Carefully go over the notes, the homework, the quizzes, and the problems in this sample midterm. Also make sure that you have a good understand of the fundamental concepts.

1. True or False:

(a) If $x(t)$ is real, its Fourier Transform $X(w) ...$
- $X(w)$ is always conjugate symmetric (T)
- $|X(w)|$ is always even (T)
- $\angle X(w)$ is always odd (T)
(b) The period of $\sin(\frac{3\pi}{5}t)$ is $T = 10/3$. (T) it has both.
(c) A periodic signal has Fourier series, but it does not have Fourier transform. (F)
(d) An aperiodic signal $(-\infty < t < \infty)$ has Fourier transform but does not have Fourier series. (T)
(e) If a signal can be represented as a linear sum (with either finite or countably infinite terms) of sinusoidal signals, then what can be concluded for this signal:
- The signal must be periodic (F) or, aperiodic for a finite time interval trig Fourier series.
- The signal must be real (T)
- The signal must be even (F)
- The signal must be odd (F)

2. Simplify the following expressions:

(a) \[ \left( \frac{1}{jw+2} \right) \delta(w + 3) \]
\[ = \frac{1}{2 - 3j} \delta(w + 3) \]

(b) \[ \left( \frac{\sin(kw)}{w} \right) \delta(w) = k \frac{\sin(kw)}{kw} \delta(w) = k \text{sinc}(kw) \delta(w) \]
\[ = k \text{sinc}(0) \delta(w) = k \delta(w) \]

(c) \[ \int_{-\infty}^{\infty} (t^3 + 4) \delta(1-t) dt = (1^3 + 4) \int_{-\infty}^{1} \delta(1-t) dt = 5 \]
3. Consider a system specified by the equation
\[ (D^2 + 2D + 1)y(t) = Df(t) \] (1)

(a) Is this system asymptotically stable, marginally stable, or unstable?
\[ \lambda_1 = \lambda_2 = -1 \quad , \quad \text{stable} \]

(b) Find the zero-input response of this system for \( t \geq 0 \), if the initial conditions are \( y_0(0) = 2 \), and \( y'_0(0) = 1 \).

characteristic modes: \( e^{-t} \), \( te^{-t} \)

\[
\begin{align*}
\dot{y}_0(t) &= A_1 e^{-t} + A_2 t e^{-t} \quad \bigg|_{t=0} = 2 \\
\dot{y}_0(t) &= -A_1 e^{-t} + A_2 e^{-t} - A_2 t e^{-t} \quad \bigg|_{t=0} = 1 \\
\end{align*}
\]

\[ \Rightarrow \left\{ \begin{array}{l}
A_1 = 2 \\
-A_1 + A_2 = 1 \Rightarrow A_1 = 2 \\
A_2 = 3 \end{array} \right. \]

\[ \Rightarrow \quad \dot{y}_0(t) = 2 e^{-t} + 3 t e^{-t} \]
(c) Find the unit impulse response of this system.

\[ b_n = 0 \]

\[ h(t) = b_n \delta(t) + \left[ P(D) \gamma_n(t) \right] u(t) = \left[ P(D) \cdot \gamma_n(t) \right] u(t) \]

\[ \gamma_n(t) = C_1 e^{-t} + C_2 t e^{-t} \]

\[ \gamma_n(t) = -C_1 e^{-t} + C_2 e^{-t} - C_2 t e^{-t} \]

\[ \Rightarrow \begin{cases} C_1 + C_2 = 0 \\ \gamma_n(t = 0) = 0 \\ \gamma_n(t = 0) = 1 \end{cases} \Rightarrow C_1 = 0, \ C_2 = 1, \gamma_n(t) = t e^{-t} \]

\[ h(t) = \left\{ D \cdot \left[ t e^{-t} \right] \right\} u(t) \]

\[ = \left( e^{-t} - t e^{-t} \right) u(t) \]

\[ = (1-t) e^{-t} u(t) \]

(d) Find the zero-state response of this system, if the input is \( f(t) = u(t) - u(t-2) \).

