"Spectral Efficiency in the Wideband Regime" *and* "Recent Results on the Capacity of Wideband Channels in the Low-Power Regime"

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Abstract

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Conventional information-theoretic study of wideband communication is often based on the "**infinite bandwidth**" analysis and assume that infinite bandwidth system is a natural extension of the "**wideband**" system. The works by Verdu, as will be reviewed below, reveal the inaccuracy of those assumptions, and that cautions should be taken when analyzing performance in the wideband regime . New analysis criterion, such as the wideband slope is proposed to give more insights into designing the wideband systems.

Backgrounds

The capacity of an AWGN channel has been formulated by Shannon in 1948 as

$$C = W \log_2(1 + \frac{P}{WN_0}) \textbf{ (b/s)}. \quad (1)$$

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- C is monotonically increasing with W, above gives the maximum C achievable by the a given power P

Backgrounds

Minimum Energy per bit Required for reliable communication:

$$(\frac{E_b}{N_0})_{\min} = \frac{P}{R_{\max}} = \frac{P}{C} = \log_e 2 = -1.59dB \qquad (3)$$

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- While the above computations are all based on the AWGN assumption, study shows that the lower of $\frac{E_b}{N_0}$ obtained above actually also hold for *any* fading channel as long as the background noise is Gaussian.
- Yet smaller lower bound of $\frac{E_b}{N_0}$ is possible under non-Gaussian background noise.

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- Flash signaling is capacity approaching optimal for wideband system;
- Availability of Channel State Information (CSI) would not affect capacity of wideband system;
- Capacity of wideband system is not affected by fading;

Is that really TRUE?

Spectrum Efficiency

Definition 1

$$\overline{C}(\frac{E_b}{N_0}) = \log_2(1 + \frac{P}{WN_0}) = \log_2(1 + \frac{E_b}{N_0}\overline{C}) \quad (4)$$

The spectral efficiency can actually also be written as a function or SNR, that is, $\widetilde{C} = \widetilde{C}(SNR)$, and we have

Definition 2

$$\frac{E_b}{N_0}\widetilde{C}(SNR) = SNR. \quad (5)$$

Wideband Slope $\mathcal{S}_0$

- The traditional paradigm that maximizes data rate for given power and bandwidth may be unapplicable in the wideband regime.
- A more sensible approach is to minimize bandwidth for a given rate and power.

Wideband Slope $\mathcal{S}_0$

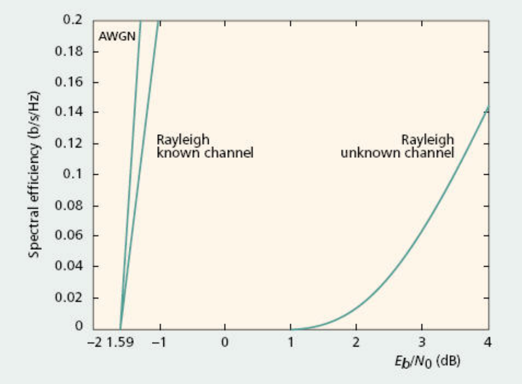

Figure: Spectral efficiency of AWGN and flat-fading Rayleigh channel, with and without CSI .

Wideband Slope $\mathcal{S}_0$

Complexity-Performance Tradeoff

As is shown in the previous figure, with/without CSI results in quite "different paths" of approaching capacity.

Illustrating Example

For a particular $\frac{E_b}{N_0} = 1.25dB$ with Rayleigh fading

- Non-coherent detection: $\overline{C}_{NC} = 0.0011$ b/s/Hz
- Coherent detection: $\overline{C}_{C} = 1.1$ b/s/Hz

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- Non-coherent detection: $\overline{C}_{NC} = 0.0011$ b/s/Hz
- Coherent detection: $\overline{C}_C = 1.1$ b/s/Hz
- Given the same data rate and power, knowing the channel buys a factor of 1000 bandwidth.

Wideband lope $\mathcal{S}_0$

Definition of Wideband Slope $\mathcal{S}_0$

$$\mathcal{S}_0 \equiv \lim_{\frac{E_b}{N_0} \to (\frac{E_b}{N_0})_{\min}} \frac{\overline{C}(\frac{E_b}{N_0})}{\frac{E_b}{N_0} - (\frac{E_b}{N_0})_{\min}} \cdot 10 \log_{10} 2, \qquad (6)$$

$\mathcal{S}_0$ indicates how "fast" the capacity is approached in terms of required bandwidth.

With Taylor expansion, we can write $\mathcal{S}_0$ as:

$$\mathcal{S}_0 = \frac{2[\dot{\tilde{C}}(0)]^2}{-\ddot{\tilde{C}}(0)}, \qquad (7)$$

Wideband Slope $\mathcal{S}_0$

Also, $\widetilde{C} = \log_2(1 + SNR)$, we can compute the optimal wideband slope for AWGN channel is

$$(\mathcal{S}_0)_{Optimal} = 2(\mathbf{b/s/Hz/3dB}). \tag{8}$$

Yet for the on-off keying with duty cycle α,

$$\mathcal{S}_0 = \mathcal{S}_0(\frac{\alpha^2}{N_0}) = \frac{2|\alpha|^4}{N_0 \exp(\frac{2|\alpha|^2}{N_0}) - 1}, \tag{9}$$

maximize over α yields

$$\mathcal{S}_0 \leq 0.3238(\mathbf{b/s/Hz/3dB}), \tag{10}$$

On-off keying is no longer optimal, it requires at least **6** times of bandwidth to achieve capacity. QPSK instead approaches $(\mathcal{S}_0)_{Optimal}$.

