# On Accuracy of Gaussian Assumption in Iterative Analysis for LDPC Codes

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***Abstract*— Iterative analysis for low-density parity-check (LDPC) codes uses the prevailing assumption that messages exchanged between the variable nodes and the check nodes follow a Gaussian distribution. However, the justification is largely pragmatic rather than being based on any rigorous theory. This paper provides a theoretic support by investigating when and how well the Gaussian distribution approximates the real message density and the far subtler why. The analytical results are verified by extensive simulations.**

## I. Introduction

The breakthrough of turbo codes in 1993 had revolutionized the coding research with new concepts for successful error correction: a paradigm of constructing long, powerful codes using short, weak component codes and decoding them using soft, iterative decoders with manageable complexity.

The rediscovery of low-density parity-check (LDPC) codes a few years later provided additional testimony to the marvel of the soft-iterative paradigm. With variations of turbo codes and LDPC codes demonstrating similar superb performance, and with softly-decoded versions of the existing block codes exceeding the error correction capability they once believed to have, the importance of both ingredients in a soft-decision decoder cannot be over-stated: "soft" enables extraction of maximal benefit from the knowledge of the channel noise statistics and refinement of useful probabilistic information through the decoding process, and "iterative" enables implementation of soft-decision decoders with efficiency and acceptable complexity.

Toward a deep theoretic understanding of soft iterative decoding, researchers have conducted active analysis. A soft iterative decoder generally consists of several component soft decoders connected in a parallel, serial or hybrid fashion, passing probabilistic message along the connecting edges between the component decoders. Message-passing algorithm, for which the *a posterior* probability decoding for turbo codes is a specialization, forms the majority of soft iterative decoding. Since almost all message-passing decoders are high-dimensional nonlinear mapping, analysis using conventional methods (such as those on the codeword space) appears ineffective. On the other hand, stochastic approaches offer a rich source for analyzing the properties of iterative decoding, enabling the modeling of the input and output of a soft decoder as random processes and the tracking of the evolution of their statistic characteristics through iterations. Density evolution (DE), proposed by Richardson *et al* in [3], was one of the pioneering stochastic methods to investigate the convergence behavior for iterative decoding. Density evolution, when applied to code graphs with asymptotically unbounded girth, can compute thresholds for the performance of LDPC codes and turbo codes with iterative decoding, but tracking the probability density function (pdf) of the messages involve infinite dimensional algebra, and is therefore computationally prohibitive.

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To simplify the analysis, researchers started to look into the widely-adapted Gaussian model. Wiberg [7] first demonstrated that the pdf of extrinsic information (exchanged between component decoders) may be approximated by a Gaussian distribution. This discovery quite simplified the stochastic analysis, since a Gaussian distribution can be completely characterized by its mean and variance. Following this approximation, [4] succeeded in estimating the thresholds for both regular and irregular LDPC codes. At the same time, [8] showed that the pdf of extrinsic information in message-passing decoding satisfies and preserves a *symmetry condition*. Realizing that a probabilistic density that is both "symmetric" and Gaussian distributed satisfies $\sigma^2 = 2m$, where $m$ and $\sigma^2$ are the mean and variance of the Gaussian distribution, researchers were able to further simplify the analysis by using a single parameter, either the mean or the variance of the message density, to track down the probabilistic evolution.

Following the success of density evolution, ten Brink [5] proposed to use extrinsic information transfer (EXIT) charts to visualize the behavior of an iterative decoder as the temporal evolution of a one-dimensional quantity: the extrinsic information exchanged between different computational units during iterative decoding. At its proposition, EXIT charts were considered an effective tool, but one providing not much more knowledge than visualizing the repeated application of the density evolution algorithm with different channel signal-to-noise ratios (SNR) and at different stages of iterative decoding. However, recent studies by Ashikmin and ten Brink, *et al* and Montanari and Urbanke *et al*, reveal surprisingly elegant and useful properties of EXIT charts, including, for example, the convergence property and the area property [6] [14].

EXIT charts and their underlying tool of density evolution both make essential use of the prevailing assumption that messages, as they pass between two component decoders at an arbitrary stage of iterative decoding, follow approximately Gaussian distribution. However, the justification of Gaussian approximation is largely pragmatic, except for messages extracted from Gaussian channel outputs which are provenly Gaussian. This seems-to-work philosophy has underlined the iterative analysis of message-passing decoding for much of the short history of the topic, and it is only very recently that [10] attempted a theoretic analysis on the accuracy of Gaussian assumption for turbo codes. In this paper, we provide a statistical support for LDPC codes by investigating when and how well the Gaussian distribution approximates the real message density, and the far subtler why. Experimental verifications are provided along with the discussion.

