An Efficient Algorithm to Compute the Euclidean Distance Spectrum of a General Intersymbol Interference Channel and Its Applications

Jing Li, *Member, IEEE*, Krishna R. Narayanan, *Member, IEEE*, and Costas N. Georghiades, *Fellow, IEEE*

***Abstract*—We present an efficient algorithm to compute the distance spectrum of a general finite intersymbol interference (ISI) channel, whose complexity is lower than those of existing methods. Closed-form expressions are derived for both input–output Euclidean distance enumerators and asymptotic distance spectrum shapes for 2-tap and 3-tap ISI channels. Coded and/or precoded ISI channels are also discussed.**

***Index Terms*—Distance spectrum, input–output Euclidean distance enumerator (IOEDE), input–output weight enumerator, intersymbol interference (ISI) channel, precoding.**

I. Introduction

We investigate the distance spectrum of a finite intersymbol interference (ISI) channel which is interpreted as a nonregular, binary input, real-valued output trellis code. Although the knowledge of distance spectrum is much desired to understand the channel and to make use of tight bounds at low signal-to-noise ratios (SNRs) [1], [23] the computation is nontrivial.

There has been rich literature investigating error events and/or distance spectra of *linear* trellis codes. In terms of code space, *linearity* is very similar to the concept of *regularity* in [2], *superlinearity* in [3], and *geometrical uniformity* in [4]. The classic approach for this class of codes is to use the state diagram and transfer function. Other approaches include the one-step transition matrix [1], [23].

When the code is nonregular, such as the equivalent code of an ISI channel, the problem becomes considerably more complex, since the distance between an incorrect path and the correct path not only depends upon the error event, but also the correct path. For simple two-state channels, it is possible to label the edges of the state diagram as the average of two typical sequences (all zeros and all ones), and treat the "equivalent code" as if it were linear [18], [19], [21]. In the general case, a nonregular code requires the extension of either the state diagram or the transfer function. For instance, the generalized pairwise-state diagram in [6] requires $(2^m)^2$ states instead of 2^m states, as for regular trellis codes (m is the memory size). Alternatively, the number of states can remain unincreased, but the edge labels need to be square matrices of dimension $(2^m) \times (2^m)$ [7] rather than scalars. Similarly, if a one-step state transition matrix is used, it will be of dimensionality $(2^{2m}) \times (2^{2m})$ [8]. Due to the complexity involved, the practical use of these approaches have been limited.

Several papers are particularly worth mentioning in the literature on nonregular trellis codes. In a series of works by Rouanne *et al.* [10]–[13], *quasi-regularity*, a property similar to *symmetry* [14], was exploited to compute the distance spectrum assuming an arbitrary correct sequence. Since the reduction in complexity relies on (weak) symmetry conditions, the method is most useful for codes that satisfy the quasi-regular condition. Similarly, Trofimov and Kudryashov showed that the generating function could be obtained by inversion of a 2^m matrix, instead of a 2^{2m} matrix, if certain constraints were imposed [15].

In [5], a general method that applies to all trellis codes was developed. By mapping binary error events to ternary error events, Forney showed that the number of states can be reduced from 2^{2m} to $3^m/2$ without complicating edge labels. This method was further developed by Altekar *et al.* [9]. Another work by Raghavan *et al.* [18] paired states into cosets, and, by labeling the edges of the modified state diagram as the average of the nonlinear mapping, closed-form transfer functions for 2-tap and 3-tap ISI channels are derived. However, since the state diagram deals with *single error events* only, to derive the entire distance spectrum, concatenation of single error events (the number of single error events, their types, lengths, and positions) need to be resolved, which significantly increases the complexity, especially at large block sizes.

The only works known to the authors that have accounted for multiple error events for an ISI channel are [16] and [17] by Oberg and Siegel, where a dicode channel is analyzed. The approach exploited the specific characteristics of the dicode channel and, hence, is not extensible to the general case.

This letter presents a new way to compute the distance spectra for general ISI channels. Like [16] and [17], we work directly on *error sequences of length* N (i.e., concatenated error events); but unlike [16] and [17], the way we enumerate and evaluate error sequences is general and extensible. The main advantage of the proposed approach is its relative efficiency, compared with many existing methods. Specifically, for 2-tap and 3-tap channels of *any finite length* N, *as well as infinite length*, closed-form expressions of the entire distance spectra (normalized) are derived. For channels with larger memories, the approach is still applicable, but the algebra will be involved.

In the latter part of the letter, coded (and/or precoded) ISI channels are also investigated to show that knowledge of distance spectra can help understand ISI systems (under maximum-

Paper approved by A. K. Khandani, the Editor for Coding and Information Theory of the IEEE Communications Society. Manuscript received October 10, 2002; revised August 8, 2003 and May 7, 2004. This work was supported in part by a Grant from the Commonwealth of Pennsylvania, Department of Community and Economic Development, through the Pennsylvania Infrastructure Technology Alliance (PITA).

J. Li is with the Department of Computer and Electrical Engineering, Lehigh University, Bethlehem, PA 18015 USA (e-mail: jingli@ece.lehigh.edu).

