CSE397/497-013
Introduction to Mobile Robotics

# Kinematics Part II

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## The Big Picture for Today

- We'll look at a forward kinematic for mobile robots
  - *Allows us to map wheel velocities to changes in robot position and orientation*
  - *Motivates the need to measure the wheel velocities well (read as "good encoder recommendations")*
- Next we'll look at what rolling and sliding constraints for each type of wheel
  - *Motions of individual wheels can be combined to compute the (constraints on the) motion of the robot as a whole*
  - *Another means to recover the forward kinematics*
- Briefly kinematic based motion control

## Introduction: Mobile Robot Kinematics

- Aim
  - *Description of mechanical behavior of the robot for* design *and* control
  - *Similar to robot manipulator kinematics*
  - *However, mobile robots can move unbound with respect to its environment*
    - *There is no direct way to measure the robot's position*
    - *Position must be integrated over time*
    - *Leads to inaccuracies of the position (motion) estimate* -> One of the primary challenges in mobile robotics
  - *Understanding mobile robot motion starts with understanding wheel* constraints *placed on the robots mobility*

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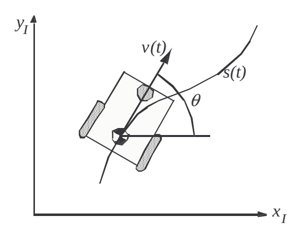
## Introduction: Kinematics Model

- Goal:
  - *Establish the robot velocity* $\dot{\xi} = [\dot{x} \quad \dot{y} \quad \dot{\theta}]^T$ *as a function of the wheel speeds* $\dot{\varphi}_i$, *steering angles* $\beta_i$, *steering speeds* $\dot{\beta}_i$ *and the geometric parameters of the robot (*configuration coordinates*).*
  - *Forward kinematics*

$$\dot{\xi} = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = f(\dot{\varphi}_1, \ldots \dot{\varphi}_n, \beta_1, \ldots \beta_m, \dot{\beta}_1, \ldots \dot{\beta}_m)$$




  - *Position estimation over time can be achieved by integrating over the velocities*

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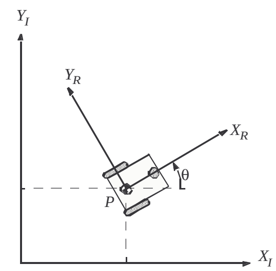
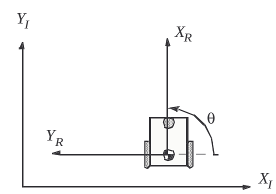
## Representing Robot Position

- Representing to robot within an arbitrary initial frame
  - *Initial frame:* $\{X_I, Y_I\}$
  - *Robot frame:* $\{X_R, Y_R\}$
  - *Robot position:* $\xi_I = [x \quad y \quad \theta]^T$
  - *Mapping between the two frames*
  - $\dot{\xi}_R = R(\theta)\dot{\xi}_I = R(\theta)\cdot[\dot{x} \quad \dot{y} \quad \dot{\theta}]^T$

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  - *Example: Robot aligned with $Y_I$*

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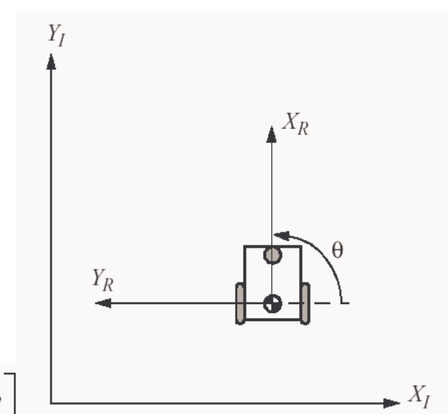
## Example

$$R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$




$$\dot{\xi}_R = R(\frac{\pi}{2})\dot{\xi}_I = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{y} \\ -\dot{x} \\ \dot{\theta} \end{bmatrix}$$

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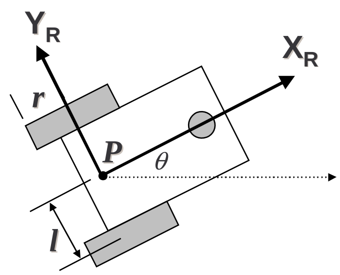
3.2.2