The remainder of the paper is organized as follows. The background of LDPC decoding and the notations used in the paper are introduced in Section II. Section III discusses lognormal distributions and establishes several properties useful for our analysis. The main results of the paper, namely, the accuracy and applicable region of the Gaussian approximation, are provided in Section IV. Finally, concluding remarks are provided in Section V.

## II. Background and Notations

LDPC codes are a class of linear codes characterized by sparse parity check matrices. Message-passing decoding for LDPC codes make essential use of graphs, known as *Tanner graphs* or their generalization *factor graphs*, to represent codes, passing probabilistic message along the edges of the graph. The Tanner graph for an $(n, k)$ LDPC code consists of $n$ variable nodes representing all the bits in the codeword, (at least) $(n-k)$ check nodes representing the parity constraints imposed to the coded bits, and multiple edges connecting the two types of nodes. The number of the edges connected to a node is termed the *degree* of this node. We will use $D_v$ and $D_c$ to represent the degree of the variable nodes and check nodes respectively.

Consider message-passing decoding over an LDPC Tanner graph, where soft extrinsic information iterates between variable nodes and check nodes, and updates itself after each iteration. Let superscript $\ell$ denote the number of decoding iterations, and subscript $i$ and $j$ denote, respectively, variable nodes and check nodes. At the $\ell$-th iteration, the extrinsic information passed from variable node $i$ to check node $j$, $m_{ij}^{\ell}$, and the information from check node $j$ to variable node $i$, $m_{ji}^{\ell}$, are updated as follows:

$$m_{ij}^{\ell} = \begin{cases} m_i, & \ell = 0, \\ m_i + \sum\limits_{j' \in N_c(i) \backslash \{j\}} m_{j'i}^{\ell}, & \ell > 0. \end{cases} \quad (1)$$

$$m_{ji}^{\ell} = \ln \left( \frac{1 + \prod_{i' \in N_v(j) \backslash \{i\}} \tanh m_{i'j}^{\ell-1}/2}{1 - \prod_{i' \in N_v(j) \backslash \{i\}} \tanh m_{i'j}^{\ell-1}/2} \right), \quad (2)$$

$$= 2 \tanh^{-1} \Big( \prod_{i' \in N_v(j) \backslash \{i\}} \tanh \frac{m_{i'j}^{\ell-1}}{2} \Big), \quad (3)$$

$$= \Big( \prod_{i' \in N_v(j) \backslash \{i\}} \operatorname{sign}(m_{i'j}^{\ell-1}) \Big) \cdot \Phi \Big( \sum_{i' \in N_v(j) \backslash \{i\}} \Phi \big( m_{i'j}^{\ell-1} \big) \Big), \quad (4)$$

where $N_c(i)$ is the set of check nodes connected with the $i$-th variable node, $N_v(j)$ is the set of variable nodes connected with the $j$-th check node, and $m_i$ is the log likelihood ratio (LLR) of signal $s_i$, extracted from the $i$th channel output $r_i$:

$$m_i = \log \frac{\Pr(s_i = +1 | r_i)}{\Pr(s_i = -1 | r_i)} \quad (5)$$

For additive white Gaussian noise (AWGN) channels with Gaussian noise of mean zero and variance $\sigma^2$ and i.i.d. input,

$$m_i = 2r_i / \sigma^2, \quad (6)$$

and hence follows a Gaussian distribution.

$\Phi$ function is defined as:

$$\Phi(x) = \ln \left( \frac{e^{|x|} + 1}{e^{|x|} - 1} \right), \quad (7)$$

where for convenience we assume $\Phi(0) = \ln(2/0) = \infty$.

The formulations in (2), (3) and (4) describe the same check update operation but in different forms. Our Gaussian analysis will be exclusively carried out on (4).

## III. Lognormal Distributions

Before providing the main results, let us first establish a few useful properties of lognormal distributions, upon which our analysis is based.

*Definition 1:* A random variable (r.v.) $X$ is said to be *lognormal* distributed if its logarithm value $\ln(X)$ follows a Gaussian distribution. Using the Jacobian rule, the lognormal pdf for $X$ can be formulated as:

$$f_X(x) = \frac{1}{\sqrt{2\pi} x \sigma} \, e^{-\frac{(\ln(x) - \mu)^2}{2\sigma^2}}, \quad \text{for } x > 0, \quad (8)$$

where $\mu$ and $\sigma^2$ are the mean and variance of $\ln(X)$.

