Fourier Transform

- A time-domain function $x(t)$ (not necessarily periodic) has a frequency-domain specification $X(w)$:

  $$X(w) = \int_{-\infty}^{\infty} x(t)e^{-jwt}dt,$$
  \hspace{1cm} (1)

  $$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(w)e^{jwt}dw$$
  \hspace{1cm} (2)

- The right hand side of 2, known as Fourier Integral, is of the nature a Fourier Series with fundamental frequency $\Delta w$ approaching zero.

- We call $X(w)$ the direct Fourier Transform of $x(t)$, $x(t)$ the inverse Fourier Transform of $X(w)$, and $x(t)$ and $X(w)$ a Fourier Transform pair. Symbolically this is expressed as:

  $$X(w) = \mathcal{F}[x(t)], \hspace{0.5cm} x(t) = \mathcal{F}^{-1}[X(w)]$$

  or

  $$x(t) \leftrightarrow X(w)$$

- Instead of using angular frequency $\Delta w$, one can also use frequency $\Delta f$ (where $w = 2\pi f$) the corresponding Fourier Transform pair is:

  $$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt,$$
  \hspace{1cm} (3)

  $$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$$
  \hspace{1cm} (4)

Spectrum

- $G(w)$ is complex $\Rightarrow$ to plot the spectrum $G(w)$ as a function of $w$, we need to plot both

  **the amplitude spectrum**, $|X(w)|$ vs $w$,

  and **the phase spectrum**, $\theta_x(w)$ vs $w$,

  where

  $$G(w) = |G(w)|e^{j\theta_x(w)}.$$
Conjugate Symmetry Property

- If \( x(t) \) is a real function of \( t \), then \( G(w) \) and \( G(-w) \) are complex conjugates, i.e.

\[
G(-w) = G^*(w)
\]

\( \implies \) the amplitude spectrum \( |G(w)| \) is an even function, and the phase spectrum \( \theta_x(w) \) is an odd function

\[
|G(-w)| = |G(w)|, \quad \theta_x(-w) = -\theta_x(w)
\]

- This **conjugate symmetry property** holds only for **real** signals.

- A similar property also holds for the Fourier series of **periodic real** signals.

Existence of Fourier Transform

- Same as the existence of Fourier series: strong Dirichlet conditions and weak Dirichlet conditions.

- Again, any signal that can be generated in practice satisfies the Dirichlet conditions, \( \implies \) the physical existence of a signal is therefore a sufficient condition for the existence of its Fourier Transform.

Relation between Fourier Transform and Fourier Series

- We can use Fourier Transform to compute the Fourier-Series coefficients, \( x_n \), for a periodic signal \( x(t) \).

  - Assume that the \( x(t) \) has period \( T_0 \). Truncate \( x(t) \) to \( x_{\text{trunc}}(t) \) such that \( x_{\text{trunc}}(t) = x(t) \) for \( -T_0/2 \leq t \leq T_0/2 \) and \( x_{\text{trunc}}(t) = 0 \) elsewhere. (You have to use the segment that is centered around the origin, that is, \([-T_0/2, T_0/2]\); other segments of duration \( T_0 \) would not work correctly.)

  - Find the Fourier Transform, \( X_{\text{trunc}}(f) \) or \( X_{\text{trunc}}(w) \), for the truncated signal \( x_{\text{trunc}}(t) \). (Make use of the Fourier Transform table and Fourier Transform theorems/properties).

  - Evaluate the Fourier transform of the truncated signal at \( f = n/T_0 \) or \( w = 2\pi/T_0 \) to obtain the \( n \)th harmonic and multiply by \( 1/T_0 \).

  - To sum up:

\[
\begin{align*}
  x(t) \implies x_{\text{trunc}}(t) \implies X_{\text{trunc}}(f) &\implies x_n = \frac{1}{T_0} X_{\text{trunc}}(n\frac{1}{T_0}), \\
  x(t) \implies x_{\text{trunc}}(t) \implies X_{\text{trunc}}(w) &\implies x_n = \frac{1}{T_0} X_{\text{trunc}}(n\frac{2\pi}{T_0})
\end{align*}
\]
Signal Bandwidth

- The **bandwidth** of a signal represents the range of frequencies present in the signal. 
  ➔ the wider the bandwidth, the larger the variations in the frequencies present.

