Minimum Probability of Error for Asynchronous Gaussian Multiple-Access Channels

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Abstract—**Consider a Gaussian multiple-access channel shared by K users who transmit asynchronously independent data streams by modulating a set of assigned signal waveforms. The uncoded probability of error achievable by optimum multiuser detectors is investigated. It is shown that the K-user maximum-likelihood sequence detector consists of a bank of single-user matched filters followed by a Viterbi algorithm whose complexity per binary decision is $O(2^K)$. The upper bound analysis of this detector follows an approach based on the decomposition of error sequences. The issues of convergence and tightness of the bounds are examined, and it is shown that the minimum multiuser error probability is equivalent in the low-noise region to that of a single-user system with reduced power. These results show that the proposed multiuser detectors afford important performance gains over conventional single-user systems, in which the signal constellation carries the entire burden of complexity required to achieve a given performance level.**

I. Introduction

CONSIDER a Gaussian multiple-access channel shared by K users who modulate simultaneously and independently a set of assigned signal waveforms without maintaining any type of synchronism among them. The coherent K-user receiver commonly employed in practice consists of a bank of optimum single-user detectors operating independently (Fig. 1). Since in general the input to every threshold has an additive component of multiple-access interference (because of the cross correlation with the signals of the other users), the conventional receiver is not optimum in terms of error probability. However, if the designer is allowed to choose a signal constellation with large bandwidth (e.g., in direct-sequence spread-spectrum systems), then the cross correlations between the signals can be kept to a low level for all relative delays, and acceptable performance can be achieved. Nevertheless, if data demodulation is restricted to single-user detection systems, then the cross-correlation properties of the signal constellation carry the entire burden of complexity required to achieve a given performance level, and when the power of some of the interfering users is dominant, performance degradation is too severe. For this reason and because of the availability of computing devices with increased capabilities, there is recent interest in investigating the degree of performance improvement achievable with more sophisticated receivers and, in particular, with the minimum error probability detector.

Optimum multiuser detection of asynchronous signals is inherently a problem of sequence detection, that is, observation of the whole received waveform is required to produce a sufficient statistic for any symbol decision, and hence one-shot approaches (where the demodulation of each symbol takes into account the received signal only in the interval corresponding to that symbol) are suboptimal. The reason is that the observation of the complete intervals of the overlapping symbols of the other users gives additional information about the received signal in the bit interval in question, and since this reasoning can be repeated with the overlapping bits, no restriction of the whole observation interval is optimal for any bit decision. Furthermore, since the transmitted symbols are not independent conditioned on the received realization, decisions can be made according to two different optimality criteria, namely, selection of the sequence of symbols that maximizes the joint posterior distribution (maximum-likelihood sequence detection), or selection of the symbol sequence that maximizes the sequence of marginal posterior distributions (minimum-probability-of-error detection). Moreover, the simultaneous demodulation of all the active users in the multiple-access channel can be regarded as a problem of periodically time-varying intersymbol interference, because from the viewpoint of the coherent K-user detector, the observed process is equivalent to that of a single-user-to-single-user system where the sender transmits K symbols during each signal period by modulating one out of K waveforms in a round-robin fashion.

Earlier work on multiuser detection includes, in the case of synchronous users (which reduces to an m-ary hypothesis testing problem) the receivers of Horwood and Gagliardi [9] and Schneider [16], and, in the asynchronous case, the one-shot baseband detector obtained by Poor [12] in the two-user case, and the detectors proposed by Van Etten [18] and Schneider [16] for interference-channel models with vector observations. Results on the probability of error have been obtained only for the conventional single-user receiver ([8], [14], and the references therein).

In Section II we obtain a K-user maximum-likelihood sequence detector which consists of a bank of K single-user matched filters followed by a Viterbi forward dynamic

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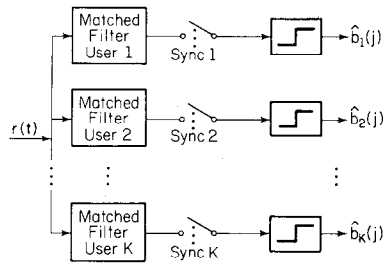


Fig. 1. Conventional multiuser detector.

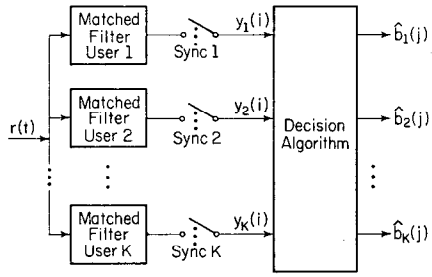


Fig. 2. Optimum K-user detector for asynchronous multiple-access Gaussian channel.

programming algorithm (Fig. 2) with 2^{K-1} states and $O(2^K)$ time complexity per bit (in the binary case). Section III is devoted to the analysis of the minimum uncoded bit error rate of multiuser detectors. This is achieved through various bounds that together provide tight approximations for all noise levels. A possible route to upper bound the error probability of the multiuser maximum-likelihood sequence detector is to generalize the approach taken by Forney in the intersymbol interference problem [3]. However, motivated by the more general structure of the multiuser problem, we introduce a different approach that, when applied to the intersymbol interference case, turns out to result in a bound that is tighter than the Forney bound. In Section IV several numerical examples illustrate the performance gains achieved by optimum multiuser detectors over conventional single-user systems.

II. Multiuser Maximum-Likelihood Sequence Detection

In this section we derive optimum decision rules for the following multiple-access model with additive linearly modulated signals in additive white Gaussian noise and scalar observations:

$$dr_t = S_t(\boldsymbol{b})\, dt + \sigma\, d\omega_t, \qquad t \in R \tag{1}$$

where

$$S_t(\boldsymbol{b}) = \sum_{i=-M}^{M} \sum_{k=1}^{K} b_k(i) s_k(t - iT - \tau_k), \tag{2}$$

the symbol interval duration is equal to T (assumed to be the same for all users), $\boldsymbol{b} = \{b(i) \in A_1 \times \cdots \times A_K,\ i = -M, \cdots, M\}$, and $A_k, s_k(t)$ $(= 0$ outside $[0, T])$, and $\tau_k \in [0, T)$ are the finite alphabet, the signal waveform, and the delay (modulo T with respect to an arbitrary reference), respectively, of the kth user, and ω_t is a standard Wiener process started at $t = -MT$. Without loss of generality, and for the sake of notational simplicity, we suppose that the users are numbered such that $0 \le \tau_1 \le \cdots \le \tau_K < T$. Note that even if all the transmitted symbols are assumed to be equiprobable and independent, there is not a unique optimality criterion due to the existence of several users. It is possible to select either the set of symbols that maximize the joint posterior distribution $P[\boldsymbol{b}|\{r_t,\ t \in R\}]$ (globally optimum or maximum-likelihood sequence detection) or those that maximize the marginal posteriori distributions $P[b_k(i)|\{r_t,\ t \in R\}]$, $i = -M, \cdots, M$, $k = 1, \cdots, K$ (locally optimum or minimum-error-probability detection). It is shown later that the maximum-likelihood sequence detector can be implemented by a signal processing front end that produces a sequence of scalar sufficient statistics, followed by a dynamic programming decision algorithm of the forward (Viterbi) type. It can be shown [22] that the multiuser detector that minimizes the probability of error has the same structure, but it uses a backward–forward dynamic programming algorithm instead [21]. The computational complexity of the various decision algorithms will be measured and compared by their *time complexity per binary decision* (TCB), that is, the limit as $M \to \infty$ of the time required by the decision algorithm to select the optimum sequence divided by the number of transmitted bits.

Since all transmitted sequences of symbols are assumed to be equiprobable, the maximum-likelihood sequence detector selects the sequence that maximizes

$$P[\{r_t,\ t \in R\}|\boldsymbol{b}] = C \exp\left(\Omega(\boldsymbol{b})/2\sigma^2\right) \tag{3}$$

where C is a positive scalar independent of $\boldsymbol{b}$ and

$$\Omega(\boldsymbol{b}) = 2\int_{-\infty}^{\infty} S_t(\boldsymbol{b})\, dr_t - \int_{-\infty}^{\infty} S_t^2(\boldsymbol{b})\, dt. \tag{4}$$

Therefore, the maximum-likelihood sequence detector selects among the possible noise realizations the one with minimum energy. Using the definition (2), we can express the first term in the right-hand side of (4) as

$$\int_{-\infty}^{\infty} S_t(\boldsymbol{b})\, dr_t = \sum_{i=-M}^{M} b^T(i) y(i), \tag{5}$$

where $y_k(i)$ denotes the output of a matched filter for the ith symbol of the kth user, that is,

$$y_k(i) = \int_{\tau_k + iT}^{\tau_k + iT + T} s_k(t - iT - \tau_k)\, dr_t. \tag{6}$$

Hence, even though $y_k(i)$ is not a sufficient statistic for the detection of $b_k(i)$, (4) and (5) imply that the whole sequence of outputs of the bank of K matched filters $\boldsymbol{y}$ is a sufficient statistic for the selection of the most likely sequence $\boldsymbol{b}$. This implies that the maximum-likelihood multiuser coherent detector consists of a front end of matched filters (one for each user) followed by a decision algorithm (Fig. 2), which selects the sequence $\boldsymbol{b}$ that maximizes (3) or, equivalently, (4). The efficient solution of this combinatorial optimization problem is the central issue in the derivation of the multiuser detector. The TCB of the ex-

haustive algorithm that computes (4) for all possible sequences has not only the inconvenient feature of being dependent on the block-size M, but it is so in an exponential way. Fortunately, $\|S(\boldsymbol{b})\|^2 = \int_{-\infty}^{\infty} S_t^2(\boldsymbol{b})\,dt$, the energy of the sequence $\boldsymbol{b}$, has the right structure to result in decision algorithms with significantly better TCB. The key to the efficient maximization of $\Omega(\boldsymbol{b})$ lies in its sequential dependence on the symbols $b_k(i)$, which allows us to put it as a sum of terms that depend only on a few variables at a time.