To provide a visual impression of how lognormal densities look like, Figure 1 plots the pdf curves for 4 lognormal distributions with $\mu = 0$ and $\sigma = 0.5, 1.0, 1.5, 3.0$, respectively.

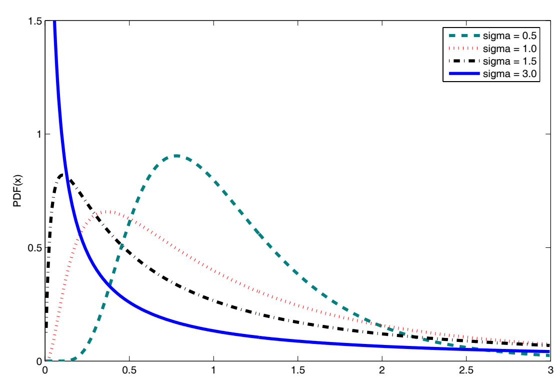


Fig. 1. Illustration of lognormal pdf's $\mu = 0$ and $\sigma = 0.5, 1.0, 1.5, 3.0$.

A long-recognized fact in statistics is that the sum of lognormal random variables is also lognormal. Works in this area can be traced back to the sixties [9] and based on this assumption, a number of models on the distribution function and moments have been developed(see [9]-[13] and the references therein). This rule is also widely applied in many science and engineering fields including, for example, the coherent channel interference model in wireless communications [11], analysis of the BCJR algorithm in coding [10] and risk measuring in finance [12]. Below we formally state the results developed in literatures by differentiating between correlated and uncorrelated random variables and between finite and infinite terms.

*Proposition 1:* The sum of a set of correlated lognormal random variables follows a lognormal distribution, regardless of whether the set is finite or countable infinite. The sum of a set of independent lognormal random variables approximates lognormal when the set size is small, transforms from lognormal to Gaussian as the set size increases, and becomes exactly Gaussian in the unlimited case.

The case of correlated random variables may be verified by simulations (see Figure 2) as well as [11] [10] [12]. For independent random variables, [13] showed that the lognormal approximation holds for a set size of 10 or smaller. Gaussianity in the unlimited case follows from the central limit theorem.

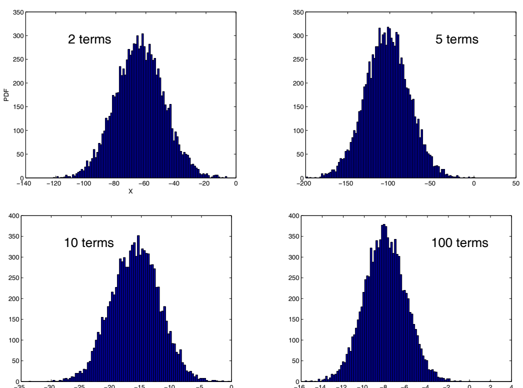


Fig. 2. Histograms for $\ln(S)$ with $k = 2, 5, 10, 100$.

*Proposition 2*: Let $X$ be a lognormal random variable, its power sum, defined as

$$S = \sum_{i=1}^{k} a_i X^{b_i}, \tag{9}$$

follows a lognormal distribution, where $\{a_i\}$ is a set of arbitrary non-zero constants, $\{b_i\}$ is a set of arbitrary negative integers and $k$ may be either finite or infinite.

*Proof:* Since $X$ follows a lognormal distribution, there exists a Gaussian random variable $Z$ that satisfies the equality $X = e^Z$. Rewrite $X^{b_i}$ as $e^{b_i Z}$. Since $b_i Z$ satisfies Gaussian distributions for $b_i < 0$, according to the definition of lognormal distributions, $e^{b_i Z}$'s, and hence $X^{b_i}$'s and $a_i X^{b_i}$'s for $a_i \neq 0$, form a set of correlated lognormal random variables. Following Proposition 1, their sum $S$ will also be lognormal. □

Since the proof of Proposition 2 uses Proposition 1 which is a statistical rule-of-thumb, we perform experimental tests to verify Proposition 2. Figure2 presents the histograms, each collected over 10000 test samples, for $\ln(S)$ with set size $k = 2, 5, 10, 100$ and randomly selected negative integers $b_i$'s. The plot shows that $\ln(S)$ looks consistently Gaussian-like regardless of the set size.

