Homework #3: Chapters 7 and 8

Problem 1: [20 pts.]
a) [8 pts.] for truth table up to, and including, KB column

<table>
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<tr>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>P ∨ R</th>
<th>P ∨ R ⇒ Q</th>
<th>¬Q</th>
<th>P ⇒ ¬Q</th>
<th>Q ∨ R</th>
<th>KB</th>
<th>¬Q ∧ R</th>
<th>P ⇒ R</th>
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From the above table, we can conclude:

b) [4 pts.] Yes, KB does entail Q, i.e., KB |= Q
   Q is true in both models where the KB is true

c) [4 pts.] No, KB does not entail ¬Q ∧ R, i.e., KB |≠ ¬Q ∧ R
   It is false in both models where the KB is true (and must be true in all)

d) [4 pts.] Yes, KB does entail P ⇒ R
   It is true in both models where the KB is true

Give partial credit for wrong answers that are due to errors in the truth table, but would otherwise be a correct definition of entailment.
Problem 2:
[10 pts.] This is just one way to prove this, similar examples are acceptable. 5pts. each part. -2 each if only one direction is proven.

a) **First direction:** if $\alpha \models \beta$ then the sentence $(\alpha \Rightarrow \beta)$ is true in all models.

*Proof:* By definition, if $\alpha \models \beta$, then every model where $\alpha$ is true, then $\beta$ is also true.
Thus, either $\alpha$ and $\beta$ are both true, or $\alpha$ is false. In the first case, the truth table for $\Rightarrow$ shows that $(\alpha \Rightarrow \beta)$ is true. For the second case, $(\alpha \Rightarrow \beta)$ is true regardless of the truth of $\beta$. Therefore, $(\alpha \Rightarrow \beta)$ is true in all models where $\alpha \models \beta$.

**Second direction:** if the sentence $(\alpha \Rightarrow \beta)$ is true in all models, then $\alpha \models \beta$.

*Proof:* By looking at the truth table of the sentence $(\alpha \Rightarrow \beta)$, it is only true if both $\alpha$ and $\beta$ are true, or if $\alpha$ is false. When determining entailment, we can ignore those models where $\alpha$ is false. Since the only models where $\alpha$ is true, $\beta$ is also true, $\alpha \models \beta$.

b) **First direction:** if $\alpha \models \beta$ then the sentence $(\alpha \land \neg \beta)$ is false in all models.

*Proof:* As above, if $\alpha \models \beta$, then every model where $\alpha$ is true, then $\beta$ is also true.
Thus, either $\alpha$ and $\beta$ are both true, or $\alpha$ is false. Consider the truth table for $\alpha \land \neg \beta$:

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\neg \beta$</th>
<th>$\alpha \land \neg \beta$</th>
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</thead>
<tbody>
<tr>
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We have used cross-hatching to rule out the only model that is inconsistent with the premise $\alpha \models \beta$. In all three of the remaining models, $\alpha \land \neg \beta$ is false.

**Second direction:** If the sentence $(\alpha \land \neg \beta)$ is false in all models then $\alpha \models \beta$.

*Proof:* Consider the truth table above. There are three models where $\alpha \land \neg \beta$ is false. Of these models, there is only one where $\alpha$ is true (the last model in the table). Since $\beta$ is also true in this model, every model in which $\alpha$ is true, $\beta$ is also true. Thus, $\alpha \models \beta$.

Alternatively, we can prove part b) by noting that if $P$ is false in all models, then $\neg P$ is true in all models (by the truth table for negation). $\neg(\alpha \land \neg \beta)$ is equivalent to $(\neg \alpha \lor \neg \beta)$ (by de Morgan’s law), which simplifies to $(\neg \alpha \lor \beta)$ which is equivalent to $(\alpha \Rightarrow \beta)$ by implication elimination (in the reverse direction). Thus, this statement is equivalent to “$\alpha \models \beta$ iff the sentence $(\alpha \Rightarrow \beta)$ is true in all models.” This is exactly what we proved in part a).
Problem 3:
[40 pts., 4 pts. each] There may be multiple right answers for each of these, especially since there are multiple ways to write semantically equivalent sentences.