- In general, we define the bandwidth of a real signal $g(t)$ as the range of *positive* frequencies present in the signal.
  ➔ $g(t) \rightarrow$ compute $G(f) \rightarrow$ find range of positive frequencies: $\text{BW} = W_{\text{max}} - W_{\text{min}}$, where $W_{\text{max}}$ is the highest positive frequency present in $G(f)$ and $W_{\text{min}}$ is the lowest positive frequency present in $X(f)$.

Useful Properties of Fourier

- **Linearity:** If $g_1(t) \Leftrightarrow G_1(w)$ and $g_2(t) \Leftrightarrow G_2(w)$, then
  $$a_1g_1(t) + a_2g_2(t) \Leftrightarrow a_1G_1(w) + a_2G_2(w)$$

- **Duality:** if $g(t) \Leftrightarrow G(f)$, then
  $$G(t) \Leftrightarrow g(-f) \quad \text{and} \quad G(-t) \Leftrightarrow g(f).$$

- **Scaling:** if $g(t) \Leftrightarrow G(f)$, then
  $$g(at) \Leftrightarrow \frac{1}{|a|}G\left(\frac{f}{a}\right), \quad a \neq 0$$
  **Comments:** If $a > 1$, then $g(at)$ is a contracted form of $g(t)$; if $a < 1$, then $g(at)$ is an expanded form of $g(t)$. When we expand a signal in the time domain, its frequency-domain representation (Fourier transform) contracts; if we contract a signal in the time domain, its frequency domain representation expands. (Intuitively, contracting a signal in the time domain makes the changes in the signal more abrupt, thus increasing its frequency content.)

- **Convolution:** if $g(t) \Leftrightarrow G(f)$ and $h(t) \Leftrightarrow H(f)$, then
  $$g(t) * h(t) \Leftrightarrow G(f)H(f)$$
  **Comments:** Finding the response of a linear time invariant (LTI) system to a given input is much easier in the frequency domain than in the time domain. This property is the basis for frequency domain analysis of LTI systems.

- **Shift in time domain:** if $g(t) \Leftrightarrow G(f)$, then
  $$g(t - t_0) \Leftrightarrow e^{-jt_0f}G(f)$$
  **Comments:** A change in the time origin does not change the magnitude of the transform; it only introduces a phase shift linearly proportional to the time shift (or delay).
- **Shift in frequency domain (Modulation Theorem):** if \( g(t) \leftrightarrow G(f) \), then

\[
g(t)e^{j2\pi f_0 t} \leftrightarrow G(f - f_0)
\]

or equivalently

\[
g(t)\cos(2\pi f_0 t) = \frac{1}{2}g(t)e^{j2\pi f_0 t} + \frac{1}{2}g(t)e^{-j2\pi f_0 t} \leftrightarrow \frac{1}{2}G(f - f_0) + \frac{1}{2}G(f + f_0)
\]

**Comments:** This relation is the basis of the operation of amplitude modulation systems.

- **Parseval’s Relation:** if \( g(t) \leftrightarrow G(f) \) and \( h(t) \leftrightarrow H(f) \), then

\[
\int_{-\infty}^{\infty} g(t)h^\ast(t)dt = \int_{-\infty}^{\infty} G(f)H^\ast(f)df
\]

Specifically, if we let \( g(t) = h(t) \), then we have the **Rayleigh Theorem**:

\[
\int_{-\infty}^{\infty} |g(t)|^2dt = \int_{-\infty}^{\infty} |G(f)|^2df
\]

**Comments:** Two ways of evaluating the energy of a signal.

- **Autocorrelation:** The (time) correlation function of the signal \( g(t) \) is denoted by \( R_g(\tau) \) and is defined by

\[
R_g(\tau) = \int_{-\infty}^{\infty} g(t)g^\ast(t - \tau)dt.
\]

If \( \mathcal{F}(g(t)) = G(f) \), then

\[
\mathcal{F}[R_g(\tau)] = |G(f)|^2
\]

**Comments:** we can conclude that the Fourier transform of the auto correlation of a signal is always a real-valued, positive function.