## Forward Kinematic Model

- Differential Drive Robot:
  - *Two drive wheels with radius* ***r***
  - *Robot frame centered at a point* ***P***
  - *Each wheel is a distance* ***l*** *from* ***P***
  - *The right and left wheel velocities are* $\dot{\varphi}_1$ and $\dot{\varphi}_2$




- Just for fun ☺ Given **r, l, θ,** $\dot{\varphi}_1$ and $\dot{\varphi}_2$ **,** let's derive the kinematic model that will allow us to predict the robot's overall speed in the global reference frame

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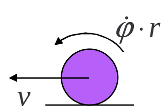
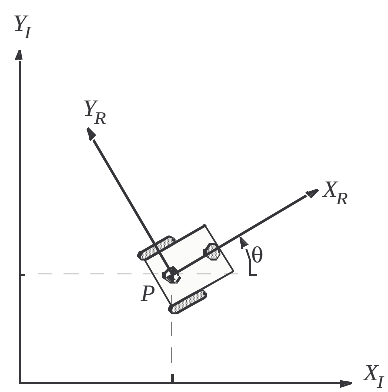
3.2.3

## Wheel Kinematic Constraints: Assumptions

- Movement on a horizontal plane
- Point contact of the wheels
- Wheels not deformable
- Pure rolling
  - v = 0 *at contact point*
- No slipping, skidding or sliding
- No friction for rotation around contact point
- Steering axes orthogonal to the surface
- Wheels connected by rigid frame (chassis)

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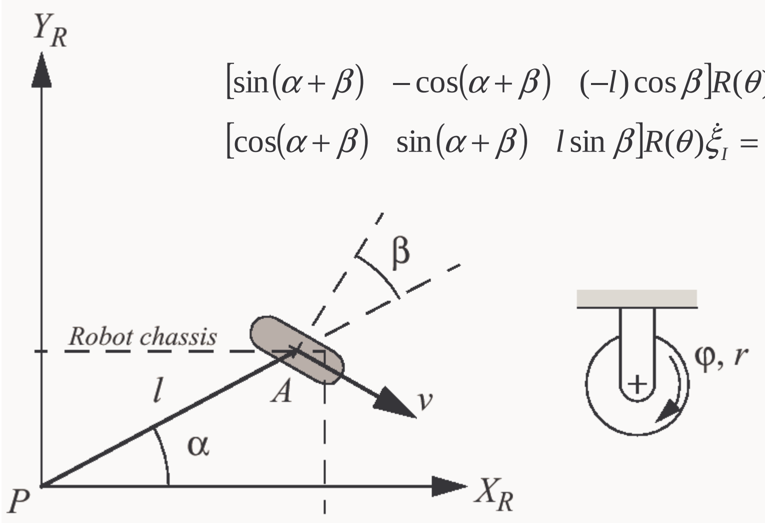
Wheel Kinematic Constraints:

## Fixed Standard Wheel




$$[\sin(\alpha+\beta) \quad -\cos(\alpha+\beta) \quad (-l)\cos\beta]R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

$$[\cos(\alpha+\beta) \quad \sin(\alpha+\beta) \quad l\sin\beta]R(\theta)\dot{\xi}_I = 0$$

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## Example

$$\left[\sin(\alpha+\beta) \; -\cos(\alpha+\beta) \; (-l)\cos\beta\right]R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

$$\left[\cos(\alpha+\beta) \; \sin(\alpha+\beta) \; l\sin\beta\right]R(\theta)\dot{\xi}_I = 0$$

- Suppose that the wheel A is in position such that α = 0 and β = 0
- This would place the contact point of the wheel on $X_I$ with the plane of the wheel oriented parallel to $Y_I$. If θ = 0, then the sliding constraint reduces to:

$$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = 0$$

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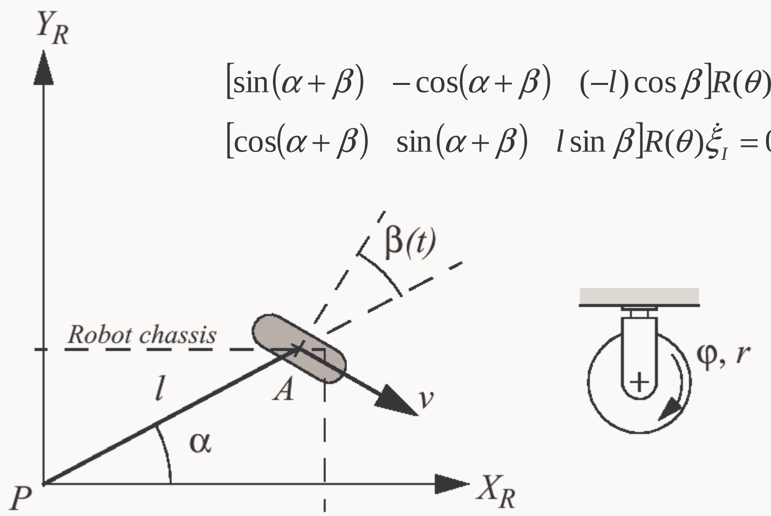
### Wheel Kinematic Constraints:

## Steered Standard Wheel

$$\left[\sin(\alpha+\beta) \quad -\cos(\alpha+\beta) \quad (-l)\cos\beta\right]R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

$$\left[\cos(\alpha+\beta) \quad \sin(\alpha+\beta) \quad l\sin\beta\right]R(\theta)\dot{\xi}_I = 0$$

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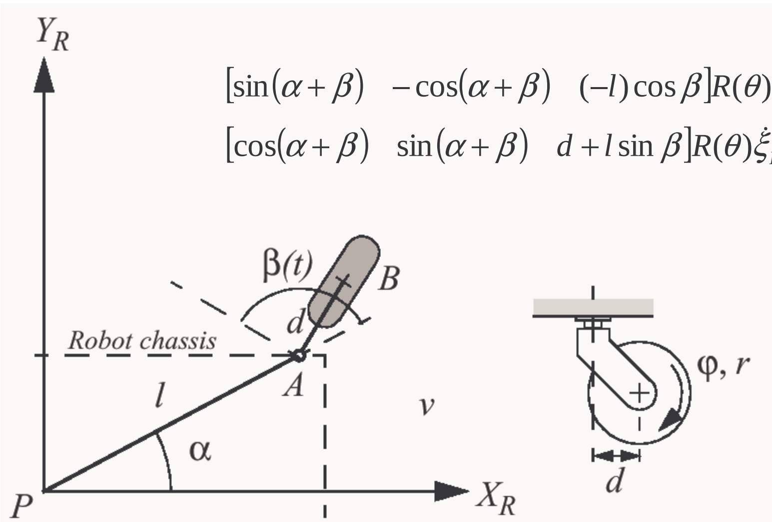
### Wheel Kinematic Constraints:

## Castor Wheel

$$\left[\sin(\alpha+\beta) \quad -\cos(\alpha+\beta) \quad (-l)\cos\beta\right]R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

$$\left[\cos(\alpha+\beta) \quad \sin(\alpha+\beta) \quad d + l\sin\beta\right]R(\theta)\dot{\xi}_I + d\dot{\beta} = 0$$

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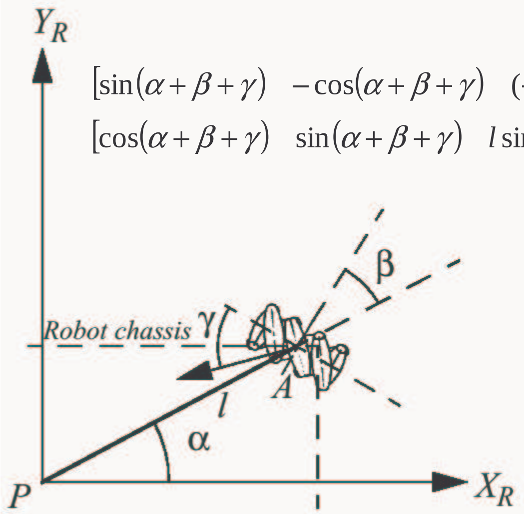
**Wheel Kinematic Constraints:**

## Swedish Wheel

$$[\sin(\alpha+\beta+\gamma) \quad -\cos(\alpha+\beta+\gamma) \quad (-l)\cos(\beta+\gamma)]R(\theta)\dot{\xi}_I - r\dot{\varphi}\cos\gamma = 0$$

$$[\cos(\alpha+\beta+\gamma) \quad \sin(\alpha+\beta+\gamma) \quad l\sin(\beta+\gamma)]R(\theta)\dot{\xi}_I - r\dot{\varphi}\sin\gamma - r_{sw}\dot{\varphi}_{sw} = 0$$