Suppose that we can find a discrete-time system $x_{i+1} = f_i(x_i, u_i)$, with initial condition x_{i_0}; a transition-payoff function $\lambda_i(x_i, u_i)$; and a bijection between the set of transmitted sequences and a subset of control sequences $\{u_i,\ i = i_0, \cdots, i_f\}$ such that $\Omega(\boldsymbol{b}) = \sum_{i=i_0}^{i_f} \lambda_i(x_i, u_i)$, subject to $x_{i+1} = f_i(x_i, u_i)$, x_{i_0}, and $\boldsymbol{b} \leftrightarrow \{u_i,\ i = i_0, \cdots, i_f\}$. Then the maximization of $\Omega(\boldsymbol{b})$ is equivalent to a discrete-time deterministic control problem with additive cost and finite input and state spaces, and therefore it can be solved by the dynamic programming algorithm either in backward or in forward fashion. Although the decision delay is unbounded because optimum decisions cannot be made until all states share a common shortest subpath, a well-known advantage in real-time applications of the forward dynamic programming algorithm (the Viterbi algorithm) is that little degradation of performance occurs when the algorithm uses an adequately chosen fixed finite decision lag. It turns out that there is not a unique additive decomposition of the log-likelihood function $\Omega(\boldsymbol{b})$, resulting in decision algorithms with very different computational complexities. It can be shown that the Viterbi algorithm suggested by Schneider [16] has 4^K states and $O(8^K/K)$ time complexity per bit, while the decision algorithm of the multiuser detector in [20] has 2^K states and TCB $= O(4^K/K)$. By fully exploiting the sequential dependence of the log-likelihood function on the transmitted symbols, it is possible to obtain an optimum decision algorithm that exhibits a lower time complexity per bit than the foregoing. This is achieved by the additive decomposition of the log likelihood function given by the following result.

Proposition 1: Define the following matrix of signal crosscorrelations:

$$G_{ij} = \begin{cases} \displaystyle\int_{-\infty}^{\infty} s_{i+j}(t - \tau_{i+j}) s_j(t - \tau_j - T)\,dt, \\ \qquad \text{if } i + j \le K \\ \displaystyle\int_{-\infty}^{\infty} s_{i+j-K}(t - \tau_{i+j-K}) s_j(t - \tau_j)\,dt, \\ \qquad \text{if } i + j > K \end{cases} \tag{7}$$

for $i = 1, \cdots, K-1$ and $j = 1, \cdots, K$ (i.e., the entries of the column[1] G^j are the correlations of the signal of the jth user with the $K-1$ preceding signals) and denote the received signal energies by $w_k = \int_0^T s_k^2(t)\,dt$, $k = 1, \cdots, K$. For any integer i, denote its modulo-K decomposition with remainder $\kappa(i) = 1, \cdots, K$, by $i = \eta(i)K + \kappa(i)$. Then we have

$$\Omega(\boldsymbol{b}) = \sum_{i=i_0}^{i_f} \lambda_i(x_i, u_i) \tag{8}$$

where $i_0 = 1 - MK$, $i_f = (M+1)K$, $u_i = b_{\kappa(i)}(\eta(i)) \in A_{\kappa(i)}$, and

$$\lambda_i(x, u) = u\left[2y_{\kappa(i)}(\eta(i)) - uw_{k(i)} - 2x^T G^{\kappa(i)}\right] \tag{9}$$

with

$$x_{i+1} = \left[x_i^2 x_i^3 \cdots x_i^{K-1} u_i\right]^T, \qquad x_{i_0} = \mathbf{0}. \tag{10}$$

Proof: Utilizing the foregoing modulo-K decomposition, it is easy to check that

$$S_t(\boldsymbol{b}) = \sum_{i=i_0}^{i_f} b_{\kappa(i)}(\eta(i)) s_{\kappa(i)}(t - \eta(i)T - \tau_{\kappa(i)}). \tag{11}$$

Hence we have

$$\begin{aligned} \|S(\boldsymbol{b})\|^2 &= \sum_{i=i_0}^{i_f} \Bigg[b_{\kappa(i)}^2(\eta(i)) \int_{-\infty}^{\infty} s_{\kappa(i)}^2(t - \eta(i)T - \tau_{\kappa(i)})\,dt \\ &\quad + 2 \sum_{j=i_0}^{i-1} b_{\kappa(i)}(\eta(i)) b_{\kappa(j)}(\eta(j)) \\ &\quad \cdot \int_{-\infty}^{\infty} s_{\kappa(i)}(t - \eta(i)T - \tau_{\kappa(i)}) \\ &\quad \cdot s_{\kappa(j)}(t - \eta(j)T - \tau_{\kappa(j)})\,dt \Bigg] \\ &= \sum_{i=i_0}^{i_f} \Bigg[b_{\kappa(i)}^2(\eta(i)) w_{\kappa(i)} \\ &\quad + 2 \sum_{l=1}^{K-1} b_{\kappa(i)}(\eta(i)) b_{\kappa(i-l)}(\eta(i-l)) \\ &\quad \cdot \int_{-\infty}^{\infty} s_{\kappa(i)}(t - \eta(i)T - \tau_{\kappa(i)}) \\ &\quad \cdot s_{\kappa(i-l)}(t - \eta(i-l)T - \tau_{\kappa(i-l)})\,dt \Bigg], \end{aligned} \tag{12}$$

where by agreement $b_{\kappa(i)}(\eta(i)) = 0$ for $i < i_0$. Now we show that the integral in the right-hand side of (12) is equal to $G_{K-l\,\kappa(i)}$. To that end we will examine separately the terms in which $\eta(i) = \eta(i-l)$ and $\eta(i) = \eta(i-l) + 1$. In the first case, $\kappa(i-l) = \kappa(i) - l$ and

$$\begin{aligned} &\int_{-\infty}^{\infty} s_{\kappa(i)}(t - \eta(i)T - \tau_{\kappa(i)}) \\ &\qquad \cdot s_{\kappa(i-l)}(t - \eta(i-l)T - \tau_{\kappa(i-l)})\,dt \\ &\quad = \int_{-\infty}^{\infty} s_{\kappa(i)}(t - \tau_{\kappa(i)}) s_{\kappa(i)-l}(t - \tau_{\kappa(i)-l})\,dt \\ &\quad = G_{K-l\,\kappa(i)}. \end{aligned} \tag{13}$$

[1] The columns of the matrix G and of the row vector x_i^T are denoted by superscripts.

In the second case, $\kappa(i-l) = \kappa(i) + K - l$ and

$$\begin{aligned}\int_{-\infty}^{\infty} & s_{\kappa(i)}(t - \eta(i)T - \tau_{\kappa(i)}) \\ & \cdot s_{\kappa(i-l)}(t - \eta(i-l)T - \tau_{\kappa(i-l)})\, dt \\ &= \int_{-\infty}^{\infty} s_{\kappa(i)}(t - T - \tau_{\kappa(i)}) s_{\kappa(i)+K-l}(t - \tau_{\kappa(i)+K-l})\, dt \\ &= G_{K-l\,\kappa(i)}. \end{aligned} \tag{14}$$

From (10) and $u_i = b_{\kappa(i)}(\eta(i))$ it is clear that $b_{\kappa(i-l)}(\eta(i-l)) = x_i^{K-l}$, and therefore

$$\|S(\boldsymbol{b})\|^2 = \sum_{i=i_0}^{i_f} b_{\kappa(i)}(\eta(i)) \cdot \left[b_{\kappa(i)}(\eta(i)) w_{\kappa(i)} + 2 \sum_{j=1}^{K-1} x_i^j G_{j\kappa(i)} \right]. \tag{15}$$

Using (15) and the fact that

$$\int_{-\infty}^{\infty} S_t(\boldsymbol{b})\, dr_t = \sum_{i=i_0}^{i_f} b_{\kappa(i)}(\eta(i)) y_{\kappa(i)}(\eta(i))$$

[see (5)], the sought-after decomposition (9) follows.