To further provide a quantifiable evaluation of how close the empirical data matches the true Gaussian distribution, we resort to a goodness-of-fit tool named Kolmogorov-Smirnov (KS) test. The KS test compares the cumulative density function (cdf) of normalized empirical data with a standard Gaussian cdf by noting the maximal difference between the two cdf's. Mathematically, the KS evaluation metric is expressed as:

$$KS = \max_x \left(|F(x) - G(x)|\right), \tag{10}$$

where $F(x)$ represents the proportion of the (normalized) experimental outcomes that are less than or equal to $x$, and $G(x)$ represents the standard Gaussian cdf evaluated at $x$.

The KS test results of the experimental data in Figure 2 are listed in Table I. That the KS values are very small confirms that $\ln(S)$ is very close to Gaussian and hence $S$ is very close to lognormal.

TABLE I

KS TEST VALUE FOR $\ln(S)$ WITH $k = 2, 5, 10, 100$

| | $KS$ test value |
|---|---|
| k = 2 | 0.0047345 |
| k = 5 | 0.0048123 |
| k = 10 | 0.0087541 |
| k = 100 | 0.0077152 |

*Proposition 3:* If $|X|$ follows an (approximate) Gaussian distribution, then $\Phi(X)$ in (7) follows an (approximate) lognormal distribution.

*Proof:* Consider an auxiliary function $\xi(z)$ defined for $z \geq 1$ as

$$\xi(z) = \ln\left(\frac{z+1}{z-1}\right), \quad z \geq 1, \tag{11}$$

Using Tailor expansions, $\xi(z)$ can be expressed as

$$\xi(z) = 2\sum_{k=1}^{\infty} \frac{1}{(2 \times k - 1)} z^{1-2\times k}. \tag{12}$$

Since $e^{|x|} \geq 1$, we substitute $z$ in (12) with $e^{|x|}$ and get

$$\Phi(X) = \xi(e^{|X|}) \tag{13}$$

$$= 2\sum_{k=1}^{\infty} \frac{1}{(2 \times k - 1)} e^{(1-2\times k)|X|} \tag{14}$$

$$= 2\,(e^{|X|})^{-1} + \frac{2}{3}\,(e^{|X|})^{-3} + \frac{2}{5}\,(e^{|X|})^{-5} + \ldots \tag{15}$$

Since $|X|$ is (approximately) Gaussian, $e^{|X|}$ satisfies an (approximate) lognormal distribution. Hence, according to Proposition 2, $\Phi(X)$, the power sum of lognormal random variable $e^{|X|}$ follows an (approximated) lognormal distribution. □

*Comment 1:* Since $|X| \geq 0$, $|X|$ cannot be exactly Gaussian. If $X$ is a Gaussian random variable such that $\Pr(X \geq 0) >> \Pr(X < 0)$ (or $\Pr(X \leq 0) >> \Pr(X > 0)$), then $|X|$ equals $X$ (or $-X$) most of the time and will follow the

Gaussian distribution closely. Hence, if we let a Gaussian random variable $X$, whose probability mass is primarily on one side of the origin, be the input to $\Phi(\cdot)$, then the output from $\Phi(\cdot)$ will follow a lognormal distribution.

*Proposition 4:* If $X$ ($X \geq 0$) follows a lognormal distribution, then $\Phi(X)$ will follow a Gaussian distribution.

*Proof:* Let $\Phi^{-1}(x)$ denote the inverse function for $\Phi(x)$. It is easy to verify that

$$\Phi^{-1}(x) = \ln(\frac{e^{|x|}+1}{e^{|x|}-1}) = \Phi(x). \tag{16}$$

Since the inverse function for $\Phi(X)$ is itself, and since Proposition 3 states that a Gaussian distribution at the input to $\Phi(X)$ will produce a lognormal distribution at the output, it follows that a lognormal distribution at the input to $\Phi(X)$ will produce a Gaussian distribution at the output. □

## IV. Accuracy of Gaussian Approximation

This section provides a theoretic validation for when and how the Gaussian assumption holds.