K. R. Narayanan and C. N. Georghiades are with the Department of Electrical Engineering, Texas A&M University, College Station, TX 77840 USA (e-mail: krn@ee.tamu.edu; georghiades@ee.tamu.edu).

Digital Object Identifier 10.1109/TCOMM.2004.838700

likelihood decoding). The rest of the letter is organized as follows. Section II presents the preliminaries. Section III discusses the proposed method and presents the results for 2-tap and 3-tap channels. Section IV applies this method to coded/precoded ISI channels, and Section V concludes the paper.

II. System Model and Preliminary

We consider a finite ISI channel in its discrete form given by

$$H(D) = f_0 + f_1 D + \cdots + f_m D^m \tag{1}$$

where m is the memory size, $f_0 \neq 0, f_m \neq 0$, and $\sum_{i=0}^{m} f_i^2 = 1$. We assume that the input to the ISI channel is binary phase-shift keying (BPSK) modulated, such that $0 \rightarrow -1$, and $1 \rightarrow +1$.

Below are some definitions and preliminaries used in the letter (superscript $H(D)$ refers to the channel response and can be omitted where there is no ambiguity).

- $A_{w,d_E^2}^{H(D)}$: The input–output Euclidean distance enumerator (IOEDE); it denotes the average number of sequence pairs with input Hamming distance w and output squared Euclidean distance d_E^2, where the average is taken over all 2^N input sequences with equal probability.
- $A_{d_E^2}^{H(D)}$: The output Euclidean distance enumerator (OEDE); $A_{d_E^2} = \sum_w A_{w,d_E^2}$.
- $\gamma_N^{H(D)}(\delta)$: The Euclidean distance spectrum shape (with block size N): $\gamma_N(\delta) = (1/N)\log(A_{d_E^2 = 4\delta N(\sum_i |f_i|)^2})$.
- $\gamma_\infty^{H(D)}(\delta)$: The asymptotic Euclidean distance spectrum shape; $\gamma_\infty^{H(D)}(\delta) = \lim_{N\rightarrow\infty} \gamma_N^{H(D)}(\delta)$.

Proposition 1: [Transformation Rules:] Let m be the memory of the ISI channel with response $H(D)$ and s be any integer

$$\begin{aligned}
&[\text{Shift}] \quad A_{w,d_E^2}^{D^s H(D)} = A_{w,d_E^2}^{H(D)} \\
&[\text{Symmetry}] \quad A_{w,d_E^2}^{H(D)} = A_{w,d_E^2}^{-H(D)} = A_{w,d_E^2}^{H(-D)} \\
&[\text{Scaling}] \quad A_{w,d_E^2}^{H(D^s)} = A_{w,d_E^2}^{H(D)} \\
&[\text{Time Reversal}] \quad A_{w,d_E^2}^{D^m H(\frac{1}{D})} = A_{w,d_E^2}^{H(D)}.
\end{aligned}$$

Proof: Proof is omitted for the sake of brevity. These rules can help simplify computation as well as revealing certain properties of ISI channels. For example, the time-reversal rule justifies the assumption of $|f_0| \geq |f_m|$, and the symmetry rule indicates that for 2-tap channels, A_{w,d_E^2} is only dependent on $|f_0/f_1|$, but irrelevant to their respective signs.

III. Computing Euclidean Distance Spectra

A. General Idea

A simple idea to evaluate the distance spectra of ISI channels is to examine codeword pairs exhaustively. Here, we first fix an error sequence, evaluate its effect on the ensemble average of input sequences, and do it for all possible error sequences. Obviously, a brute-force application of this idea is prohibitively complex. The complexity reduction is made possible by two means: 1) enumerating all length-N error sequences via a simple classification and 2) computing the distance between an error sequence and the ensemble average of all input sequences by extending the nonlinear mapping method introduced in [18].

TABLE I
Effect of Error Event for 2-Tap ISI Channels

$E = [e_t e_{t-1}]$	$X = [x_t x_{t-1}]$	$d_E^2(X, E)$
00	00, 01, 10, 11	0
01	00, 01, 10, 11	$4f_1^2$
10	00, 01, 10, 11	$4f_0^2$
11	00, 11	$4(f_0 + f_1)^2$
	01, 10	$4(f_0 - f_1)^2$

Let $\bar{e} = [e_N, e_{N-1}, \ldots, e_2, e_1] \in \{0,1\}^N$ be an error sequence of length N (regardless of memory m), where "1" denotes a difference in position with the input sequence. Clearly, $\bar{e}$ is formed by alternating segments of *1-run*'s and *0-run*'s. It belongs to one of the following four categories: I starts with *1-run* and ends with *0-run*, II starts with *1-run* and ends with *1-run*, III starts with *0-run* and ends with *1-run*, IV starts with *0-run* and ends with *0-run*.

Error sequences in the same category can be characterized in a unified way, permitting an efficient enumeration of all possible error sequences (which will be illustrated through examples). The complexity here is independent of channel memory m.

To examine the effect of an error sequence on the ensemble average of input sequences, we make use of the $(m+1)$*-tuple nonlinear mappings* for an ISI channel of memory size m [18]. The idea is actually simple (and will be explained through examples), but now the complexity increases with channel memory m (at the speed of, at the most, $O(2^m)$). For 2-tap and 3-tap channels, the complexity is moderate (at both finite and infinite lengths). Channels with longer memories incur a higher complexity, but it is still simpler than existing methods.