The algorithm resulting from the decomposition of Proposition 1 performs K times the number of stages of that derived in [20]; however, its asymptotic complexity is considerably better since it exploits fully the separability of the log-likelihood function. In this case the dimensionality of the state space is equal to 2^{K-1}, and each state is connected to two states in the previous stage (if all the users employ binary modulation), resulting in TCB = $O(2^K)$.[2] This decomposition was introduced in [19] and shows that the part of the transition metric that is independent of the matched filter outputs is periodically time-varying with a period equal to the number of users. The nature of this behavior can be best appreciated by particularizing it to the intersymbol interference problem, in which $s_i(t) = s_j(t)$ for all $t \in [0, T]$, $i \neq j$, and $\tau_{i+1} - \tau_i = T/K$. In this case, $w_i = w_j$ and $G^i = G^j$ for $i \neq j$, and (9) reduces to the Ungerboeck metric [17, eq. (27)] when each symbol suffers the interference of $K-1$ signals. So, (9) can be viewed as the generalization of the Ungerboeck metric to a problem of periodically time-varying intersymbol interference equivalent from the receiver viewpoint, to the asynchronous multiuser model (1)–(2). The assumption that the signals of all users have the same duration can be relaxed, resulting in a decomposition similar to (9); however, the periodicity of the transition metric is lost unless the ratio between every pair of signal periods is rational.

The Viterbi algorithm resulting from the additive decomposition of Proposition 1 requires knowledge of the partial cross correlations between the signals of every pair of active users, and, unless binary antipodal modulation is employed, they also require the received signal energies. However, the signal cross correlations depend on the received relative delays, carrier phases, and amplitudes and hence cannot be determined *a priori* by the receiver. Nevertheless, the basic assumption is that the signal waveform of each user is known and that the K-user coherent receiver locks to the signaling interval and phase of each active user. Then the required parameters G_{ij} can be generated internally by cross-correlating the normalized waveform replicas stored in the receiver with the adequate delays and phases supplied by the synchronization system and by multiplying the resulting normalized cross correlations by the received amplitudes of the corresponding users. Hence the only requirement (beyond synchronization) imposed by the need for the partial cross correlations is the availability (up to a common scale factor) of the K received signal amplitudes.

The decomposition in Proposition 1 can be generalized to the case where the modulation is not necessarily linear, that is, each symbol $u \in A_k$ is mapped into a different waveform $s_k(t; u)$. It can be shown, analogously to the proof of Proposition 1, that in this case the transition metric is

$$\lambda_i(x, u) = 2 y_{\kappa(i)}(\eta(i); u) - w_{\kappa(i)}[u] - 2 \sum_{l=1}^{K-1} G_{l\kappa(i)}[x^l, u], \tag{16}$$

where $y_k(i, u)$ is the output of the matched filter of the uth waveform of the kth user; $w_k[u] = \int_0^T s_k^2(t; u)\, dt$; and

$$G_{lk}[a, u] = \begin{cases} \int s_{l+k}(t - \tau_{l+k}; a) s_k(t - \tau_k - T; u)\, dt, & \text{if } l + k \le K \\ \int s_{l+k-K}(t - \tau_{l+k-K}; a) s_k(t - \tau_k; u)\, dt, & \text{if } l + k > K. \end{cases} \tag{17}$$

III. Error Probability Analysis of Optimum Multiuser Detectors

A. Upper and Lower Bounds

This section is devoted to the analysis of the minimum uncoded bit error probability of multiuser detectors for antipodally modulated signals in additive white Gaussian noise, that is, attention is focused on the multiple-access model (1)–(2) in the case $A_1 = \cdots = A_K = \{-1, 1\}$. Denote the most likely transmitted ith symbol by the kth user given the observations by

$$\hat{b}_k^M(i) \in \arg \max_{b \in \{-1,1\}} P\left[b_k(i) = b \,|\, \cdot dr_t = S_t^M(b) + \sigma\, d\omega_t,\ t \in R \right] \tag{18}$$

where $\boldsymbol{b} = \{b(i) \in \{-1,1\}^K,\ i = -M, \cdots, M\}$ is the transmitted sequence. The goal is to obtain the finite and

[2] If the alphabet sizes are arbitrary, then (see [22]) TCB = $O(K \prod_{i=1}^K |A_i| / \sum_{i=1}^K \log |A_i|)$. If there are S clusters of synchronized users, then it is possible to reduce this complexity by a factor of S/K by taking into account that some of the cross correlations are equal to zero.

infinite horizon error probabilities,

$$P_k^M(i) = P\left[b_k(i) \neq \hat{b}_k^M(i)\right], \qquad i = -M, \cdots, M \quad (19)$$

$$P_k = \lim_{M \to \infty} P_k^M(i), \quad (20)$$

for arbitrary signal waveforms and relative delays. The existence of the limit for any given i in the right-hand side of (20) follows because $P_k^M(i)$ is equal to probability of error of an optimum detector for a multiuser signal with horizon equal to $M+1$ and complete information about $b(-M-1)$ and $b(M+1)$; hence $P_k^{M+1}(i) \geq P_k^M(i)$ for any positive integer M, $i \in \{-M, \cdots, M\}$ and $k \in \{1, \cdots, K\}$. The independence of the limit on i readily follows from the assumed stationarity of the noise and of the priors.

Our approach is to derive lower and upper bounds on the error rate for each user, which are tight in the low and high SNR regions and give a close approximation over the whole SNR range. The upper bounds are based on the analysis of two detectors that are suboptimum in terms of error probability, namely, the conventional single-user coherent detector and the K-user maximum-likelihood sequence detector. The lower bounds are the error probabilities of two optimum binary tests derived from the original problem by allowing certain side information.

The normalized difference between any pair of distinct transmitted sequences will be referred to as an error sequence, that is, the set of nonzero error sequences is[3]

$$E = \left\{\boldsymbol{\epsilon} = \left\{\epsilon(i) \in \{-1,0,1\}^K, i = -M, \cdots, M; \right.\right.$$
$$\left.\left. \epsilon(j) \neq \mathbf{0} \text{ for some } j\right\}\right\}.$$

The set of error sequences that are admissible conditioned on $\boldsymbol{b}$ being transmitted, that is, those that correspond to the difference between $\boldsymbol{b}$ and the sequence selected by the detector, is denoted by

$$A(\boldsymbol{b}) = \{\boldsymbol{\epsilon} \in E, 2\boldsymbol{\epsilon} - \boldsymbol{b} \in D\},$$

where

$$D = \left\{\boldsymbol{b} = \left\{b(i) \in \{-1,1\}^K, i = -M, \cdots, M\right\}\right\}.$$

The admissible error sequences that affect the ith bit of the kth user of $\boldsymbol{b}$ are

$$A_k(\boldsymbol{b}, i) = \{\boldsymbol{\epsilon} \in A(\boldsymbol{b}), \epsilon_k(i) = b_k(i)\}.$$

The number of nonzero components of an error sequence and the energy of a hypothetical multiuser signal modulated by the sequence $\boldsymbol{\epsilon}$ are denoted, respectively, by

$$w(\boldsymbol{\epsilon}) = \sum_{i=-M}^{M} \sum_{k=1}^{K} |\epsilon_k(i)|$$

and

$$\|S(\boldsymbol{\epsilon})\|^2 = \int_{-\infty}^{\infty} \left(\sum_{i=-M}^{M} \sum_{k=1}^{K} \epsilon_k(i) s_k(t - iT - \tau_k) \right)^2 dt.$$

[3]For the sake of notational simplicity, the explicit dependence of the sets of sequences on M is dropped when this causes no ambiguity.

Proposition 2: Define the following minimum distance parameters:

$$d_k^M(\boldsymbol{b}, i) = \inf_{\boldsymbol{\epsilon} \in A_k^M(\boldsymbol{b}, i)} \|S(\boldsymbol{\epsilon})\|$$

and

$$d_{k,\min}^M(i) = \inf_{\boldsymbol{b} \in D} d_k^M(\boldsymbol{b}, i).$$

Then, the minimum error probability of the ith bit of the kth user is lower-bounded by

$$P_k \geq P_k^M(i)$$
$$\geq P\left[d_k^M(\boldsymbol{b}, i) = d_{k,\min}^M(i)\right] Q\left(d_{k,\min}^M(i)/\sigma\right) \quad (21)$$

and by

$$P_k \geq P_k^M(i) \geq Q(w_k/\sigma), \quad (22)$$

where $P[d_k(\boldsymbol{b}, i) = d_{k,\min}(i)]$ is the *a priori* probability that the transmitted sequence is such that one of its congruent error sequences affects the ith bit of the kth user and has the minimum possible energy.

Proof: The basic technique for obtaining lower bounds on the minimum error probability is to analyze the performance of an optimum receiver that, in addition to observing $\{r_t, t \in R\}$, has certain side information.

To obtain the first lower bound (21), the following reasoning, analogous to that of Forney [2], can be employed. Suppose that if the transmitted sequence $\boldsymbol{b}$ is such that $d_k(\boldsymbol{b}, i) = d_{k,\min}(i)$, the detector is told by a genie that the true sequence is either $\boldsymbol{b}$ or $\boldsymbol{b} - 2\boldsymbol{\delta}$, where $\boldsymbol{\delta}$ is arbitrarily chosen by the genie (independently of the noise realization) from the set

$$\arg \min_{\boldsymbol{\epsilon} \in A_k(\boldsymbol{b}, i)} \|S(\boldsymbol{\epsilon})\|.$$

Under these conditions, the minimum error probability detector and the optimum sequence detector coincide and both reduce to a binary hypothesis test $\Omega(\boldsymbol{b}) \gtrless \Omega(\boldsymbol{b} - 2\boldsymbol{\delta})$. The conditional probability of error given that $\boldsymbol{b}$ is transmitted is given by

$$P[\Omega(\boldsymbol{b} - 2\boldsymbol{\delta}) > \Omega(\boldsymbol{b}) | \boldsymbol{b} \text{ transmitted}]$$
$$= P\left[\sigma \int S_t(\boldsymbol{\delta})\, d\omega_t + \|S(\boldsymbol{\delta})\|^2 < 0\right]$$
$$= Q(\|S(\boldsymbol{\delta})\|/\sigma) \quad (23)$$

where the foregoing equalities follow from (4) and the fact that $\int S_t(\boldsymbol{\delta})/\|S(\boldsymbol{\delta})\|\, d\omega_t$ is a zero-mean unit-variance Gaussian random variable. If the transmitted sequence $\boldsymbol{b}$ is such that $d_k(\boldsymbol{b}, i) > d_{k,\min}(i)$, then the error probability of the receiver with side information is trivially bounded by zero, and the lower bound (21) follows. The lower bound (22) is the kth user minimum error probability if no other user was active or, equivalently, if the receiver knew the transmitted bits of the other users.