\[ \gamma_n(t) = f(t) \ast h(t) = \int_{-\infty}^{\infty} f(\tau) \cdot h(t-\tau) d\tau \]

\[ = \int_{-\infty}^{t} (u(\tau) - u(\tau-2)) \cdot (1-t+\tau) e^{-t+\tau} u(t-\tau) d\tau \]

\[ = \begin{cases} \int_{0}^{2} (1-t+\tau) e^{-t+\tau} d\tau, & \text{if } t \geq 2 \\ \int_{0}^{t} (1-t+\tau) e^{-t+\tau} d\tau, & \text{if } 0 \leq t < 2 \end{cases} \]

\[ = \int_{0}^{2} e^{\tau} (1-t+\tau) e^{-t} d\tau + e^{-t} \int_{0}^{2} \tau e^{\tau} d\tau \]

\[ = e^{-t} (1-t) e^{\tau} + e^{-t} \tau e^{\tau} (\tau-1) = e^{-t} (\tau-t) \]

\[ = \begin{cases} e^{2-t} + te^{-t}, & \text{if } t \geq 2 \\ te^{-t}, & \text{if } 0 \leq t < 2 \end{cases} \]
4. A certain periodic signal is given by \( f(t) = 3 + \sqrt{3} \cos(2t) + \sin(2t) + \sin(3t) - 0.5 \cos(5t + \pi/3) \)

(a) What is the fundamental period?

\[
\omega_0 = 1 \quad \text{and} \quad T_0 = \frac{2\pi}{\omega_0} = 2\pi
\]

(b) Write the compact trigonometric Fourier series (Hint: combine the sine and cosine terms of the same frequency. All terms must appear in the cosine form with positive amplitudes. This can always be done by suitably adjusting the phase.)

\[
f(t) = 3 + 2 \cos \left( 2t + \frac{\pi}{6} \right) + \cos \left( 3t - \frac{\pi}{2} \right) + 0.5 \cos \left( 5t + \frac{\pi}{3} - \pi \right)
\]

\[
C_0 = 3, \quad C_2 = 2, \quad C_3 = 1, \quad C_5 = 0.5
\]

\[
\Omega_2 = \frac{\pi}{6}, \quad \Omega_3 = -\frac{\pi}{2}, \quad \Omega_5 = -\frac{2\pi}{3}, \quad \text{all others} = 0
\]

(c) Sketch the trigonometric Fourier spectra

(d) Write the exponential Fourier series for \( f(t) \).

\[
\mathbf{f}(t) = D_0 + \sum_{n=\pm 1}^{\infty} D_n e^{in\Delta t}
\]

where

\[
D_0 = C_0 = 3
\]

\[
D_n = \frac{1}{2} C_n e^{i\Omega_n} \implies D_2 = e^{\frac{i\pi}{6}}, \quad D_3 = \frac{1}{2} e^{-\frac{i\pi}{2}}, \quad D_5 = \frac{1}{4} e^{-\frac{2i\pi}{3}}
\]

\[
D_{-n} = D_n^* \implies D_{-2} = e^{-\frac{i\pi}{6}}, \quad D_{-3} = \frac{1}{2} e^{\frac{i\pi}{2}}, \quad D_{-5} = \frac{1}{4} e^{\frac{2i\pi}{3}}
\]
5. Find the Fourier transform of the unit impulse function $\delta(t)$.

\[
\mathcal{F}[\delta(t)] = \int_{-\infty}^{+\infty} \delta(t) \ e^{-j\omega t} \ dt
\]

\[
= e^{-j \cdot 0 \cdot t} = 1
\]

\[
\therefore \delta(t) \leftrightarrow 1
\]

\[
\therefore \int_{-\infty}^{+\infty} \delta(t) f(t) \ dt = f(0)
\]

Textbook. Page 248. Example 4.3

6. Prove the Fourier transform pair $e^{j\omega t} \leftrightarrow 2\pi \delta(w - w_0)$

- From the last problem: $\delta(t) \leftrightarrow 1$
- Symmetry (time-frequency duality): $F(t) \leftrightarrow 2\pi f(-w)$
- 1 $\leftrightarrow 2\pi \delta(-w)$
- Time-frequency inversion (scaling by $\alpha$, $\alpha = -1$):
- Frequency shifting:
- $f(-t) \leftrightarrow F(-w)$
- 1 $\leftrightarrow 2\pi \delta(w)$
- $f(t) e^{j\omega t} \leftrightarrow F(w - w_0)$
- $e^{j\omega t} \leftrightarrow 2\pi \delta(w - w_0)$
7. Prove the time convolutional property: if $f_1(t) \leftrightarrow F_1(w)$ and $f_2(t) \leftrightarrow F_2(w)$, then $f_1(t) * f_2(t) \leftrightarrow F_1(w)F_2(w)$

Textbook page 263