The best way to discuss the proposed method in detail is through the examples of 2-tap and 3-tap channels.

B. Example: A General 2-Tap ISI Channel

1) Finite Length N: The discrete channel model of a 2-tap ISI channel is given by $H(D) = f_0 + f_1 D$, where $f_0 \neq 0, f_1 \neq 0, |f_0| \geq |f_1|$, and $f_0^2 + f_1^2 = 1$. The nonlinear mapping of 2-tuples for this channel can be found in Table I [18].

An error sequence belongs to one of the four categories as mentioned before. For case I, we have

$$\bar{e} = \overbrace{0\cdots0}^{\beta_k}\underbrace{1\cdots1}_{\alpha_k}\overbrace{0\cdots0}^{\beta_{k-1}}\cdots\overbrace{0\cdots0}^{\beta_1}\underbrace{1\cdots1}_{\alpha_1}\ \overbrace{(0)}^{\text{preamble}} \tag{2}$$

where w is the (input) Hamming weight, and positive integers α_i and β_i denote the lengths of *1-run*'s and *0-run*'s such that $\sum_{i=1}^{k} \alpha_i = w$ and $\sum_{i=1}^{k} \beta_i = N - w$.

Let us first count the number of 2-tuples in $\bar{e}$ that contribute to nonzero Euclidean distances, which, according to Table I, are (01), (10), and (11). The computation is simple; e.g., since each *0-run* followed by a *1-run* generates one (01) tuple, there are altogether $k\,(01)$ tuples. Likewise, there are $k\,(10)$ tuples (the preamble is a *0-run*), and $\sum_{\alpha_i}(\alpha_i - 1) = w - k\,(11)$ tuples. Since an (11) tuple yields a squared Euclidean distance of either $4(f_0+f_1)^2$ or $4(f_0-f_1)^2$, let us use p and $(w-k-p)$ to denote the number of (11) tuples that result in distance $4(f_0+f_1)^2$ and

$4(f_0-f_1)^2$, respectively. The squared Euclidean weight of the whole sequence is, thus, given by

$$\begin{aligned} d_E^2 &= 4kf_1^2+4kf_0^2+4p(f_0+f_1)^2 \\ &\quad +4(w-k-p)(f_0-f_1)^2 \\ &= 4w+8f_0f_1(2p-w+k). \end{aligned} \tag{3}$$

For a fixed k, there are

$$\binom{w-1}{k-1}\binom{N-w-1}{k-1}$$

sequences with length N and weight w, starting with *1-run* and ending with *0-run*. Rewriting (3) as $p=(d_{E^2}/4-w+2f_0f_1(w-k))/(4f_0f_1)$, we can get the total number of error sequences with input Hamming weight w and average output square Euclidean distance d_E^2

$$A^{\mathrm{I}}_{w,d_E^2}=\sum_{k=1}^{w}2^{k-w}\binom{w-k}{p}\binom{w-1}{k-1}\binom{N-w-1}{k-1}. \tag{4}$$

Since all input sequences are taken with equal probability, the value of A^I_{w,d_E} may or may not be an integer.

The other three cases can be evaluated in a similar way

$$A^{\mathrm{II}}_{w,d_E^2}=\sum_{k=1}^{w}2^{k-w}\binom{w-k}{p_1}\binom{w-1}{k-1}\binom{N-w-1}{k-2} \tag{5}$$

$$A^{\mathrm{III}}_{w,d_E^2}=\sum_{k=1}^{w}2^{k-w}\binom{w-k}{p_1}\binom{w-1}{k-1}\binom{N-w-1}{k-1} \tag{6}$$

$$A^{\mathrm{IV}}_{w,d_E^2}=\sum_{k=1}^{w}2^{k-w}\binom{w-k}{p_2}\binom{w-1}{k-1}\binom{N-w-1}{k} \tag{7}$$

where $p_2=(d_{E^2}/4-w+2f_0f_1(w-k))/(4f_0f_1)$ and $p_1=p2+f_1/(4f_0)$. Combining all four cases leads to the following result.

Proposition 2: The IOEDE of a general 2-tap ISI channel with channel response $H(D)=f_0+f_1D$, where $f_0^2+f_1^2=1$ and $|f_0|\geq|f_1|$, is given by

$$A^{2\text{-tap}}_{w,d_E^2}=\sum_{k=1}^{w-1}2^{k-w}\binom{w-k}{p}\binom{w-1}{k-1}\binom{N-w}{k-I} \tag{8}$$

where N is the sequence length, $I=0$ or $1, p=(d_E^2-4w+4If_1^2)/(16f_0f_1)+(w-k)/2$, and valid output squared Euclidean distances take the form of $d_E^2=4w+8jf_0f_1-4If_1^2$, where $j=-(w-1),-(w-3),-(w-5),\ldots,(w-3),(w-1)$.