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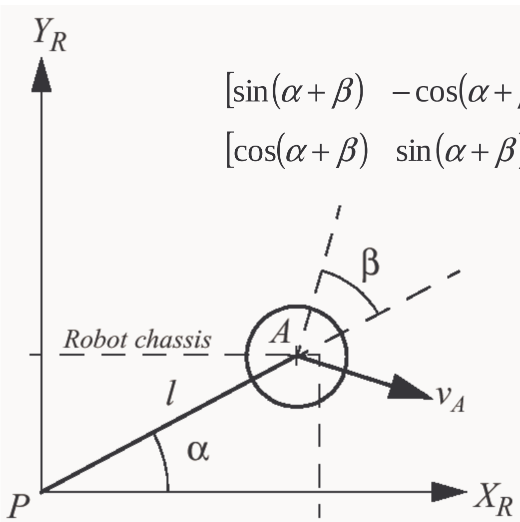
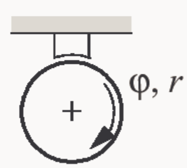
**Wheel Kinematic Constraints:**

## Spherical Wheel

$$[\sin(\alpha+\beta) \quad -\cos(\alpha+\beta) \quad (-l)\cos\beta]R(\theta)\dot{\xi}_I - r\dot{\varphi} = 0$$

$$[\cos(\alpha+\beta) \quad \sin(\alpha+\beta) \quad l\sin\beta]R(\theta)\dot{\xi}_I = 0$$

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## Robot Kinematic Constraints

- Given a robot with *M* wheels
  - *each wheel imposes zero or more constraints on the robot motion*
  - *only fixed and steerable standard wheels impose constraints*
- What is the maneuverability of a robot considering a combination of different wheels?
- Suppose we have a total of $N=N_f + N_s$ standard wheels
  - *We can develop the equations for the constraints in matrix forms:*
  - *Rolling*

$$J_1(\beta_s)R(\theta)\dot{\xi}_I + J_2\dot{\varphi} = 0 \qquad \varphi(t) = \begin{bmatrix} \varphi_f(t) \\ \varphi_s(t) \end{bmatrix}_{(N_f+N_s)\times 1} \qquad J_1(\beta_s) = \begin{bmatrix} J_{1f} \\ J_{1s}(\beta_s) \end{bmatrix}_{(N_f+N_s)\times 3} \qquad J_2 = diag(r_1 \cdots r_N)$$

  - *Lateral movement*

$$C_1(\beta_s)R(\theta)\dot{\xi}_I = 0 \qquad C_1(\beta_s) = \begin{bmatrix} C_{1f} \\ C_{1s}(\beta_s) \end{bmatrix}_{(N_f+N_s)\times 3}$$

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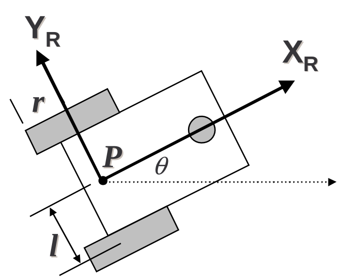
## Example: Differential Drive Robot

- Just for laughs ☺ Let's derive the kinematic constraints for a differential drive robot.
  - *How many fixed wheels?*
  - *How many steerable wheels?*
  - *Do we have to model the Castor wheel?*



## Mobile Robot Maneuverability: Degree of Mobility

- To avoid any lateral slip the motion vector $R(\theta)\dot{\xi}_I$ has to satisfy the following constraints:

$$C_{1f} R(\theta)\dot{\xi}_I = 0 \qquad C_1(\beta_s) = \begin{bmatrix} C_{1f} \\ C_{1s}(\beta_s) \end{bmatrix}$$

$$C_{1s}(\beta_s) R(\theta)\dot{\xi}_I = 0$$

- Mathematically:
  - ➢ $R(\theta)\dot{\xi}_I$ *must belong to the* ***null space*** *of the projection matrix* $C_1(\beta_s)$
  - ➢ Null space *of* $C_1(\beta_s)$ *is the space* **N** *such that for any vector* **n** *in* **N**

$$C_1(\beta_s) \cdot n = 0$$

  - ➢ *Geometrically this can be shown by the* **Instantaneous Center of Rotation (ICR)**