- **Differentiation in time domain:**

\[
\frac{d}{dt}g(t) \leftrightarrow j2\pi fG(f)
\]

**Comments:** With repeated application of the differentiation theorem, we obtain the relation

\[
\frac{d^n}{dt^n}g(t) \leftrightarrow (j2\pi f)^nG(f)
\]

- **Differentiation in frequency domain:**

\[
tg(t) \leftrightarrow \frac{j}{2\pi} \frac{d}{df}G(f)
\]

**Comments:** With repeated application, we have

\[
t^n g(t) \leftrightarrow \left(\frac{j}{2\pi}\right)^n \frac{d^n}{df^n}G(f)
\]
**Integration:** if \( g(t) \leftrightarrow G(f) \), then

\[
\mathcal{F} \left[ \int_{-\infty}^{t} g(\tau)d\tau \right] = \frac{G(f)}{j2\pi f} + \frac{1}{2}G(0)\delta(f)
\]

Proof: first show \( \int_{-\infty}^{t} g(\tau)d\tau = g(t) * u_{-1}(t) \); then use the convolution theorem and the fact that the Fourier transform of \( u_{-1}(t) \) is \( \frac{1}{j2\pi f} + \frac{1}{2}\delta(f) \). \( u_{-1}(t) \) is the unit step function. \( u_{-1}(t) = 1 \) when \( t > 0 \), \( u_{-1}(t) = 0 \) when \( t < 0 \), and \( u_{-1}(t) = 1/2 \) when \( t = 0 \).

**Moments:** If \( g(t) \leftrightarrow G(f) \), then \( \int_{-\infty}^{\infty} t^n g(t)dt \), the \( n \)th moment of \( g(t) \), can be obtained from the relation

\[
\int_{-\infty}^{\infty} t^n g(t)dt = \left. \left( \frac{j}{2\pi} \right)^n \frac{d^n}{df^n} G(f) \right|_{f=0}
\]

Proof: First, using the differentiation in the frequency domain result, we have

\[
\mathcal{F}[t^n g(t)] = \left( \frac{j}{2\pi} \right)^n \frac{d^n}{df^n} G(f)
\]

This means that

\[
\int_{-\infty}^{\infty} t^n g(t)e^{j2\pi ft}dt = \left( \frac{j}{2\pi} \right)^n \frac{d^n}{df^n} G(f).
\]

Letting \( f = 0 \) on both sides, we obtain the desired result.

Comments: For the special case of \( n = 0 \), we obtain this simple relation for finding the area under a signal, i.e.

\[
\int_{-\infty}^{\infty} g(t)dt = G(0)
\]

**Summary Comments on Fourier Series and Fourier Transforms**

- Fourier representation is a way of expressing a signal in terms of everlasting sinusoids and exponentials.

- The Fourier spectrum of a signal indicates the relative amplitude and phases of the sinusoids/exponentials that are required to synthesize that signal.

- If \( x(t) \) is periodic, the spectrum is discrete, and \( x(t) \) can be expressed as a sum of discrete exponentials with finite amplitudes

\[
x(t) = \sum_{n} x_n e^{jnw_0 t}.
\]

That is, the Fourier spectrum of a periodic signal has finite amplitudes and exists at discrete frequencies (\( w_0 \) and its multiples).

- An aperiodic signal has continuous spectrum that exists at every frequency (i.e. every value of \( w \)), but the amplitude of each component in the spectrum is zero (infinitesimal). The meaningful measure here is not the amplitude of a component of some frequency but the spectral density per unit bandwidth.