### A. Validation of Gaussian Assumption in Message-Passing

Since the transmit channel is symmetric and memoryless, i.e. $\Pr(r_i = q|x_i = 1) = \Pr(r_i = -q|x_i = -1)$, and since LDPC codes are linear codes, without loss of generality, we assume that the all-zero codeword, which is mapped to $s_i = +1$ for all $i$, is transmitted. Following the conventions, we assume the message-passing algorithm is operated on a Tanner graph with asymptotically unbounded girth. Hence, the messages passed through different edges from variable nodes to check nodes (as well as from check nodes to variable nodes) are considered to follow the same distribution but independent. We focus our discussion on additive white Gaussian noise (AWGN) channels.

Consider the variable node update in (1) and the check node update in (4). Initially, $m_{ij}^0 = m_i$ for all $i$ and $j$. According to (6), the LLR information $m_i$ extracted from the Gaussian channel is Gaussian distributed. Hence, the first set of messages, $m_{ij}^0$, passed from variable nodes to check nodes, follow a Gaussian density.

Now assuming that the messages exchanged at the $(\ell-1)$th iteration are Gaussian, we wish to show whether or when Gaussianity is preserved through the variable node update and the check node update in the $\ell$th iteration. We state our main results below:

*Theorem 1:* The outbound messages from check nodes to variable nodes at the $\ell$th iteration, $m_{ji}^{\ell}$, can preserve Gaussianity from the previous iteration, provided that the inbound messages are reasonably reliable and the degrees of check nodes are small.

*Proof:* Consider the check node update in (4). Since $+1$'s are transmitted, reasonably reliable inbound messages imply that majority of $m_{i'j}^{\ell-1}$'s take positive values. Following Proposition 3 and Comment 1, $|m_{i'j}^{\ell-1}|$ will then approximate a Gaussian distribution and so $\Phi(m_{i'j}^{\ell-1})$ will follow an (approximate) lognormal distribution. Further, $\Phi(m_{i'j}^{\ell-1})$'s are independent from each other, since $m_{i'j}^{\ell}$'s, transmitted through different edges, are independent. Now Proposition 2 states that only the sum of a small set of independent lognormal random variables will continue to be lognormal. Hence, $\sum_{i'\in N_c(j)\setminus\{i\}} \Phi(m_{i'j}^{\ell-1})$ will be lognormal when (and only when) the check node degree ($D_c$) is small. Finally, using Proposition 4 that lognormal distribution at the input to $\Phi(\cdot)$ makes the output Gaussian, we get that $\phi(\sum_{i'\in N_c(j)\setminus\{i\}} \Phi(m_{i'j}^{\ell-1}))$ and subsequently $m_{ji}^{\ell}$ follow Gaussian distributions.

The proof is best summarized as

$$m_{ji}^{\ell} = \Big(\prod_{i'\in N_v(j)\setminus\{i\}} \text{sign}(m_{i'j}^{\ell-1})\Big) \cdot \underbrace{\Phi\Big(\overbrace{\sum_{i'\in N_v(j)\setminus\{i\}} \overbrace{\Phi(\underbrace{m_{i'j}^{\ell-1}}_{Gaussian\ 1})}^{lognormal\ 2}}^{lognormal\ 3}\Big)}_{Gaussian\ 4}, \tag{17}$$

where from "Gaussian 1" to "lognormal 2" it requires $m_{i'j}^{\ell-1}$ to be a Gaussian r.v. with a small tailing probability (i.e. reliable messages), and from "lognormal 2" to "lognormal 3" it requires the terms in the summation to be small (i.e. small check degrees). □

*Theorem 2:* The outbound messages from variable nodes to check nodes at the $\ell$th iteration, $m_{ij}^{\ell}$, preserves Gaussianity from the previous iteration.

*Proof:* Since all the inbound messages, $m_{j'i}^{\ell}$, and the message extracted from the channel, $m_i$, are independent and Gaussian, and since the sum of independent Gaussian is also Gaussian, $m_{ij}^{\ell}$ is Gaussian.

Additionally, even when the inbound messages to the variable nodes $m_{j'i}^{\ell}$'s are not exactly Gaussian, as long as the variable node degree $D_v$ is large, $\sum_{j'\in N_v(i)\setminus\{j\}} m_{j'i}^{\ell}$ tends to be Gaussian according to the central limit theorem. Hence outbound messages from variable nodes help enforce the Gaussian assumption when the code rate is small. □

### B. Comments and Simulation Verifications

It is clear from Theorem 1 and Theorem 2 that in order for the message density to approximate a Gaussian distribution well, the following conditions need to be satisfied:

First, the messages passed along the edges are reasonably reliable to start with. In general, the message reliability improves with iterations (may stuck or saturate at a certain stage); but to ensure reliability in the first few iterations, the operating channel SNR needs to be reasonably high.