2) Asymptotic Case: The asymptotic spectrum shape can be derived from the IOEDE using the Stirling's formula [22]. First, the OEDE of a 2-tap channel can be derived from $A^{2\text{-tap}}_{w,d_E^2}$

$$A^{2\text{-tap}}_{d_E^2}=\sum_{w=1}^{N}\sum_{k=1}^{w-1}2^{k-w}\binom{w-k}{p}\binom{w-1}{k-1}\binom{N-w}{k-I} \tag{9}$$

which leads to

$$A^{2\text{-tap}}_{d_E^2}\leq N^2\max_{w,k}2^{k-w}\binom{w-k}{p}\binom{w-1}{k-1}\binom{N-w}{k-I} \tag{10}$$

$$A^{2\text{-tap}}_{d_E^2}\geq \max_{w,k}2^{k-w}\binom{w-k}{p}\binom{w-1}{k-1}\binom{N-w}{k-I} \tag{11}$$

where $I=0$ or 1, and $p=(d_E^2-4w+4f_1^2I)/(16f_0f_1)+(w-k)/2$.

Recall the following fact (see, for example, [22]):

$$\lim_{N\to\infty}\frac{1}{N}\log\binom{\alpha N+a}{\beta N+b}=\alpha H\left(\frac{\beta}{\alpha}\right) \tag{12}$$

where $0\leq\alpha\leq1, 0\leq\beta\leq1, a$ and b are arbitrary numbers, and $H(x)=-x\log x-(1-x)\log(1-x)$. Defining $u\triangleq w/N\in[0,1], v\triangleq k/N\in[0,u]$ and $\delta=d_E^2/(4N(|f_0|+|f_1|)^2)$, we have $w=uN, k=vN, d_E^2=4\delta(|f_0|+|f_1|)^2N$, and $p_1=((\delta(|f_0|+|f_1|)^2-u)/(4f_0f_1)+(u-v)/2)N+If_0/f_1$.

Substituting them to the right-hand side of (10), we have

$$\begin{aligned} &\gamma_\infty^{2\text{-tap}}(\delta) \\ &\quad\leq \lim_{N\to\infty}\frac{1}{N}\log\left(A^{2\text{-tap}}_{w,d_E^2=4\delta N(|f_0|+|f_1|)^2}\right) \\ &\quad=\max_{0\leq v\leq u\leq 1}\left\{uH\left(\frac{v}{u}\right)+(1-u)H\left(\frac{v}{1-u}\right)\right. \\ &\qquad\left.+(u-v)H\left(\frac{1}{2}+\frac{\delta(|f_0|+|f_1|)^2-u}{4f_0f_1(u-v)}-\log 2\right)\right\}. \end{aligned} \tag{13}$$

On the other side, from (11), we get

$$\begin{aligned} \gamma_\infty^{2\text{-tap}}(\delta)\geq \max_{0\leq v\leq u\leq 1}&\left\{uH\left(\frac{v}{u}\right)+(1-u)H\left(\frac{v}{1-u}\right)\right. \\ &\left.+(u-v)H\left(\frac{1}{2}+\frac{\delta(|f_0|+|f_1|)^2-u}{4f_0f_1(u-v)}-\log 2\right)\right\}. \end{aligned} \tag{14}$$

Combining (10), (13), and (14), we have the following.

Proposition 3: The Euclidean distance spectrum shape of a general 2-tap ISI channel is given by ($\delta\triangleq(d_E^2)/(4N(|f_0|+|f_1|)^2)\in[0,1]$)

$$\begin{aligned} &\gamma_\infty^{2\text{-tap}}(\delta) \\ &\quad=\max_{0\leq v\leq u\leq 1}\left\{uH\left(\frac{v}{u}\right)+(1-u)H\left(\frac{v}{1-u}\right)\right. \\ &\qquad\left.+(u-v)\left(H\left(\frac{1}{2}+\frac{\delta(|f_0|+|f_1|)^2-u}{4f_0f_1(u-v)}\right)-\log 2\right)\right\} \end{aligned} \tag{15}$$

$$\begin{aligned} &\gamma_N^{2\text{-tap}}(\delta) \\ &\quad\leq\frac{2\log N}{N}+\gamma_\infty^{2\text{-tap}}(\delta). \end{aligned} \tag{16}$$

We note that the upper bound in (16) is actually quite loose. Empirical results (Fig. 1) show that the spectrum shape of a finite-length ISI channel, $\gamma_N(\delta)$, monotonously increases with N (for valid values of δ) and converges to the asymptotic case of $N=\infty$. This suggests that $\gamma_N^{2\text{-tap}}(\delta)\leq\gamma_\infty(\delta)$.

Fig. 2 plots the spectrum shapes of several 2-tap ISI channels. We see that the dicode channel (or the PR1 channel) has

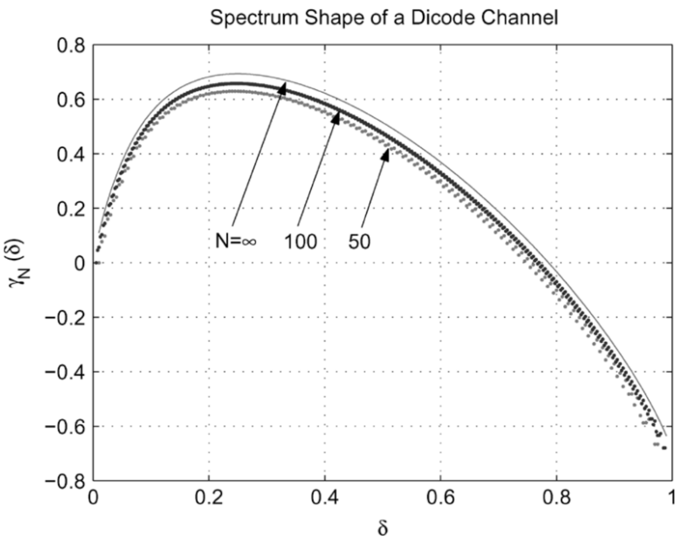

Fig. 1. Comparing the spectrum shapes of dicode channels with finite length and infinite length.