CSI available at the receiver

Transmitter also knows H, but no power control

$$\mathcal{S}_0 = \frac{2\ell}{m\mathcal{K}(\sigma_{\max}(\mathbf{H}))} \tag{11}$$

(provable by water-filling), where $\sigma_{\max}(\mathbf{H})$ is the maximal singular value of the channel matrix $\mathbf{H}$, ℓ is the multiplicity of $\sigma_{\max}(\mathbf{H})$.
The **kurtosis** of a random variable Z is defined as:

$$\mathcal{K}(Z) = \frac{\mathbb{E}[Z^4]}{\mathbb{E}^2[Z^2]}. \tag{12}$$

The required minimum bandwidth of a fading channel is proportional to the kurtosis of the fading amplitude distribution.

CSI available at the receiver

Transmitter does not know the channel

$$\mathcal{S}_0 = \frac{2(\mathrm{Tr}(\mathbb{E}[\mathbf{H}^\dagger \mathbf{H}]))^2}{m(\mathrm{Tr}(\mathbb{E}[(\mathbf{H}^\dagger \mathbf{H})^2]))}, \tag{13}$$

If the entries of **H** are independent Rayleigh variables,

$$\mathcal{S}_0 = \frac{2nm}{n+m} \ \textbf{(b/s/Hz/3dB)}. \tag{14}$$

m, n represent the number of receive and transmit antennas.

The bandwidth required to approach wide-band capacity is closely related to the number of receive/**transmit** antennas.

CSI not available

If neither the receiver nor the transmitter has the channel information, which *often* indicates that

$$\lambda_{\max}(\mathbb{E}[\mathbf{H}]^{\dagger}\mathbb{E}[\mathbf{H}]) < \lambda_{\max}(\mathbb{E}[\mathbf{H}^{\dagger}\mathbf{H}]), \tag{15}$$

the wideband slope is then

$$\mathcal{S}_0 = 0 \tag{16}$$

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Multiple Access: TDMA

It has long been established that although TDMA can not achieve the full rate region (Pareto-optimal polygon), it does achieve a "single-user" capacity in both MAC and broadcast channels. Also, TDMA approach the full rate region in the limit of low SNR. In that case, there seems to be not much justification for adopting complex superposition technique (such as CDMA) for wide-band system.

- **Question**: Is TDMA so optimal?

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- **Question**: Is TDMA so optimal?
- : No, if bandwidth expenditure is taken into consideration.

Theme example: Broadcast Channels

$$\mathbf{Y_1} = \mathbf{X} + \mathbf{N_1},\ \mathbf{Y_2} = \mathbf{X} + \mathbf{N_2},\ \ \texttt{with}\ \mathbb{E}[|\mathbf{N_i}|] \leq \sigma_i^2. \tag{17}$$

Assume $R_1/R_2 = \theta$, the slope region achievable by TDMA is:

$$\{(\mathcal{S}_1, \mathcal{S}_2) : 0 \leq \mathcal{S}_1 \leq \frac{2\theta}{1+\theta}, 0 \leq \mathcal{S}_2 \leq \frac{2}{1+\theta}\}. \tag{18}$$

In comparison, optimum slope region achievable by superposition technique (such as CDMA) is:

$$\{(\mathcal{S}_1, \mathcal{S}_2) : \quad 0 \leq \mathcal{S}_1 \leq \frac{2\theta(\theta+\sigma_2^2/\sigma_1^2)}{\theta^2+2\theta+\sigma_2^2/\sigma_1^2} \quad 0 \leq \mathcal{S}_2 \leq \frac{2(\theta+\sigma_2^2/\sigma_1^2)}{\theta^2+2\theta+\sigma_1^2/\sigma_2^2} \quad \}. \tag{19}$$

Theme example: Broadcast Channels

The two regions are only identical when $\sigma_1^2 = \sigma_2^2$, in the general case of $\sigma_1^2 \neq \sigma_2^2$, CDMA can be much more bandwidth efficiency when approaching capacity.

Illustrating Example

Assume $\sigma_1^2 = 10\sigma_2^2$ and $\theta = 3$, we can show that

$$\{(\mathcal{S}_1, \mathcal{S}_2)_{\textbf{TDMA}} : 0 \leq \mathcal{S}_1 \leq \frac{3}{2}, 0 \leq \mathcal{S}_2 \leq \frac{1}{2}\}, \quad (20)$$

and

$$\{(\mathcal{S}_1, \mathcal{S}_2)_{\textbf{CDMA}} : 0 \leq \mathcal{S}_1 \leq \frac{78}{25}, 0 \leq \mathcal{S}_2 \leq \frac{260}{151}\}, \quad (21)$$

TDMA require more than twice the bandwidth to achieve capacity.

Closing Remarks

The study of the wideband low-power region can not follow the analysis result obtained directly from "infinite bandwidth" analysis. New criterion, such as the wideband slope may be more insightful than the conventional $\frac{E_b}{N_0}$ measure when designing system with "*wide-yet-still-precious*" bandwidth.

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Question Time!