For any error sequence $\boldsymbol{\epsilon}$ such that $\epsilon(j) = \epsilon(n) \neq \mathbf{0}$, for $j \neq n$, the sequence

$$\epsilon'(m) = \begin{cases} \epsilon(m), & m \leq j \\ \epsilon(m + n - j), & m > j \end{cases}$$

satisfies $\|S(\epsilon')\| \le \|S(\epsilon)\|$. (Otherwise, one could construct a sequence with negative energy.) This implies that the infinite-horizon minimum distances, $d_k(\boldsymbol{b})$ and $d_{k\,\min}$ (i.e., one-half of the minimum rms of the difference between the signals of any pair of transmitted sequences that differ in any bit of the kth user), are achieved by finite-length error sequences, and the error rate of the kth user can be lower-bounded by

$$P_k \ge P\left[d_k(\boldsymbol{b}) = d_{k,\min}\right] Q(d_{k,\min}/\sigma). \tag{24}$$

Note that since $d_{k,\min} \le d_{k,\min}^M(i)$, the bound (24) is at least as tight as (21) in the low-noise region.

The following upper bound, the error probability of the conventional single-user coherent receiver, is mainly useful in the low SNR region.

Proposition 3: Let $R_{ij} = G_{K-i\kappa(k+i)}$, for $i = 1, \cdots, K-1$ and $j = 1, \cdots, K$, that is, the entries of the column R^j are the correlations of the signal of the jth user with the $K-1$ posterior signals, and denote $I_k(\alpha, \beta) = \sum_{j=1}^{K-1}(\alpha_j G_{jk} + \beta_j R_{jk})/w_k$. The minimum error probability of the kth user is upper-bounded by

$$P_k^M(i) \le P_k \le E\left[Q\left(\sqrt{w_k}\left[1 + I_k(\alpha, \beta)\right]/\sigma\right)\right], \tag{25}$$

where the expectation is over the ensemble of independent uniformly distributed $\alpha \in \{-1, 1\}^{K-1}$, $\beta \in \{-1, 1\}^{K-1}$.

Proof: Suppose that $|i| < M$ and that the transmitted sequence is such that $b_k(i) = 1$. The kth matched filter output corresponding to the ith bit is a conditionally Gaussian random variable with variance equal to $w_k\sigma^2$ and mean given by

$$w_k + \sum_{l=k+1}^{K} b_l(i-1)G_{l-k\,k} + \sum_{l=1}^{k-1} b_l(i)G_{l-k+K\,k} + \sum_{l=k+1}^{K} b_l(i)R_{l-k\,k} + \sum_{l=1}^{k-1} b_l(i+1)R_{l-k+K\,k}.$$

Since the transmitted bits are assumed to be equiprobable and independent, we have

$$P_k^M(i) \le E\left[Q\left(\sqrt{w_k}\left[1 + I_k(\alpha, \beta)\right]/\sigma\right)\right], \tag{26}$$

and because the right-hand side does not depend on M, (25) follows.

We turn to the derivation of an upper bound on the error probability of the K-user maximum-likelihood sequence detector. Our approach hinges on the following definition. An error sequence $\epsilon \in E$ is *decomposable* into $\epsilon' \in E$ and $\epsilon'' \in E$ if

1) $\epsilon = \epsilon' + \epsilon''$,
2)[4] $\epsilon' < \epsilon$, $\epsilon'' < \epsilon$,
3) $< S(\epsilon'), S(\epsilon'') > \ge 0$.

As an illustration consider the following simple two-user example: $s_1(t) = s_2(t) = 1$, $0 \le t \le T/2$; $s_1(t) = -s_2(t) = -1$, $T/2 < t \le T$; and $\tau_2 - \tau_1 = T/2$. Then it is easy to check that the error sequence $\epsilon = \cdots, \mathbf{0}, [1\ 1]^T, [1\ 0]^T, \mathbf{0}, \cdots$ is decomposable into $\epsilon' = \cdots, \mathbf{0}, [1\ 0]^T, [1\ 0]^T, \mathbf{0}, \cdots$, and $\epsilon'' = \cdots, \mathbf{0}, [0\ 1]^T, [0\ 0]^T, \cdots$.

Proposition 4: Denote the set of error sequences that affect the ith bit of the kth user by $Z_k(i) = \{\epsilon \in E, \epsilon_k(i) \ne 0\}$. Let $F_k(i)$ be the subset of *indecomposable* sequences in $Z_k(i)$. Then the minimum-error probability of the ith bit of the kth user is upper-bounded by

$$P_k^M(i) \le \sum_{\epsilon \in F_k(i)} 2^{-w(\epsilon)} Q(\|S(\epsilon)\|/\sigma). \tag{27}$$

Proof: Formula (27) is an upper bound on the probability of error of the K-user maximum-likelihood sequence detector, that is, the receiver whose output is the sequence that maximizes $\Omega(\boldsymbol{b})$. This is the only property of the detector that we use for the proof of (27); it is not necessary to assume any specific decision algorithm, in particular that of Section II. Define the following sets of error sequences:

$$L = \left\{\epsilon \in E,\ \sigma \int S_t(\epsilon)\, dw_t \le -\|S(\epsilon)\|^2\right\} \tag{28}$$

and

$$ML(\boldsymbol{b}) = \{\epsilon \in A(\boldsymbol{b}), \Omega(\boldsymbol{b} - 2\epsilon) \ge \Omega(\boldsymbol{d}), \text{for all } \boldsymbol{d} \in D\}. \tag{29}$$

If $\boldsymbol{b}$ is the transmitted sequence and $\epsilon \in A(\boldsymbol{b})$, then it can be shown that $\Omega(\boldsymbol{b} - 2\epsilon) \ge \Omega(\boldsymbol{b})$ if and only if $\epsilon \in L$. Hence it follows that $ML(\boldsymbol{b}) \subset L$ and that if $A(\boldsymbol{b}) \cap L \ne \varnothing$, then the sequence detector outputs an erroneous sequence $\boldsymbol{b} - 2\epsilon$, where $\epsilon \in ML(\boldsymbol{b})$. Consider the following inclusions between events in the probability space on which the transmitted sequence $\boldsymbol{b}$, and the Wiener process $\{\omega_t, t \in R\}$ are defined:

$$\{b_k(i) \ne b_k^*(i)\} \subset \bigcup_{\epsilon \in E} \{\epsilon \in A_k(\boldsymbol{b}, i) \cap ML(\boldsymbol{b})\} \subset \bigcup_{\epsilon \in F_k(i)} \{\epsilon \in A_k(\boldsymbol{b}, i) \cap L\}, \tag{30}$$

where $\boldsymbol{b}^*$ is the sequence selected by the detector. The first inclusion follows from the definitions of $A_k(\boldsymbol{b}, i)$ and $ML(\boldsymbol{b})$ (the converse holds if there are no ties in the maximization of $\Omega(\cdot)$). The key to the tightness of the bound in (27) is the second inclusion in (30); to verify it, we show that for every $\epsilon \in Z_k(i)$ there exists $\epsilon' \in F_k(i)$ such that

1) $\{\epsilon \in A_k(\boldsymbol{b}, i)\} \subset \{\epsilon' \in A_k(\boldsymbol{b}, i)\}$
2) $\{\epsilon \in ML(\boldsymbol{b})\} \subset \{\epsilon' \in L\}$.

If $\epsilon \in F_k(i)$, then $\epsilon' = \epsilon$ satisfies 1) and 2). Otherwise, $\epsilon \in Z_k(i) - F_k(i)$. We now show by induction on $w(\epsilon)$, the weight of the sequence ϵ, that there exists $\epsilon^* \in F_k(i)$ such that ϵ is decomposable into $\epsilon^* + (\epsilon - \epsilon^*)$. If a sequence of weight two is decomposable, then it is so into its two components, both of which are indecomposable since they have unit weight. Now suppose that the claim is true for any sequence whose weight is strictly less than $w(\epsilon)$.

[4] We denote $\epsilon' < \epsilon$ if $|\epsilon_j'(i)| \le |\epsilon_j(i)|$ for all $j = 1, \cdots, K$ and $i = -M, \cdots, M$.