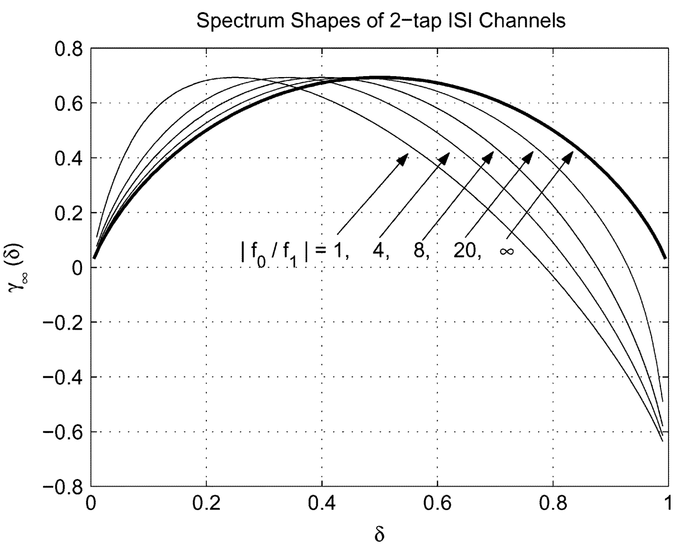

Fig. 2. Spectrum shapes of 2-tap ISI channels with different amounts of ISI.

the worst spectrum shape, due to severe ISI. As ISI decreases ($|f_0/f_1|$ increases), there is a noticeable improvement in spectrum shape in that the portion of the low distance end drops. In the limiting case where there is no ISI ($|f_0/f_1| = \infty$), we expect the spectrum shape to converge to that of the rate-1 binary uniform random code (i.e., $H(\delta)$).

3) Complexity Comparison: Among the many existing methods, the ones based on extended state diagrams like [6]–[8] are applicable to general trellis codes, but the complexity is very high. Others that are of lower complexity, including [10]–[17], exploit certain properties/constraints and are useful only to particular code sets. To give a feel of the relative complexity of the proposed method, we refer to [18], where closed-form transfer functions for 2-tap and 3-tap ISI channels are derived using an efficient approach applicable to general ISI cases.

The main result in [18] on 2-tap ISI channels is the average transfer function (for single error events) given below [18]

$$T(w, d_E, L) = \frac{4L^2 d_E^{\alpha_1+\alpha_2} w}{1 - L\left(d_E^{\alpha_3} + d_E^{\alpha_4}\right) w} \tag{17}$$

where $\alpha_1 = 4f_0^2, \alpha_2 = 4f_1^2, \alpha_3 = 4(f_0+f_1)^2, \alpha_4 = 4(f_0-f_1)^2$, and w, d_E, and L denote the input Hamming distance, the output Euclidean distance, and the error-event length. Multiple error events formed from k single error events (disregarding block size) can be conveniently obtained by raising $T(d_E, L, w)$ to the power of k; but to account for the fact that these k single error events need to occur within a block of size N, the computation immediately becomes involved. First, long division needs to be performed on (17) to obtain all coefficients (denoted as $A^{(1)}_{w,d_E,L}$). Second, all valid concatenations of k single error events need to be enumerated subject to their positions, lengths, and input/output weights. Finally, multiple error events having the same input–output weights (but different combined error-event lengths) need to be combined. This is mathematically expressed as shown in (18) and (19) at the bottom of the page. Note that the computation of $A^{(k)}_{w,d_E,L}$ in (19) involves a search for a set of k points in a three-dimensional space of size $(N/2)^3$, subject to three constraints. This requires a complexity of $O((N/2)^3)$. To obtain $A^{2-\text{tap}}_{w,d_E}$, another two levels of summation are needed, making the overall complexity on the order of $(N/2)^5$, which is computationally prohibitive for large lengths. Comparatively, (8) requires only linear complexity in N, which is significantly simpler. This allows the evaluation of any finite-length and infinite-length case, which is not possible with [18] (and many other approaches).

C. Example: A General 3-Tap ISI Channel

A general 3-tap ISI channel can be treated in a similar way. Due to the space limitation, we only present the results here. To ease exposition, we introduce a new operation $x = \phi(\Delta; \delta_1, \delta_2, \ldots)$ which is defined as "$x = \Delta + \delta_i$, if $\Delta + \delta_i$ is an integer, and x is not defined otherwise."