∀d, p Daughter(d, p) ⇔ Child(d, p) ∧ Female(d)

∀s, p Son(s, p) ⇔ Child(s, p) ∧ Male(s)

∀w, h Wife(w, h) ⇔ Spouse(w, h) ∧ Female(w)

∀x, z Grandchild(x, z) ⇔ ∃y Child(x, y) ∧ Child(y, z)

∀a, d GreatGrandParent(a, d) ⇔ ∃b Parent(a, b ) ∧ Grandparent(b, d)
  or
  ∀a, d GreatGrandParent(a, d) ⇔ ∃b, c Parent(a, b ) ∧ Parent(b, c) ∧ Parent(c, d)

∀b, x Brother(b, x) ⇔ Sibling(b, x) ∧ Male(b)

∀s, x Sister(s, x) ⇔ Sibling(s, x) ∧ Female(s)

(Aunt is defined as the sister of one's father or mother, or the wife of one's uncle.)
∀a, c Aunt(a, c) ⇔ (∃p Sibling(a, p) ∧ Female(a) ∧ Parent(p, c)) ∨ (∃u Spouse(a, u) ∧ Uncle(u, c))
  or
  ∀a, c Aunt(a, c) ⇔ (∃p Sibling(a, p) ∧ Parent(p, c)) ∨ (∃u Spouse(a, u) ∧ Uncle(u, c))

(Uncle is defined as a brother of one's father or mother, or an aunt's husband.)
∀u, c Uncle(u, c) ⇔ (∃p Sibling(u, p) ∧ Male(u) ∧ Parent(p, c)) ∨ (∃a Spouse(u, a) ∧ Aunt(a, c))
  or
  ∀u, c Uncle(u, c) ⇔ (∃p Brother(u, p) ∧ Parent(p, c)) ∨ (∃a Spouse(u, a) ∧ Aunt(a, c))

(First cousin is defined as a child of one’s aunt or uncle.)
∀c, y FirstCousin(c, y) ⇔ ∃p, x Child(c, x) ∧ Sibling(x, p) ∧ Parent(p, y)
  or
  ∀c, y FirstCousin(c, y) ⇔ ∃x Child(c, x) ∧ (Aunt(x, y) ∨ Uncle(x, y))

  Note this definition is probably more lax than it should be. If someone who is an aunt or uncle by marriage to one of your blood-relations has a child by a prior marriage, then this definition considers that child to be your first-cousin (some dictionaries include a definition such as “Persons who have a grandparent in common are called first cousins” to rule this situation out). Nevertheless, I’ll accept this axiom as is. The stricter definition would be:

∀c, x FirstCousin(c, x) ⇔ c ≟ x ∧ ¬Sibling(c, x) ∧
  ∃g GrandParent(g, x) ∧ GrandParent(g, c)
Problem 4: [20 pts, 4 pts. each]

a) \( \exists x \ knows(x, \ Tim) \land \ loves(x, \ Tim) \)

b) \( \forall x \ knows(x, \ Sue) \Rightarrow avoids(x, \ Sue) \)

c) \( \exists x \ \forall y \ loves(y, x) \)

d) \( \neg \exists x \ \forall y \ knows(x, y) \) or
\( \forall x \ \neg \forall y \ knows(x, y) \) or
\( \forall x \ \exists y \ \neg knows(x, y) \)

e) \( \exists x \ loves(x, x) \land \ \forall y \ loves(x, y) \Rightarrow x = y \ or\)
\( \exists x \ loves(x, x) \land \ \forall y \ x \neq y \Rightarrow \neg loves(x, y) \)

Problem 5: [10 pts]

\( \forall s \ NearbyMines(s,2) \Rightarrow (\exists r,t \ r \neq t \land Adjacent(s,r) \land Mine(r) \land Adjacent(s,t) \land Mine(t) \land (\forall u \ u = r \lor u = t \lor \neg Mine(u) \lor \neg Adjacent(s,u)) ) \)

or

\( \forall s \ NearbyMines(s,2) \Rightarrow (\exists r,t \ r \neq t \land Adjacent(s,r) \land Mine(r) \land Adjacent(s,t) \land Mine(t) \land (\forall u \ Adjacent(s,u) \land Mine(u) \Rightarrow u = r \lor u = t)) \)

Other variations are possible.