Find the (not necessarily unique) decomposition $\epsilon = \epsilon^1 + \epsilon^2$, $\epsilon^1 \in Z_k(i)$, with largest inner product, that is,

$$\langle S(\epsilon^1), S(\epsilon^2)\rangle \geq \langle S(\epsilon^a), S(\epsilon^b) > \geq 0$$

for any decomposition $\epsilon = \epsilon^a + \epsilon^b$. If $\epsilon^1 \in F_k(i)$, we have found the sought-after decomposition of ϵ. Otherwise, we can decompose ϵ^1 into $\epsilon^3 + \epsilon^4$ such that $\epsilon^3 \in F_k(i)$ because of the induction hypothesis. However, ϵ is indeed decomposable into ϵ^3 and $(\epsilon^2 + \epsilon^4)$, for otherwise the right-hand side of the equation

$$\begin{aligned}\langle S(\epsilon^2) + S(\epsilon^3), S(\epsilon^4)\rangle &- \langle S(\epsilon^1), S(\epsilon^2)\rangle \\ &= 2\langle S(\epsilon^3), S(\epsilon^4)\rangle - \langle S(\epsilon^3), S(\epsilon^4) + S(\epsilon^2)\rangle \quad (31)\end{aligned}$$

is strictly positive, contradicting the choice of $\epsilon^1 + \epsilon^2$ as the largest-inner-product decomposition of ϵ.

Since $\epsilon^* < \epsilon$, it follows that $\epsilon' = \epsilon^*$ satisfies property 1); to see that 2) is also fulfilled, let $\epsilon'' = \epsilon - \epsilon'$ and consider

$$\begin{aligned}\Omega(\boldsymbol{b} - 2\epsilon) &- \Omega(\boldsymbol{b} - 2\epsilon'') \\ &= 2\sigma\int[S_t(\boldsymbol{b} - 2\epsilon) - S_t(\boldsymbol{b} - 2\epsilon'')]\, dw_t \\ &\quad + \|S(\boldsymbol{b} - 2\epsilon'') - S(\boldsymbol{b})\|^2 - \|S(\boldsymbol{b} - 2\epsilon) - S(\boldsymbol{b})\|^2 \\ &= -4\left(\sigma\int S_t(\epsilon')\, dw_t + \|S(\epsilon')\|^2\right) - 8\langle S(\epsilon''), S(\epsilon')\rangle \quad (32)\end{aligned}$$

where (32) follows from the fact that $\boldsymbol{b}$ is the transmitted sequence. If $\epsilon \in ML(\boldsymbol{b})$, then it is necessary that $\Omega(\boldsymbol{b} - 2\epsilon) \geq \Omega(\boldsymbol{b} - 2\epsilon'')$; moreover, the decomposition of ϵ into $\epsilon' + \epsilon''$ implies that $\langle S(\epsilon'), S(\epsilon'')\rangle \geq 0$. Therefore, (32) indicates that $\epsilon' \in L$, and property 2) is satisfied (see Fig. 3).

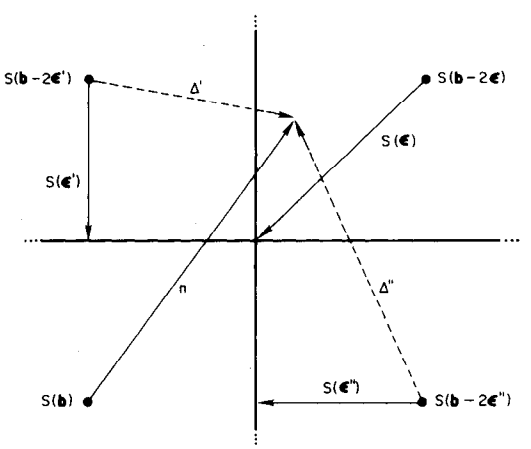


Fig. 3. If $\boldsymbol{b} - 2\epsilon$ is most likely sequence *a posteriori* ($\epsilon \in ML(\boldsymbol{b})$) and $\langle S(\epsilon'), S(\epsilon'')\rangle \geq 0$, then $\|\Delta'\| \leq \|n\|$ and $\|\Delta''\| \leq \|n\|$, i.e., $\epsilon' \in L$ and $\epsilon'' \in L$. $S(\boldsymbol{b})$ is transmitted signal. n is projection of noise realization.

In order to take probabilities of the events in the right-hand side of (30), note that $\{\epsilon \in A_k(\boldsymbol{b}, i)\}$ depends on the transmitted sequence but not on the noise realization; hence it is independent of the event $\{\epsilon \in L\}$. Finally,

$$P[\epsilon \in A_k(\boldsymbol{b}, i)] = 2^{-w(\epsilon)} \quad (33)$$

and

$$P[\epsilon \in L] = Q(\|S(\epsilon)\|/\sigma) \quad (34)$$

are immediate from the respective definitions of $A_k(\boldsymbol{b}, i)$ and L, and since the probability of the union of the events in the right-hand side of (3) is not greater than the sum of their probabilities, the upper bound of (27) follows.

One of the features of the foregoing proof is that it remains valid in the general case where the modulation is not necessarily linear, that is,

$$S_t(\boldsymbol{b}) = \sum_{i=-M}^{M}\sum_{k=1}^{K} s_k(t - iT - \tau_k; b_k(i)), \qquad \boldsymbol{b} \in D. \quad (35)$$

The only modification needed to state and prove Proposition 4 is to substitute $S_t(\epsilon)$ by

$$\begin{aligned}\hat{S}_t(\epsilon) = \frac{1}{2}\sum_{i=-M}^{M}\sum_{k=1}^{K} &s_k(t - iT - \tau_k; \epsilon_k(i)) \\ &- s_k(t - iT - \tau_k; -\epsilon_k(i)). \quad (36)\end{aligned}$$

It is interesting to particularize the foregoing result to the intersymbol interference problem. The Forney bound [3] corresponds to the sum of $2^{-w(\epsilon)}Q(\|S(\epsilon)\|/\sigma)$ over all *simple* sequences, that is, those containing no more than $L - 1$ consecutive zeros amid nonzero components (where L is the number of overlapping symbols). It turns out that a great proportion of simple sequences are, in fact, decomposable; however, all indecomposable sequences are simple. Hence the analysis of maximum-likelihood sequence detection of signals subject to intersymbol interference via decomposition of error sequences results in a bound that is tighter than Forney's result. This issue and the question of how to compute (27) up to any prespecified degree of accuracy are examined in [24].

In the case of bit-synchronous users ($\tau_1 = \cdots = \tau_K$), the derivation of optimum decision rules is, in contrast to Section II, a simple exercise. However, the analysis of the optimum synchronous receiver has basically the same complexity as the general case presented in this section. In particular, even though the one-shot model ($M = 0$) is sufficient, the approach of decomposition of error sequences (K-vectors in this case) is the most effective one.

B. Convergence of the Bounding Series

The limit of the right-hand side of (27) as $M \to \infty$ is an infinite series that bounds the multiuser infinite-horizon error probability. Since this upper bound is monotonic in the noise level, we can distinguish three situations depending on the actual energies, crosscorrelations, and relative delays of the signals—namely, divergence of the series for all noise levels, convergence for sufficiently low noise levels, and global convergence. Although most cases fall into the second category, we will show examples of the other two. We begin by proving a sufficient condition for local convergence of a series that overbounds (27).

Proposition 5: Partition the real line with the nontrivial semi-open intervals $R = \bigcup_{i \in Z} \Lambda_i$ defined by the points $\tau_k + iT$, $k = 1, \cdots, K$, $i \in Z$. Define $N(\epsilon)$ as the union of all intervals Λ_i for which $\int_{\Lambda_i} S_t^2(\epsilon)\, dt = 0$. Let G be a set of simple sequences such that for every pair of distinct finite sequences $\epsilon^1, \epsilon^2 \in G$ that satisfy $N(\epsilon^1) = N(\epsilon^2)$, there exist $j \in \{1, \cdots, K\}$ and $i \in Z$ such that $\epsilon_j^1(i) \neq \epsilon_j^2(i)$ and $(iT + \tau_j, (i+1)T + \tau_j) \not\subset N(\epsilon^1)$. Then there exists $\sigma_0 > 0$ such that

$$\sum_{\epsilon \in G} 2^{-w(\epsilon)} \exp\left(-\|S(\epsilon)\|^2/2\sigma^2\right) < \infty \tag{37}$$

for $0 < \sigma < \sigma_0$.

