On the Duality of Gaussian Multiple-Access and Broadcast Channels

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- Capacity Region of the Gaussian BC is equal to the *union* of capacity regions of the dual MAC, where the union is taken over all individual power constraint that sum up to the BC power constraint.

\[ C_{BC} = \bigcup \{ C_{MAC}(P_1, \ldots, P_K; h) \} \]

\[ \{ P_i \}_1^K : \sum_{i=1}^K P_i = \bar{P} \]

- MAC capacity region is equal to the *intersection* of dual BC capacity regions.

\[ C_{MAC}(\bar{P}_1, \ldots, \bar{P}_K; h) = \bigcap \{ C_{BC} \left( \sum_{i=1}^K \frac{\bar{P}_i}{\alpha_i}; \alpha_1 h_1, \ldots, \alpha_K h_K \right) \} \]

\[ \{ \alpha_i \}_1^K : \alpha_i > 0 \]
Outline

- System Model
- Capacity Region Analysis
  - Capacity Region of the MAC
  - Capacity Region of the BC
- Constant MAC and BC
  - MAC to BC
  - BC to MAC
- Fading MAC and BC
  - MAC to BC
  - BC to MAC
- Conclusion
System Model

BC: \( Y_j[i] = \sqrt{h_j[i]}X[i] + n_j[i] \)

MAC: \( Y[i] = \sum_{j=1}^{K} \sqrt{h_j[i]}X_j[i] + n[i] \)

The Broadcast Channel

\[ \mathbb{E}[n^2_1[i]] = \mathbb{E}[n^2_2[i]] = \ldots = \mathbb{E}[n^2_K[i]] = \mathbb{E}[n^2[i]] = \sigma^2 \]
The capacity region of a Gaussian MAC with channel gains $h = (h_1, \ldots, h_K)$ and power constraints $P = (P_1, \ldots, P_K)$ is

$$\begin{align*}
\{ R : \sum_{j \in S} R_j &\leq \frac{1}{2} \log(1 + \frac{1}{\sigma^2} \sum_{j \in S} h_j P_j) \quad \forall S \subseteq \{1, \ldots, K\} \}, \\
\end{align*}$$

With success decoding with interference cancellation, the corner points are

$$R_{\pi(j)} = \frac{1}{2} \log \left(1 + \frac{h_{\pi(j)} P_{\pi(j)}}{\sigma^2 + \sum_{i=j+1}^{K} h_{\pi(i)} P_{\pi(i)}} \right), j = 1, \ldots, K.$$
The capacity region of a Gaussian BC with channel gains $h = (h_1, \ldots, h_K)$ and power constraints $\overline{P}$ is

$$\left\{ R : R_j \leq \frac{1}{2} \log \left( 1 + \frac{h_j \overline{P}_j^B}{\sigma^2 + h_j \sum_{k=1}^{K} P_k^B \mathbf{1}[h_k > h_j]} \right) , j = 1, \ldots, K \right\}$$
Theorem

The capacity region of a constant Gaussian BC with power constraint $\overline{P}$ is equal to the union of capacity regions of the dual MAC with power constraint $(P_1, \ldots, P_K)$ such that $\sum_{j=1}^{K} P_j = \overline{P}$

$$C_{BC} = \bigcup_{P: \mathbf{1} \cdot P = \overline{P}} C_{MAC}(P; \mathbf{h}).$$

Corollary

The capacity region of a constant Gaussian MAC with power constraints $\overline{P} = (\overline{P}_1, \cdots, \overline{P}_K)$ is a subset of the capacity region of the dual BC with power constraint $P = \mathbf{1} \cdot P$

$$C_{MAC}(P; \mathbf{h}) \subseteq C_{BC}(\mathbf{1} \cdot P; \mathbf{h}).$$
MAC to BC (Constant)
MAC to BC (Constant)

- Channel Scaling:
  \[
  \text{from } (P; h) \text{ to } \left( \frac{P}{\alpha}; \alpha h \right)
  \]
The capacity region of a constant Gaussian MAC is equal to the intersection of the capacity regions of the scaled dual BC over all possible channel scalings:

$$
C_{MAC}(P; h) = \bigcap_{\alpha > 0} C_{BC}(1 \cdot \frac{P}{\alpha}; \alpha h)
$$
BC to MAC (Constant)

\[ \alpha \rightarrow 0 \]
\[ \alpha = \left( \frac{h_2}{h_1} \right)^2 \]

\[ \alpha \rightarrow \infty \]
Fading MAC and BC

Theorem

MAC to BC: The capacity region of a fading Gaussian BC with power constraint $\bar{P}$ is equal to the union of ergodic capacity regions of the dual MAC with power constraint $(P_1, \ldots, P_K)$ such that $\sum_{j=1}^{K} P_j = \bar{P}$

$$\mathcal{C}_{BC} = \bigcup_{P: 1 \cdot P = \bar{P}} \mathcal{C}_{MAC}(P; h).$$

Theorem

BC to MAC: The capacity region of a fading Gaussian MAC is equal to the intersection of the ergodic capacity regions of the dual BC over all scalings

$$\mathcal{C}_{MAC}(P; h) = \bigcap_{\alpha > 0} \mathcal{C}_{BC}(1 \cdot \frac{P}{\alpha}; \alpha h).$$
Fading MAC and BC

(a) Fading MAC to BC

(b) Fading BC to MAC
Duality between the Gaussian MAC and BC is defined by establishing fundamental relationships between the capacity regions of the MAC and BC with the same channel gains and the same noise power at all receivers.