Proposition 4: The average IOEDE of a general 3-tap channel with the channel response $H(D) = f_0 + f_1 D + f_2 D^2$,

$$A^{2-\text{tap}}_{w,d_E} = \sum_{L=1}^{N/2} \sum_{k=1}^{N/2} \binom{N-L+k}{k} k!\, A^{(k)}_{w,d_E,L} \tag{18}$$

$$\text{where } A^{(k)}_{w,d_E,L} = \underset{L_1+L_2+\cdots+L_k=L,\, w_1+w_2+\cdots+w_k=w,\, d^2_{E_1}+d^2_{E_2}+\cdots+d^2_{E_k}=d^2_E}{\prod_{i=1}^{k}} A^{(1)}_{w_i,d_{E_i},L_i} \tag{19}$$

where $f_0 \neq 0, f_2 \neq 0, |f_0| \geq |f_2|$ and $\sum_{i=0}^{2} f_i^2 = 1$, is given by

$$
\begin{aligned}
&A_{w,d_E^2}^{3-\mathrm{tap}} \\
&\quad = \sum_{k=1}^{w-1}\sum_{t=0}^{k-1}\sum_{s=0}^{k-1}\sum_{p_1=0}^{k-t}\sum_{p_2=0}^{k-t}\sum_{p_3+p_4+p_5+p_6=w+t-2k} D \\
&\qquad \cdot 2^{t-w-s}\frac{(w+t-2k)!}{p_3!p_4!p_5!p_6!}\binom{k-t}{p_1}\binom{k-t}{p_2}\binom{s}{q} \\
&\qquad \cdot \binom{w-k-1}{k-t-1}\binom{k}{t}\binom{N-w-k+1}{k-s-1}\binom{k-1}{s}
\end{aligned}
$$

where

$$
\begin{aligned}
D &= 1 + ((A-B+2)(A-B+1)(A+B))/(AB(A+1)) \\
q &= \phi((d_E^2-4w)/(16f_0f_2) + (s-C_3)/(2) \\
&\quad - (f_1(t-k+2p_2+C_1))/(2f_2) \\
&\quad - (f_1(t-k+2p_1+C_2))/(2f_0); 0 \\
&\quad (f_2^2)/(4f_0f_2) \\
&\quad - (f_0^2)/(4f_0f_2) \\
&\quad (f_2^2 - f_0^2)/(4f_0f_2) \\
&\quad (f_1^2 + f_2^2)/(4f_0f_2) \\
&\quad (f_2^2 + (f_1-f_2)^2)/(4f_0f_2) \\
&\quad (f_1^2 + 2f_2^2)/(4f_0f_2) \\
&\quad ((f_1-f_2)^2 + 2f_2^2)/(4f_0f_2) \\
&\quad (f_1^2 + f_2^2 + (f_0-f_2)^2)/(4f_0f_2) \\
&\quad (f_2^2 + (f_0-f_2)^2 + (f_1-f_2)^2)/(4f_0f_2)) \\
A &= N - w - k \\
B &= k - s \\
C_1 &= p_3 + p_4 - p_5 - p_6 \\
C_2 &= p_3 - p_4 - p_5 + p_6 \\
C_3 &= p_3 - p_4 + p_5 - p_6.
\end{aligned}
$$

Proposition 5: The Euclidean distance spectrum shape of a 3-tap ISI channel is given by

$$
\begin{aligned}
\gamma_\infty^{3-\mathrm{tap}}(\delta) = \max_{\mathcal{R}} \Bigg\{ & (\rho_1 - \rho_2 - u)\log 2 \\
& + v\left(H\left(\frac{\rho_1}{v}\right) + H\left(\frac{\rho_2}{v}\right)\right) \\
& + (v-\rho_1)\left(H\left(\frac{\tau_1}{v-\rho_1}\right) + H\left(\frac{\tau_2}{v-\rho_1}\right)\right) \\
& + (u-v)H\left(\frac{v-\rho_1}{u-v}\right) \\
& + (1-u-v)H\left(\frac{v-\rho_2}{1-u-v}\right) + \rho_2 H\left(\frac{\kappa}{\rho_2}\right) \\
& + (u+\rho_1-2v) \\
& \times H\left(\frac{\tau_3}{u+\rho_1-2v}, \frac{\tau_4}{u+\rho_1-2v}, \right. \\
& \left. \frac{\tau_5}{u+\rho_1-2v}, \frac{\tau_6}{u+\rho_1-2v}\right)\Bigg\} \\
\gamma_N^{3-\mathrm{tap}}(\delta) \leq \frac{9\log N}{N} & + \gamma_\infty^{3-\mathrm{tap}}(\delta)
\end{aligned}
$$