Proof: Define $G_{n,l} = \{\epsilon \in G$, for some $j \in Z$ the point of arrival of the first nonzero component of ϵ and the point of departure of the last nonzero component of ϵ define the interval $\bigcup_{i=j+1}^{j+n} \Lambda_i$ and exactly l of these intervals are such that $\int_{\Lambda_i} S_t^2(\epsilon)\, dt > 0\}$. For every $\epsilon \in G_{n,l}$, we have

$$(K-1)w(\epsilon) \geq n \tag{38}$$

because ϵ is simple and we cannot have more than $K-2$ zeros surrounded by nonzeros. Define

$$\alpha^2 = \min\left\{ r > 0 \mid \text{there exists } i \in Z \text{ and } \epsilon \in G \text{ with } r = \int_{\Lambda_i} S_t^2(\epsilon)\, dt \right\}. \tag{39}$$

Notice that the existence of α is guaranteed because the signal energies are nonzero and there is a finite number of distinct signal waveforms in each interval Λ_i. From the definition of $G_{n,l}$, (38), and (39), we have

$$\sum_{\epsilon \in G_{n,l}} 2^{-w(\epsilon)} \exp\left(-\|S(\epsilon)\|^2/2\sigma^2\right) \leq 2^{(n/1-K)} |G_{n,l}| \exp\left(-l\alpha^2/2\sigma^2\right). \tag{40}$$

Now we use the assumption in the theorem to overbound the cardinality of $G_{n,l}$ by

$$|G_{n,l}| \leq n \binom{n}{l} 3^{Kl}. \tag{41}$$

To show (41) let us see how many different sequences are congruent with every choice of j in the definition of $G_{n,l}$ (for which there are n possibilities, since $\epsilon_k(0) \neq 0$ for all $\epsilon \in G$) and with every distribution of the l nonzero-energy intervals (at most $\binom{n}{l}$ possibilities). The assumption of the result states that any pair of sequences whose nonzero-energy intervals coincide cannot differ only in symbols whose intervals have zero energy. Since $S_t(\epsilon)$, $t \in \Lambda_i$ depends on ϵ through K elements at most, the number of error sequences congruent with the aforementioned choice is bounded by 3^{Kl}, and (41) follows. Substituting (41) into (40), we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{l=0}^{n} \sum_{\epsilon \in G_{n,l}} & 2^{-w(\epsilon)} \exp\left(-\|S(\epsilon)\|^2/2\sigma^2\right) \\ &\leq \sum_{n=1}^{\infty} n 2^{n/1-K} \sum_{l=0}^{n} \binom{n}{l} \left(3^{Kl} \exp\left(-l\alpha^2/2\sigma^2\right)\right) \\ &= \sum_{n=1}^{\infty} n 2^{n/1-K} \left(1 + 3^K \exp\left(-\alpha^2/2\sigma^2\right)\right)^n, \end{aligned} \tag{42}$$

which converges for

$$0 < \sigma^2 < \frac{\alpha^2}{2} \left[K \ln 3 - \ln\left(2^{1/K-1} - 1\right)\right]^{-1}. \tag{43}$$

Local convergence of the Forney bound for any intersymbol interference problem was proved by Foschini [5]. Not every multiuser problem, however, results in a locally convergent bounding series. Admittedly, the implications of the sufficient condition of Proposition 5 on the waveforms and delays of the signal set are not readily apparent. As is shown by the next result, the sufficient condition of Proposition 5 is satisfied except in certain pathological cases where some degree of synchronism exists along with heavy correlation between the signals.

Proposition 6: Define the times of effective arrival and departure of the ith signal of the kth user as

$$\lambda_{iK+k}^a = \tau_k + iT + \sup\left\{\tau \in [0, T),\ \int_0^\tau s_k^2(t)\, dt = 0\right\}$$

and

$$\lambda_{iK+k}^d = \tau_k + iT + \inf\left\{\tau \in (0, T],\ \int_\tau^T s_k^2(t)\, dt = 0\right\},$$

respectively.

Suppose that a pair of distinct finite sequences $\epsilon^1, \epsilon^2 \in E$ exists such that $N(\epsilon^1) = N(\epsilon^2)$ and such that $\epsilon_k^1(i) \neq \epsilon_k^2(i)$ implies $(iT + \tau_k, (i+1)T + \tau_k) \subset N(\epsilon^1)$. Then the following two statements are true:

1) $S_t(\epsilon^1) = S_t(\epsilon^2)$, a.e.;
2) a pair of reals $\rho < \xi$ exists such that
   - $\rho = \lambda_i^a = \lambda_j^a$ for $i \neq j$,
   - $\xi = \lambda_i^d = \lambda_j^d$ for $i \neq j$, and
   - if $\epsilon_k^1(i) \neq \epsilon_k^2(i)$ then $\rho \leq \lambda_{iK+k}^a < \lambda_{iK+k}^d \leq \xi$.

Proof: 1) If $t \notin N(\epsilon^1) = N(\epsilon^2)$, then $S_t(\epsilon^1)$, $S_t(\epsilon^2)$ depend on their arguments only through those symbols that coincide; on the other hand,

$$\int_{N(\epsilon^1)} S_t^2(\epsilon^1)\, dt = 0 = \int_{N(\epsilon^2)} S_t^2(\epsilon^2)\, dt.$$

2) We show first that the effective arrival of the first symbol that differs, say $\epsilon_q(j)$, must be a point of effective multiarrival, that is, there exists $i > i_1 = jK + q$, $\lambda_i^a = \lambda_{i_1}^a$. If $\lambda_{i_1}^a$ is not a point of effective multiarrival, then we can select λ such that $\lambda_{i_1}^a < \lambda < \lambda_{iK+k}^a$, $\epsilon_k^1(i) \neq \epsilon_k^2(i)$ and

$$\int_{\lambda_{i_1}^a}^{\lambda} s_q^2(t - \tau_q - jT)\, dt > 0. \tag{44}$$

On the other hand, if $t \in (\lambda_{i_1}^a, \lambda)$, then

$$S_t(\epsilon^1) = \epsilon_q^1(j) s_q(t - \tau_q - jT) + \delta^1(t) \tag{45a}$$

and

$$S_t(\epsilon^2) = \epsilon_q^2(j) s_q(t - \tau_q - jT) + \delta^2(t), \tag{45b}$$

where $\delta^1(t) = \delta^2(t)$, a.e. in $(\lambda_{i_1}^a, \lambda)$ because the effective arrival of the rest of the unequal symbols is posterior to λ. Using 1) and (45), we obtain that $\epsilon_q^1(j) s_q(t - \tau_q - jT) = \epsilon_q^2(j) s_q(t - \tau_q - jT)$ a.e. in $(\lambda_{i_1}^a, \lambda)$, which contradicts (44) since $\epsilon_q^1(j) \neq \epsilon_q^2(j)$. Similarly, it can be shown that the effective departure of the last symbol that differs between

ϵ' and ϵ'' must be a point of effective multidepartture, and 2) follows.

Proposition 6 implies that for asynchronous models, where the delays are independent and uniformly distributed, an interval of convergence exists for the bounding series with probability one. On the other hand, it is easy to see that for bit-synchronous models, the set $F_k(i)$ is finite so that (27) is finite for all noise levels in that case. Hence, the necessary ingredients for the everywhere divergence of (27) are the partial effective synchronism and the heavy cross correlation of the signal constellation. To illustrate this, consider the following example of divergence of the bounding series (27) for all noise levels.

Let $K = 6$, $s_k(t) = 1$, $t \in [0,1]$, $k = 1,\cdots,6$; $\tau_k - \tau_1 = 1/2$, $k = 2,\cdots,6$; $\tau_i - \tau_j = 0$, $i, j = 2,\cdots,6$. Consider the set of error sequences: $A_n = \{\epsilon \in E, \epsilon(i) = 0, i < 0$ and $i > n$; $\epsilon(i) \in \{[1\ -1\ 0\ 0\ 0\ 0]^T, [1\ 0\ -1\ 0\ 0\ 0]^T, [1\ 0\ 0\ -1\ 0\ 0]^T, [1\ 0\ 0\ 0\ -1\ 0]^T, [1\ 0\ 0\ 0\ 0\ -1]^T\}$, $i = 0,\cdots,n-1$; $\epsilon(n) = [1\ 0\ 0\ 0\ 0\ 0]^T\}$. Notice that $\|S(\epsilon)\| = 1$, $w(\epsilon) = (2n+1)$, for all $\epsilon \in A_n$, and $|A_n| = 5^n$. It is straightforward to show that every sequence $\epsilon \in A_n$ is indecomposable; thus $A_n \in F_1$, for all $n > 1$, and the following inequality is true:

$$\begin{aligned}\sum_{\epsilon \in F_1} 2^{-w(\epsilon)} Q(\|S(\epsilon)\|/\sigma) &> \sum_{n=1}^{\infty} \sum_{\epsilon \in A_n} 2^{-w(\epsilon)} Q(\|S(\epsilon)\|/\sigma) \\ &= Q(1/\sigma) \sum_{n=1}^{\infty} \sum_{\epsilon \in A_n} 2^{-w(\epsilon)} \\ &= \frac{1}{2} Q(1/\sigma) \sum_{n=1}^{\infty} (5/4)^n.\end{aligned}$$

It follows that (27) diverges for any noise level.

C. Asymptotic Probability of Error

In this section we show that whenever the error probability upper bound (27) converges for sufficiently low noise, both bounds (24) and (27) are asymptotically tight as $\sigma \to 0$. In particular, we prove that for any $\delta > 0$ there exists $\sigma_0 > 0$ such that for all $\sigma < \sigma_0$,

$$C_k^L Q(d_{k,\min}/\sigma) \le P_k \le C_k^U (1+\delta) Q(d_{k,\min}/\sigma), \quad (46)$$

where

$$C_k^L = P[d_k(\boldsymbol{b}) = d_{k,\min}] = P\left[\bigcup_{\substack{\epsilon \in F_k \text{ s.t.} \\ \|S(\epsilon)\| = d_{k,\min}}} \{\epsilon \in A(\boldsymbol{b})\}\right] \quad (47)$$

and

$$C_k^U = \sum_{\substack{\epsilon \in F_k \text{ s.t.} \\ \|S(\epsilon)\| = d_{k,\min}}} 2^{-w(\epsilon)} = \sum_{\substack{\epsilon \in F_k \text{ s.t.} \\ \|S(\epsilon)\| = d_{k,\min}}} P[\epsilon \in A(\boldsymbol{b})]. \quad (48)$$

The left-hand inequality of (46) was obtained in (24). Expression (47) follows because if $\epsilon \notin F_k$ has $\|S(\epsilon)\| = d_{k,\min}$, then it is decomposable into $\epsilon' + \epsilon''$ such that $\epsilon' \in F_k$, $\|S(\epsilon')\| = d_{k,\min}$ and $\{\epsilon \in A(\boldsymbol{b})\} \subset \{\epsilon' \in A(\boldsymbol{b})\}$. C_k^U defined in (48) is the sum of the coefficients in the bounding series (27) that correspond to the sequences with minimum energy.[5] The right-hand inequality of (46) follows from the following proposition.

