Solutions  HW 3

2.4-16

(a) \[ f(t) \]

width \[ \frac{AB+1}{2} \]
height \[ f_1 \times f_2 \]
area \[ t \]

(b) \[ f_1(t) \]

width \[ \frac{AB+2}{2} \]
height \[ f_1 \times f_2 \]
area \[ t \]

\[ f_2(t - \tau) \quad (\tau = 0) \]

\[ f_2(t - 2\tau) \]

\[ f_2(-2\tau) \]
\(f_1(\tau)\) \(f_2(t-\tau)\) \(t=0\)

\(f_2(-4-\tau)\) \(t=4\)  
(+) move to left!

\(f_1(t) \times f_2(t)\)

Note area

Note: time \(t\)
(a) \( \lambda^2 + 8\lambda + 12 = (\lambda+2)(\lambda+6) \)

roots are \(-2 \\& \ -6\)

\(\therefore\) system is asymptotically stable

(b) \( \lambda(\lambda^2 + 3\lambda + 2) = \lambda(\lambda+1)(\lambda+2) \)

roots are 0, \(-1, -2\) one root on imaginary axis, system is marginally stable

(c) \( \lambda^2(\lambda^2+2) = \lambda^2(\lambda+j\sqrt{2})(\lambda-j\sqrt{2}) \)

roots are \(\lambda_1=0, \lambda_2=0, \lambda_3=-j\sqrt{2}, \lambda_4=j\sqrt{2} \)

\(\lambda_1\) and \(\lambda_2\) are repeated

\(\therefore\) system is unstable

(d) \( \lambda+1)(\lambda^2 - 6\lambda + 5) = (\lambda+1)(\lambda-1)(\lambda-5) \)

\(\lambda_1=-1, \lambda_2=1, \lambda_3=5\) 2 roots are in RHP

\(\therefore\) system is unstable
\[ c = \frac{1}{E_x} \int_0^1 f(t)x(t) \, dt = \frac{1}{E_x} \int_0^1 t \, dt = \frac{1}{E_x} \left[ \frac{1}{2} t^2 \right]_0^1 = \frac{1}{2} E_x \] 

but \( E_x = \int_0^1 dt = 1 \) \quad \Rightarrow \quad \boxed{c = 0.5} \]

Thus \( f(t) \approx 0.5 x(t) \quad [0 \leq t \leq 1] \)

\[ e(t) = f(t) - c x(t) = t - 0.5 \quad [0 \leq t \leq 1] \]

(is orthogonal) \quad = 0 \quad \text{outside of } [0 \leq t \leq 1]

\[ e(t) \parallel x(t) \quad \text{because} \]

\[ \int_0^1 \frac{(t-0.5)^2}{e(t)} x(t) \, dt = \left. \frac{t^2}{2} - 0.5t \right|_0^1 = 0 \]

\[ E_e = \int_0^1 (t-0.5)^2 \, dt = \frac{1}{2} \quad E_f = \int_0^1 f^2(t) \, dt = \int_0^1 t^2 \, dt = \frac{1}{3} \]

Note that \( E_f = c^2 E_x + E_e \)
\[ E_f = \int_0^1 f^2(t) \, dt = \int_0^1 t^2 \, dt = \frac{1}{3} \]

\[ C = \frac{1}{E_f} \int_0^1 x(t) f(t) \, dt = 3 \int_0^1 t \, dt = \frac{3}{2} = 1.5 \]

Thus, \( x(t) \approx 1.5 f(t) \)

\( e(t) = x(t) - 1.5f(t) = 1 - 1.5t \quad [0 \leq t \leq 1] \)

\[ e(t) = 0 \quad \text{outside } [0, 1] \]

\[ E_e = \int_0^1 (1 - 1.5t)^2 \, dt = \int_0^1 (1 - 1.5t)^2 + 2.25t^2 \, dt \]

\[ E_e = \frac{1}{4} \]
$$E_{f_1} = \int_0^1 \sin^2(4\pi t) \, dt = \frac{t}{2} - \frac{\sin(2.4\pi t)}{4.4\pi} \bigg|_0^1$$

$$E_{f_1} = \frac{1}{2} \quad \text{similarly} \quad E_{f_2} = E_{f_3} = E_{f_4} = 0.5$$

Now $$E_X = \int_0^1 \sin^2(2\pi t) \, dt = 0.5$$

Use eq. 3.25 to calculate the correlation coefficients for the four cases:

$$c_n = \frac{1}{\sqrt{E_f E_X}} \int_{-\infty}^{+\infty} f(t) x(t) \, dt$$

(a) $$c_n = \frac{1}{\sqrt{0.5(0.5)}} \int_0^1 \sin(2\pi t) \sin(4\pi t) \, dt = 0$$

(b) $$c_n = \frac{1}{\sqrt{0.5(0.5)}} \int_0^1 \sin(2\pi t)(-\sin 2\pi t) \, dt = -1 \quad \text{(maximum protection)}$$

(c) $$c_n = \frac{1}{\sqrt{0.5(0.5)}} \int_0^1 0.707 \sin 2\pi t \, dt = 0$$

(d) $$c_n = \frac{1}{\sqrt{0.5(0.5)}} \int_0^{1/2} 0.707 \sin 2\pi t \, dt = \frac{1.414}{\pi}$$