the 128-point FFT was done experimentally. Thus, the FFT was implemented by the use of the CLNS, and the SNR performance subject to the changes of base b was measured. Additionally, a study of the SNR performance of the fixed-point representation was conducted and a comparison between the two implementation is presented.

In Table I the optimal bases, b_{opt}, that a (k,l) two's complement CLNS representation must adopt in order to achieve the maximum achievable SNR, $\text{SNR}_{\text{CLNS,max}}$ are presented. For the fixed-point representation random imaginary numbers were used that their real and imaginary part followed a Gaussian distribution of expected value zero and of standard deviation σ_{fxp}, and the same values were used for the CLNS implementation. For each (k,l) there is an optimal value of the inputs standard deviation, $\sigma_{\text{CLNS,opt}}$, that must be used in order the maximum possible SNR to be achieved.

For the implementation of a digital system with the use of either the LNS [5], or the CLNS [3], the base $b = 2$

(k,l)	Fixed Point		CLNS			
	SNR_{max}	σ_{opt}	SNR_{max}	σ_{opt}	b_{opt}	vs. $b=2$
(3,5)	22.3	.160	31.5	1.11	2.28	17.7%
(3,6)	25.6	.145	35.3	1.14	2.35	20.8%
(3,7)	31.8	.140	42.0	1.18	2.44	22.6%
(3,8)	37.9	.137	48.0	1.19	2.50	27.4%
(3,9)	43.7	.135	52.8	1.20	2.70	32.1%
(3,10)	49.2	.135	58.4	1.30	2.73	33.3%
(3,11)	56.1	.134	65.2	1.50	2.75	33.1%
(3,12)	60.6	.133	71.1	1.70	2.81	34.5%
(3,13)	68.3	.132	78.0	1.73	2.92	36.7%

TABLE I

COMPARISON BETWEEN THE FIXED-POINT AND THE CLNS REPRESENTATION IN TERMS OF SNR PERFORMANCE.

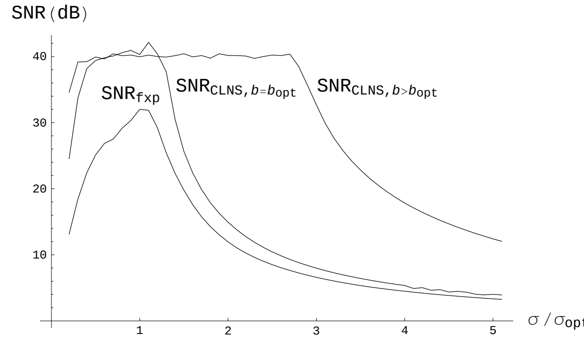

Fig. 4. The CLNS representation has weaker dependence on input's standard deviation variations.

was used, without considering the particular application. For example, in [3], Arnold *et.al* use $b = 2$ for the study of the absolute squared error of an FFT implementation. However, from Table I, it is apparent that the best value for the base of the representation is different to the value of $b = 2$. Additionally, the optimal base is bigger than the value $b = 2$, which results in a reduction to the number of words required for the CLNS addition, since $\partial w_{CLNS}(b, l)/\partial b < 0$.

The reduction of the memory requirements is presented in the last column of the Table I. The memory reduction is of great importance because the part of a logarithmic circuit that occupies the most area, and adds the most delay in the critical path is the LUTs used for logarithmic addition. Thus, the proper choice of base results in reduction of the area, reduction of the power dissipation, increase of the frequency, and better SNR performance of a CLNS implementation.

Additionally, the dependence of SNR_{CLNS} on the variations of inputs standard deviation is weaker than a fixed-point implementation. This property was presented in [10], but the SNR only of the representation was considered. In Fig. 4 the variations of the SNR of the 128-point FFT with regard to the input standard deviation is depicted. It is apparent that the SNR of the CLNS implementation is preserved constant for a bigger interval compared to the fixed point implementation. This interval can be increased even more by increasing the value of the base b. Although, this reduces the SNR performance, at the same time it reduces the memory requirements for logarithmic addition.

V. CONCLUSIONS

The CLNS was utilized for the implementation of a 128-point FFT suitable for an UWB OFDM receiver. Additionally, an optimization of this representation, in terms of SNR behavior, was presented. It has been shown that the optimal base b selection leads to the maximum achievable SNR of the output, decreased memory requirements for CLNS addition, and weaker dependence of SNR on input's standard deviation variations. Although, similar results were demonstrated in [10] for real-valued LNS, SNR data gathered for the 128-point FFT implemented with CLNS are novel. The results presented in Table I can be used for the comparison between the two different representations, and for the exploration of the design space of the CLNS implementation.

Further study of the function $S_b(Z_{\mathrm{CP}})$ proves to offer more potential for memory savings. In a work-in-progress [12] a parallel-search algorithm combined with co-transformation [1] is used, which results in a decrease in the access speed of the memory, with a concurrent reduction in its size.

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