Proposition 7: If $\sigma_0 > 0$ exists such that for all $0 < \sigma \le \sigma_0$

$$\sum_{\epsilon \in F_k} 2^{-w(\epsilon)} \exp(-\|S(\epsilon)\|^2/2\sigma^2) < \infty, \quad (49)$$

then

$$\lim_{\sigma \to 0} \sum_{\epsilon \in F_k} 2^{-w(\epsilon)} Q(\|S(\epsilon)\|/\sigma)/Q(d_{k,\min}/\sigma) = C_k^U. \quad (50)$$

Proof: First, we show that for any set $G \subset E$ and any constant $r \ge 0$ that satisfy

1) $\inf_{\epsilon \in G} \|S(\epsilon)\| > r$,
2) there exists σ_0 such that for all $0 < \sigma \le \sigma_0$, $\sum_{\epsilon \in G} 2^{-w(\epsilon)} \exp(-\|S(\epsilon)\|^2/2\sigma^2) < \infty$,

we have

$$\lim_{\sigma \to 0} \sum_{\epsilon \in G} 2^{-w(\epsilon)} Q(\|S(\epsilon)\|/\sigma)/Q(r/\sigma) = 0. \quad (51)$$

Consider the following inequalities:

$$\begin{aligned}&\sum_{\epsilon \in G} 2^{-w(\epsilon)} Q(\|S(\epsilon)\|/\sigma)/Q(r/\sigma) \\ &\le \sum_{\epsilon \in G} 2^{-w(\epsilon)} \exp([r^2 - \|S(\epsilon)\|^2]/2\sigma^2) \\ &\le \exp\left(\left[\inf_{\epsilon \in G} \|S(\epsilon)\|^2 - r^2\right]\left[1/2\sigma_0^2 - 1/2\sigma^2\right]\right) \\ &\quad \cdot \sum_{\epsilon \in G} 2^{-w(\epsilon)} \exp([r^2 - \|S(\epsilon)\|^2]/2\sigma_0^2) \quad (52)\end{aligned}$$

where the inequalities follow from $Q(\sqrt{x+y}) \le \exp(-x/2) Q(\sqrt{y})$ if $x \ge 0$ and $y \ge 0$, and 1) respectively. The left-hand side of (52) vanishes as $\sigma \to 0$ because as a result of 2) the series therein converges.

Particularizing this result to $r = \inf_{\epsilon \in F_k} \|S(\epsilon)\|$ and $G = \{\epsilon \in F_k$, such that $\|S(\epsilon)\| > r\}$, we see that (50) follows. Note that the infimum of $\|S(\epsilon)\|$ over both F_k and the subset $\{\epsilon \in F_k$, such that $\|S(\epsilon)\| > r\}$ are achieved because both sets have a finite subset whose elements have no greater energy than the rest of the elements in F_k and G. This implies that $r = d_{k,\min}$, and condition 1) is satisfied.

The high SNR-upper and lower bounds (46) to the kth user error probability differ by a multiplicative constant independent of the noise level. From (47) and (48) it can be seen that this constant is related to the degree of overlapping of the events $\{\{\epsilon \in A(\boldsymbol{b})\}, \epsilon \in F_k$ and $\|S(\epsilon)\| = d_{k,\min}\}$. Typically, there exists only a pair of elements in F_k, $\{\epsilon, -\epsilon\}$, that achieve the minimum energy. Since $\{\epsilon \in A(\boldsymbol{b})\} \cap \{-\epsilon \in A(\boldsymbol{b})\} = \varnothing$ for all $\epsilon \in E$, it follows that in such case $C_k^L = C_k^U$.

An important performance measure for multiuser detectors in high SNR situations is the SNR degradation due to the existence of other active users in the channel, i.e., the

[5] It can be shown that the error probability of the multiuser maximum-likelihood sequence detector can be lower bounded by $C_k^U Q(d_{k,\min}/\sigma)$.

limit as $\sigma \to 0$ of the ratio between the effective SNR (that required by a single-user system to achieve the same asymptotic error probability) and the actual SNR. We denote this parameter as η_k, the kth user asymptotic efficiency[6] defined formally as

$$\eta_k = \sup\left\{0 \le r \le 1;\ P_k(1/\sigma) = O\left(Q\left(\sqrt{rw_k}/\sigma\right)\right)\right\}. \quad (53)$$

This parameter depends both on the signal constellation and on the multiuser detector employed. In the case of the minimum-error probability detector, (46) indicates that in the high SNR region the behavior of the kth user error probability coincides with that of an antipodal single-user system with bit-energy equal to $d^2_{k,\min}$. Therefore, the maximum achievable asymptotic efficiency is given by

$$\eta_k = d^2_{k,\min}/w_k. \quad (54)$$

Since the upper bound (46) is actually an upper bound on the error probability of the multiuser maximum-likelihood sequence detector, this detector achieves the maximum asymptotic efficiency, although it is not optimum in terms of error probability. The set of K-user asymptotic efficiencies emerge as the parameters that determine optimum performance for all practical purposes in the SNR region of usual interest. In a sequel to this paper, we derive analytical expressions, bounds, and numerical methods for the computation of these parameters which play a central role in the analysis and comparison of multiuser detectors.

IV. Numerical Examples

Two pairs of lower and upper bounds to the kth user minimum-error probability have been presented in Section III, and they have been shown to be tight asymptotically; nonetheless, it remains to ascertain the SNR level for which such asymptotic approximation is sufficiently accurate. In the sequel this question is illustrated by several examples of the computation of averages and extreme cases of the foregoing bounds with respect to the relative delays of asynchronous users. The first example is a baseband asynchronous system with two equal-energy users that employ a simple set of signal waveforms (Fig. 4). In this figure the upper bounds on the best and worst cases of the optimum detector are indistinguishable from each other, and for SNR higher than about 6 dB, from the single-user lower bound (which is also the minimum energy lower bound since $\eta_1 = 1$). Note also that the maximum interference coefficient, $\bar{I}_k = \max_{\alpha,\beta} I_k(\alpha, \beta)$ (recall Proposition 3) is one-third for all delays, and the performance of the conventional receiver varies only slightly with the relative delay.

In the next examples, we employ a set of spread-spectrum signals: three maximal-length signature sequences of length 31 generated to maximize a signal-to-multiple-access interference functional [7, table 5]. The average probability of error of the conventional receiver for equal-energy users

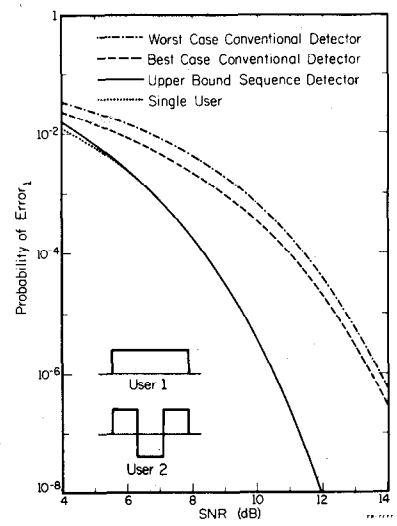


Fig. 4. Best and worst cases of error probability of user 1 achieved by conventional and optimum detectors.