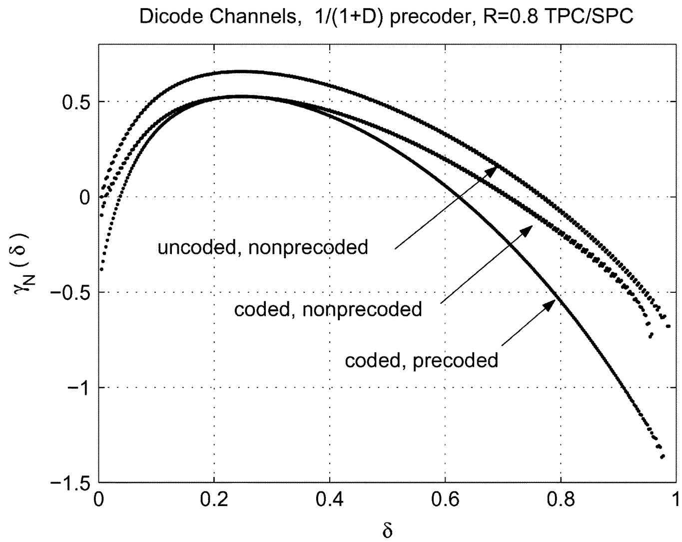

Fig. 3. Spectrum shapes of uncoded dicode channels. 2-D TPC/SPC code with rate $R = 0.8, N = 100$, precoder $1/(1 \oplus D)$.

where $\delta \triangleq d_E^2/(4N(|f_0|+|f_1|+|f_2|)^2)$, $\kappa = (\delta(|f_0|+|f_1|+|f_2|)^2 - u)/(4f_0f_2) + (u - 2v + \rho_1 + \rho_2 + 2\tau_3 + 2\tau_5)/2 + f_1(-u+v+2\tau_2+2\tau_3+2\tau_4)/(2f_2) + f_1(u-3v+2\rho_1+2\tau_2-2\tau_4-2\tau_5)/(2f_2)$, and the max operation is taken over the region $\mathcal{R} \triangleq \{0 \leq \frac{\rho_1}{\rho_2} \leq v \leq u \leq 1, 0 \leq \frac{\tau_1}{\tau_2} \leq v - \rho_1, \tau_3 + \tau_4 + \tau_5 + \tau_6 = u + \rho_1 - 2v\}$.

IV. Coded ISI Channels

Coded/precoded ISI channels are of more interest in practical systems. Below, we discuss a few examples to show how the results derived above can help understand such systems.

The overall distance spectrum of coded ISI systems can be computed using the weight/distance enumerators of the channel code and of the ISI channel (assuming uniform interleaver)

$$
A_{d_E^2}^{\mathrm{coded-ISI}} = \sum_{l=1}^{n}\frac{A_l^{(\mathrm{code})}A_{l,d_E}^{(\mathrm{ISI})}}{\binom{N}{l}} = \sum_{l=1}^{n}\sum_{w=1}^{k}\frac{A_{w,l}^{(o)}A_{l,d_E^2}^{(\mathrm{ISI})}}{\binom{N}{l}}.
$$

For simple codes like single-parity check (SPC) codes, Hamming codes and SPC turbo product codes (TPC/SPC) [20], [21], $A_h^{(\mathrm{code})}$'s can be easily computed (details omitted).

1) Coding vs. Precoding: Fig. 3 demonstrates the spectrum shapes of TPC/SPC-coded dicode channels, both precoded and nonprecoded. The figure clearly shows the different roles an error-correction code (ECC) and a precoder assume on ISI channels. An ECC, through "code-space expansion" (a 2^K space to a 2^N space), *uniformly* increases the distances among *all* valid codewords, whereas a precoder affects primarily the *short-distance codeword pairs* by mapping them to high-distance pairs (a phenomenon known as "spectrum thinning").

2) Effect of Code Rate and Precoding: Fig. 4 shows the spectrum shapes of ISI channels coded by SPC codes of different rates. We see that the lower the code rate, the more spaced apart the codewords and the better the spectrum shape. The case of $R = 0.8$ (20 blocks of (5, 4) SPC codewords combined together) is particularly interesting. Since the all-ones sequence

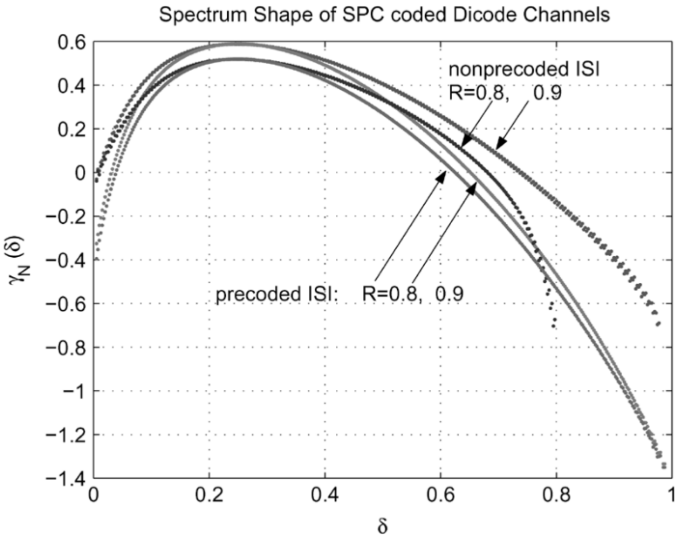

Fig. 4. Spectrum shapes of SPC-coded dicode channels. Either 20 blocks of (5, 4) SPC codewords or 10 blocks of (10, 9) codewords are concatenated.

is not a valid codeword in this case, the normalized Hamming weight of the SPC codeword, and subsequently, the normalized squared Euclidean distance of the output sequence from the SPC-coded nonrecursive ISI channel, can only reach $\delta = 0.8$ (instead of $\delta = 1$). However, when the channel is recursive, the spectrum-thinning effect makes it possible for some sequences to reach $\delta = 1$. This again shows the impact of a precoder on a coded ISI channel.

