ECE 108 Signals and Systems
Spring 2007, Instructors: Prof. Tiffany Jing Li
Midterm (50 minutes, March 14, 9:10-10:00am)

- You are allowed to bring one piece (double-sided ok) of letter-size note. Otherwise, the exam is closed-book and closed-note.

- Disclaimer: By signing below, I testify that the work in the turned pages is my independent work. I did not cheat in any form.

Name:

ID:

Date:

Signature:

| Problem 1  | 30 points |
| Problem 2  | 40 points |
| Problem 3  | 30 points |
| Bonus      | 10 points |
| Total      | 110 points |
1. (30 points) Given \( (D^2 + 4D + 3)y(t) = (D + 5)f(t) \), calculate the impulse response \( h(t) \).

\[
Q(D) = D^2 + 4D + 3, \quad n = 2 \quad ; \quad P(D) = D + 5, \quad m = 1.
\]

\[
\Rightarrow b_n = 0
\]

\[
h(t) = b_0 \delta(t) + \left[ P(D) \gamma_1(t) \right] u(t)
\]

\[
Q(\lambda) = \lambda^2 + 4\lambda + 3 = (\lambda + 1)(\lambda + 3) = 0, \quad \lambda_1 = -1, \quad \lambda_2 = -3
\]

characteristic modes \( e^{-t}, e^{-3t} \)

\[
\gamma_1(t) = c_1 e^{-t} + c_2 e^{-3t}, \quad \gamma_2(t) = -c_1 e^{-t} - 3c_2 e^{-3t}
\]

for \( n = 2 \),

\[
\begin{align*}
\gamma_n(t=0) &= 0 \\
\gamma_n'(t=0) &= 1
\end{align*}
\]

\( \leftarrow \) known initial conditions

\[
\begin{align*}
0 &= c_1 + c_2 \\
1 &= -c_1 - 3c_2
\end{align*}
\]

\[
\Rightarrow \begin{cases} 
\quad c_1 = \frac{1}{2} \\
\quad c_2 = -\frac{1}{2}
\end{cases}
\]

\[
\gamma_n(t) = \frac{1}{2} \left( e^{-t} - e^{-3t} \right)
\]

\[
h(t) = (D + 5) \left[ \frac{1}{2} \left( e^{-t} - e^{-3t} \right) \right] u(t)
\]

\[
= \left[ -\frac{1}{2} e^{-t} + \frac{3}{2} e^{-3t} + \frac{5}{2} e^{-t} - \frac{5}{2} e^{-3t} \right] u(t)
\]

\[
= \left[ 2e^{-t} - e^{-3t} \right] u(t)
\]
2. (40 points)

(a) (26 points) Find the compact Trigonometric Fourier series for an ever-lasting periodic signal \( f(t) \) whose period is \( T_0 = 2\pi \), and whose value is \( f(t) = \frac{1}{2\pi} \) for \( 0 \leq t \leq 2\pi \).

Hints:
- \( \int \sin(ax)dx = -\frac{1}{a}\cos(ax) \),
- \( \int \cos(ax)dx = \frac{1}{a}\sin(ax) \),
- \( \int x\sin(ax)dx = \frac{1}{a^2}(\sin(ax) - ax\cos(ax)) \),
- \( \int x\cos(ax)dx = \frac{1}{a^2}(\cos(ax) + ax\sin(ax)) \),
- \( \int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax - 1) \).

\[ T_0 = 2\pi, \quad \therefore \omega_0 = 1. \]

\[ f(t) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos n\pi t + b_n \sin n\pi t \right). \]

\[ a_0 = \frac{1}{2} \quad \text{by inspection,} \]

\[ a_n = \frac{1}{\pi} \int_{0}^{2\pi} \frac{f(t)}{\sin n\pi t} dt = 0 \]

\[ b_n = \frac{1}{\pi} \int_{0}^{2\pi} \frac{f(t)}{\cos n\pi t} dt = -\frac{1}{\pi n} \]

\[ f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \left( -\frac{1}{\pi n} \right) \sin(n\pi t) \]

Note the relation:

\[
\begin{align*}
C_0 &= a_0 = \frac{1}{2} \\
C_n &= \sqrt{a_n^2 + b_n^2} = \sqrt{0 + b_n^2} = |b_n| \\
\theta_n &= \tan^{-1}\frac{b_n}{a_n} = \tan^{-1}\frac{\frac{1}{\pi n}}{\frac{1}{2\pi}} = \frac{\pi}{2}
\end{align*}
\]
(b) (14 points) Sketch the amplitude and phase spectra.
(c) (30 points) Prove the time-convolution property:

\[ f_1(t) * f_2(t) \iff F_1(u)F_2(u) \]

[Textbook p263]
3. (10 Bonus points) True or False:

CAUTION: You earn 2 bonus points for each correct answer, and lose 1 point for each incorrect answer.

(a) Consider \( x(t) = s(t) + g(t) \), where \( x(t) \), \( s(t) \) and \( g(t) \) are all real signals. Let \( X(w) \), \( S(w) \) and \( G(w) \) be their respective Fourier transform.

- \(|X(w)| = |S(w)| + |G(w)| \) (\( \neg \))
- \( \angle X(w) = \angle S(w) + \angle G(w) \) (\( \neg \))

(b) The Fourier transform of the signal \( x(t) \) is \( X(w) = j2\delta(w - 2) - j2\delta(w + 2) \)

- \( x(t) \) must be odd (\( \neg \))
- \( x(t) \) must be periodic (\( \neg \))
- \( x(t) \) must be a sinusoidal function (\( \neg \))

\[
X(w) = S(w) + G(w), \text{ but } |X(w)| \neq |S(w)| + |G(w)|
\]

- Complex values

\[
\angle X(w) \neq \angle S(w) + \angle G(w)
\]

If \( X(t) \) is real, then \( X(w) \) is conjugate symmetric.

If \( X(t) \) is real \& even, then \( X(w) \) is purely real \& even.

If \( X(t) \) is real \& odd, then \( X(w) \) is purely imaginary \& odd.

The reverse is also true.

Further if \( X(t) \) is periodic, then \( X(w) \) consists of only discrete spectral lines; and vice versa.