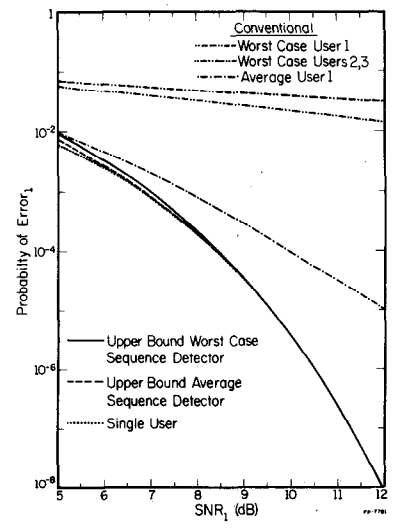
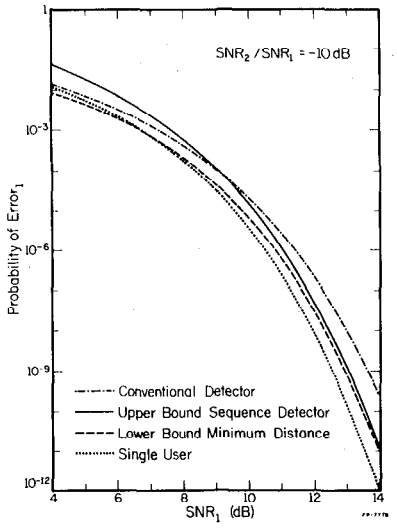


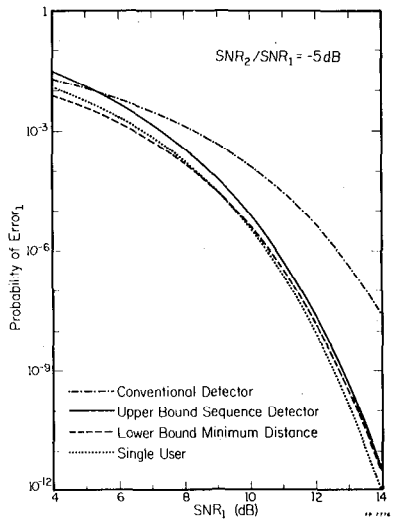
Fig. 5. Worst-case and average error probabilities achieved by conventional and optimum multiuser detectors with three active users employing m-sequences of length 31.

employing this signal set has been thoroughly studied previously [5], [14], and in Fig. 5 we reproduce (from [8, fig. 2]) the average error probability of user 1 achieved by the coherent conventional detector. Also shown in Fig. 5 are the worst cases of the conventional detector and upper bounds to the baseband worst-case and average minimum error probabilities for user 1. From the observation of Fig. 5, we can conclude that for error probabilities of 10^{-2} the average performance of the conventional detector is fairly close to the single-user lower bound, but the worst-case error probability is notably poor for the whole SNR range considered in the figure. Note, however, that since the signal set has good cross correlations for most of the relative delays, error probabilities close to the worst-case curve will occur with low probability. The worst-case and, especially, the average upper bounds on the optimum sequence detector performance are remarkably close to the single-user lower bound and show that the minimum-error

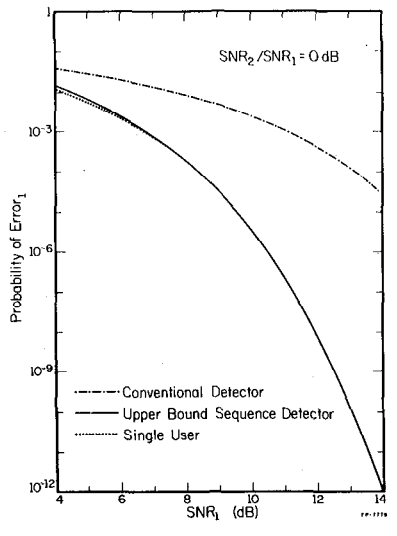
[6] This is not to be confused with Pitman's asymptotic measure of detection efficiency.



(a)



(b)



(c)

Fig. 6. Bounds on minimum error probability of user 1. Worst-case delays and two active users. (a) $w_2/w_1 = -10$ dB. (b) -5 dB. (c) 0 dB.

probability not only has a low average (around one order of magnitude better than that of the conventional detector, at 9 dB), but its dependence on the delays is negligible.

The next example investigates the *near–far* problem (i.e., the effects of unequal received energies) for two users that employ a subset of the previous set of maximal-length signature sequences. Bounds on the error probabilities corresponding to this example (worst-case relative delay between users 1 and 2) are calculated in Fig. 6 for three relative energies, namely, $\mathrm{SNR}_2/\mathrm{SNR}_1 = -10$ dB, -5 dB, and 0 dB. It is interesting to observe in the graphs corresponding to $\mathrm{SNR}_2/\mathrm{SNR}_1 = -10$ dB, -5 dB that all four bounds derived in this chapter play a role in some SNR interval; in particular, the error probability of the conventional detector is lower than the upper bound on the optimum sequence detector for small SNR. The opposite effect of an increase in the energy of the interfering users on the minimum and conventional probabilities of error is apparent: while the optimum sequence detector bounds become tighter and closer to the single-user lower bound, the conventional error probability grows rapidly until it becomes multiple-access limited (for $\mathrm{SNR}_2/\mathrm{SNR}_1 =$ 6.3 dB).

The results presented here open the possibility of a trade-off between the complexities of the receiver and the signal constellation in order to achieve a fixed level of performance; the actual compromise being dictated by the relative power of each user at the receiver. In multipoint-to-multipoint problems, when some active users need not be demodulated at a particular location, such a trade-off is likely to favor a multiuser detector that takes into account only those unwanted users that are not comparatively weak. If the signal constellation has moderate cross-correlation properties and the energy of the kth user is not dominant, then its minimum distance is achieved by an error sequence with only one nonzero component, and the minimum error probability approaches asymptotically the single-user bit error rate. This implies that contrary to what is sometimes conjectured, the performance of the conventional receiver is not close to the minimum error even if signals with low cross correlations are employed.

References

[1] P. R. Chevillat, "*N*-User trellis coding for a class of multiple-access channels," *IEEE Trans. Inform. Theory*, vol. IT-27, no. 1, pp. 114–120.

[2] G. D. Forney, "Lower bounds on error probability in the presence of large intersymbol interference," *IEEE Trans. Commun.*, vol. COM-20, pp. 76–77, Feb. 1972.

[3] ——, "Maximum likelihood sequence estimation of digital sequences in the presence of intersymbol interference," *IEEE Trans. Inform. Theory*, vol. IT-18, pp. 363–378, May 1972.

[4] ——, "The Viterbi algorithm," *Proc. IEEE*, vol. 61, pp. 268–278, Mar. 1973.

[5] G. J. Foschini, "Performance bound for maximum-likelihood reception of digital data," *IEEE Trans. Inform. Theory*, vol. IT-21, pp. 47–50, Jan. 1975.

[6] R. Gagliardi, "*M*-link multiplexing over the quadrature communications channel," *IEEE Commun. Mag.*, vol. 22, pp. 22–30, Sep. 1984.

[7] F. D. Garber and M. B. Pursley, "Optimal phases of maximal

sequences for asynchronous spread-spectrum multiplexing," *Electron. Lett.*, vol. 16, pp. 756–757, Sep. 11, 1980.

[8] E. A. Geraniotis and M. B. Pursley, "Error probability for direct-sequence spread-spectrum multiple-access communications-Part II: Approximations," *IEEE Trans. Commun.*, vol. COM-30, pp. 985–995, May 1982.

[9] D. Horwood and R. Gagliardi, "Signal design for digital multiple access communications," *IEEE Trans. Commun.*, vol. COM-23, pp. 378–383, Mar. 1975.

[10] J. K. Omura, "Performance bounds for Viterbi algorithms," in *1981 IEEE Int. Commun. Conf. Rec.*, pp. 2.21–2.25.

[11] C. H. Papadimitrou and K. Steiglitz, *Combinatorial Optimization: Algorithms of Complexity.* Englewood Cliffs, NJ: Prentice-Hall, 1982.

[12] H. V. Poor, "Signal detection in multiple access channels," U.S. Army Research Office Proposal (Contract DAAG29-81-K-0062 to Coordinated Science Laboratory, Univ. of Illinois), 1980.

[13] J. G. Proakis, *Digital Communications.* New York: McGraw-Hill, 1983.

[14] M. B. Pursley, D. V. Sarwate, and W. E. Stark, "Error probability for direct-sequence spread-spectrum multiple-access communications—part I: Upper and lower bounds," *IEEE Trans. Commun.*, vol. COM-30, pp. 975–984, May 1982.

[15] J. E. Savage, "Signal detection in the presence of multiple-access noise," *IEEE Trans. Inform. Theory*, vol. IT-20, pp. 42–49, Jan. 1974.

[16] K. S. Schneider, "Optimum detection of code division multiplexed signals," *IEEE Trans. Aerosp. Electron. Syst.*, vol. AES-15, pp. 181–185, Jan. 1979.

[17] G. Ungerboeck, "Adaptive maximum likelihood receiver for carrier-modulated data transmission systems," *IEEE Trans. Commun.*, vol. COM-22, pp. 624–636, May 1974.

[18] W. Van Etten, "Maximum likelihood receiver for multiple channel transmission systems," *IEEE Trans. Commun.*, vol. COM-24, pp. 276–283, Feb. 1976.

[19] S. Verdú, "Optimum sequence detection of asynchronous multiple-access communications," in *Abstracts of Papers: IEEE 1983 Int. Symp. Inform. Theory*, Sep. 1983, p. 80.

[20] ——, "Minimum probability of error for asynchronous multiple-access communication systems," in *Proc. 1983 IEEE Mil. Commun. Conf.*, vol. 1, Nov. 1983, p. 80, pp. 213–219.

[21] S. Verdú and H. V. Poor, "Backward, forward and backward-forward dynamic programming models under commutativity conditions," in *Proc. 23rd IEEE Conf. Decision Contr.*, Dec. 1984, pp. 1081–1086.

[22] S. Verdú, *Optimum Multi-user Signal Detection*, Ph.D. dissertation, Dep. Elec. Comput. Eng., Univ. of Illinois, Urbana-Champaign, Aug. 1984.

[23] A. J. Viterbi and J. K. Omura, *Principles of Digital Communication and Coding.* New York: McGraw-Hill, 1979.

[24] S. Verdú, "New bound on the error probability of maximum likelihood sequence detection of signals subject to intersymbol interference," in *Proc. 1985 Conf. Inform. Sci. Syst.*, Mar. 1985, pp. 413–418.