V. Conclusion

We have proposed an efficient way to compute the Euclidean distance spectrum of a general ISI channel. Closed-form expressions are derived for 2-tap and 3-tap channels of any finite length, as well as infinite length. Examples of coded and/or precoded systems are also discussed to show the usefulness of such knowledge.

References

[1] D. Divsalar, "A simple tight bound on error probability of block codes with application to turbo codes," *IEEE Trans. Commun.*, submitted for publication.

[2] A. R. Calderbank and N. J. A. Sloane, "New trellis codes based on lattices and cosets," *IEEE Trans. Inform. Theory*, vol. IT-33, pp. 177–195, Mar. 1987.

[3] S. Benedetto, M. A. Marsan, G. Albertengo, and E. Giachin, "Combined coding and modulation: Theory and applications," *IEEE Trans. Inform. Theory*, vol. 34, pp. 223–236, Mar. 1988.

[4] G. D. Forney, Jr., "Geometrically uniform codes," *IEEE Trans. Inform. Theory*, vol. 37, pp. 1241–1260, Sept. 1991.

[5] ——, "Maximum likelihood sequence estimation of digital sequences in the presence of intersymbol interference," *IEEE Trans. Inform. Theory*, vol. IT-18, pp. 363–378, May 1972.

[6] E. Biglieri, "High-level modulation and coding for nonlinear satellite channels," *IEEE Trans. Commun.*, vol. COM-32, pp. 616–626, May 1984.

[7] Y. J. Liu, I. Oka, and E. Biglieri, "Error probability for digital transmission over nonlinear channels with application to TCM," *IEEE Trans. Inform. Theory*, vol. 36, pp. 1101–1110, Sept. 1990.

[8] T. M. Duman and E. Kurtas, "Performance bounds for high rate linear codes over partial-response channels," *IEEE Trans. on Inform. Theory*, vol. 47, pp. 1201–1205, Mar. 2001.

[9] S. A. Altekar, M. Berggren, B. E. Moision, P. H. Siegel, and J. K. Wolf, "Error-event characterization on partial-response channels," *IEEE Trans. Inform. Theory*, vol. 45, pp. 241–247, Jan. 1999.

[10] M. Rouanne and D. J. Costello, "An algorithm for computing the distance spectrum of trellis codes," *IEEE J. Select. Areas Commun.*, vol. 7, pp. 929–940, Aug. 1989.

[11] ——, "A lower bound on the minimum Euclidean distance of trellis-coded modulation schemes," *IEEE Trans. Inform. Theory*, vol. 34, pp. 1011–1020, Sept. 1995.

[12] C. Schlegel and D. J. Costello, "Bandwidth efficient coding for fading channels: Code construction and performance analysis," *IEEE J. Select. Areas Commun.*, vol. 7, pp. 1356–1368, Dec. 1989.

[13] C. Schlegel, "Evaluating distance spectra and performance bounds of trellis codes on channels with intersymbol interference," *IEEE Trans. Inform. Theory*, vol. 37, pp. 627–634, May 1995.

[14] E. Zehavi and J. K. Wolf, "On the performance evaluation of trellis codes," *IEEE Trans. Inform. Theory*, vol. IT-33, pp. 196–202, Mar. 1987.

[15] A. N. Trofimov and B. D. Kudryashov, "Distance spectra and upper bounds on error probability for trellis codes," *IEEE Trans. Inform. Theory*, vol. 41, pp. 561–572, Mar. 1995.

[16] M. Oberg and P. H. Siegel, "Performance analysis of the turbo-equalized dicode partial-response channel," in *Proc. Allerton Conf. Communications, Control, Computing*, Sept. 1998, pp. 230–239.

[17] ——, "Performance analysis of turbo-equalized partial response channel," *IEEE Trans. Commun.*, vol. 49, pp. 436–444, Mar. 2001.

[18] S. A. Raghavan, J. K. Wolf, and L. B. Milstein, "On the performance evaluation of ISI channels," *IEEE Trans. Inform. Theory*, vol. 39, pp. 957–965, May 1993.

[19] L. L. McPheters, S. W. McLaughlin, and K. R. Narayanan, "Precoded PRML, serial concatenation, and iterative (turbo) decoding for digital magnetic recording," *IEEE Trans. Magn.*, vol. 35, pp. 2325–2327, Sept. 1999.

[20] G. Caire and C. Taricco, "Weight distribution and performance of the iterated product of single parity-check codes," in *Proc. GLOBECOM*, 1994, pp. 206–211.

[21] J. Li, K. R. Narayanan, E. Kurtas, and C. N. Georghiades, "On the performance of high-rate TPC/SPC codes and LDPC codes over partial-response channels," *IEEE Trans. Commun.*, vol. 50, pp. 723–734, May 2002.

[22] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. New York: Wiley, 1991.

[23] D. Divsalar, "A simple tight bound on error probability of block codes with application to turbo codes," JPL TMO Prog. Rep. 42–139, Nov. 1999.