To demonstrate the impact of channel SNR on the message density, we demonstrate in Figure 3 the histograms of messages passed from check nodes to variable nodes during the first iteration. It is evident from the figure that the message density is close to Gaussian at high SNRs, but deviates severely from Gaussian as the SNR drops low.

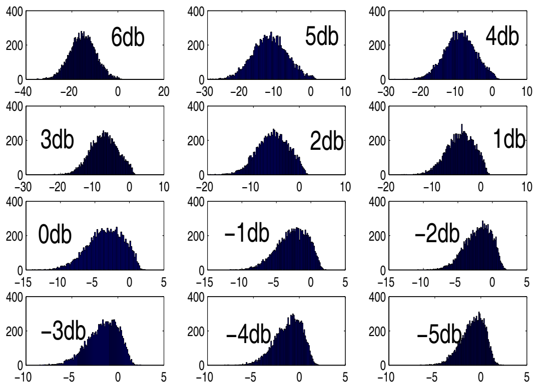


Fig. 3. Histogram of messages at different SNRs with code rate 1/2.

Second, the degrees of the check nodes cannot be large. A regular LDPC code with variable node degree of 3 will require check node degree to be 6 for rate $1/2$, 9 for rate $2/3$, 12 for rate $3/4$, 15 for rate $4/5$ and so on. The implication here is that the Gaussian approximation does work well for high-rate codes (such as rates above 0.8). To illustrate, Figure 4 plots the histograms of messages for different check node degrees at the channel SNR of 5db. A check node degree of 30 and above has clearly caused a large discrepancy from Gaussian density (at this SNR).

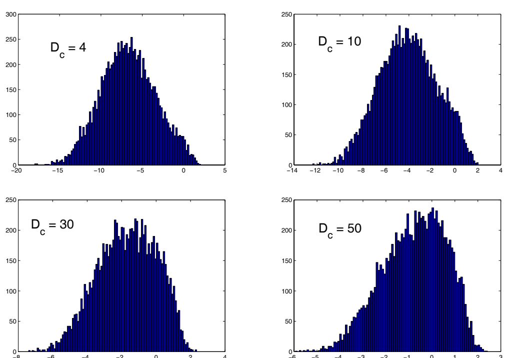


Fig. 4. Histograms of messages for different check node degrees at SNR=5db.

It should be noted that the two conditions we just discussed speak for different dimensions of the problem, and a favorable condition for one may mitigate the negative impact of the other. To evaluate the effect with both conditions combined, we show in Figure 5 the KS test values of the first round check node messages, $m_{ji}^1$, for different SNRs and check node degrees. A KS value below a threshold of 0.04, marked out in black horizontal line, indicates a close approximation to the Gaussian distribution. Not surprisingly, “a high SNR” points to different db values for different check node degrees. For a check degree of 4, 0db appears to be adequate, whereas for a check degree of 30, it takes 4.5db to start with to make the Gaussian assumption a reasonable one.

## V. Conclusion

While the prevailing assumption of Gaussian density, and the simplicity it brings to density evolution and EXIT charts,

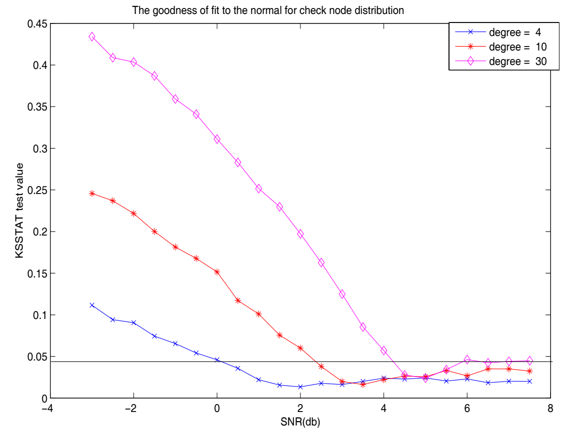


Fig. 5. Goodness-to-fit of the distribution of check node with Gaussian distribution

contribute substantially to the flourishing of iterative analysis, its theoretic justification is largely lacking. This paper fills the gap for LDPC codes by performing a theoretic analysis for when, how and how well the Gaussian distribution approximates the real message densities in message-passing decoding. The analytical results are verified by extensive simulations.

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