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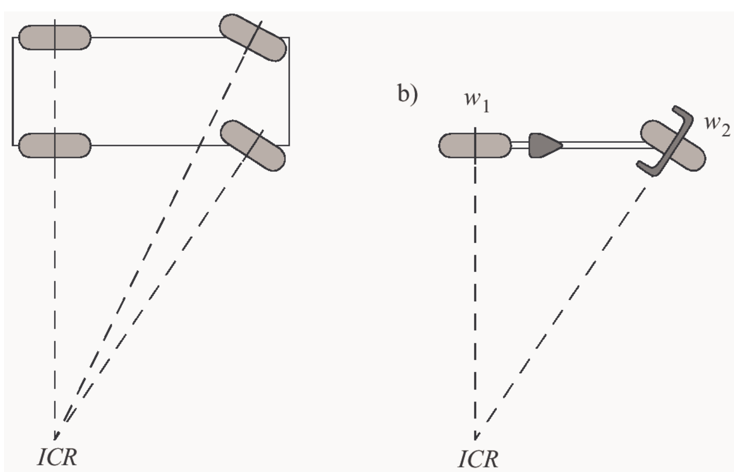
## Mobile Robot Maneuverability: Instantaneous Center of Rotation

- Ackermann Steering Bicycle

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## Mobile Robot Maneuverability: More on Degree of Mobility

- Robot chassis kinematics is a function of the set of *independent constraints* $\equiv rank[C_1(\beta_s)]$
  - *the greater the rank of* , $C_1(\beta_s)$ *the more constrained is the mobility*

- Mathematically

$$\delta_m = \dim N[C_1(\beta_s)] = 3 - rank[C_1(\beta_s)] \qquad 0 \le rank[C_1(\beta_s)] \le 3$$

  - *no standard wheels* $rank[C_1(\beta_s)] = 0$
  - *all direction constrained* $rank[C_1(\beta_s)] = 3$
- Examples:
  - *Unicycle: One single fixed standard wheel*
  - *Differential drive: Two fixed standard wheels*
    - *wheels on same axle*
    - *wheels on different axle*

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## Mobile Robot Maneuverability: Degree of Steerability

- Indirect degree of motion

$$\delta_s = rank[C_{1s}(\beta_s)]$$

  - *The particular orientation at any instant imposes a kinematic constraint*
  - *However, the ability to change that orientation can lead additional degree of maneuverability*

- Range of $\delta_s$ : $0 \le \delta_s \le 2$

- Examples:
  - *one steered wheel: Tricycle*
  - *two steered wheels: No fixed standard wheel*
  - *car (Ackermann steering):* $N_f = 2$, $N_s = 2$ *-> common axle*

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## Mobile Robot Maneuverability: Robot Maneuverability

- Degree of Maneuverability

$$\delta_M = \delta_m + \delta_s$$

  - *Two robots with same* $\delta_M$ *are not necessary equal*
  - *Example: Differential drive and Tricycle (next slide)*

  - *For any robot with* $\delta_M = 2$ *the ICR is always constrained to* lie on a line
  - *For any robot with* $\delta_M = 3$ *the ICR is not constrained an can* be set to any point on the plane

- The Synchro Drive example: $\delta_M = \delta_m + \delta_s = 1 + 1 = 2$

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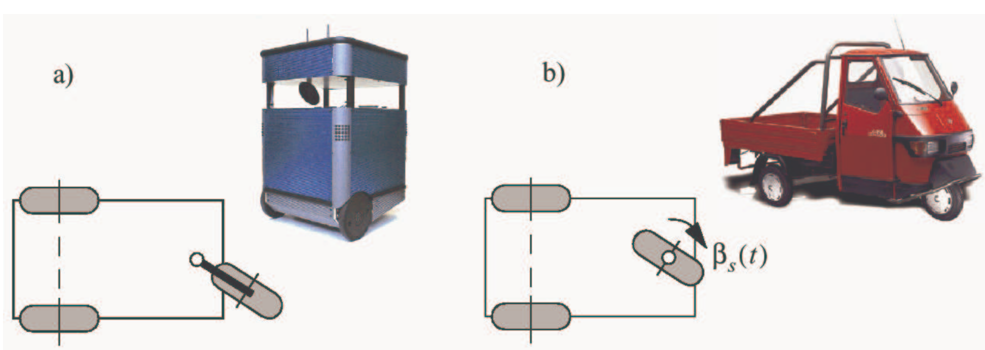
## Mobile Robot Maneuverability: Wheel Configurations

- Differential Drive Tricycle

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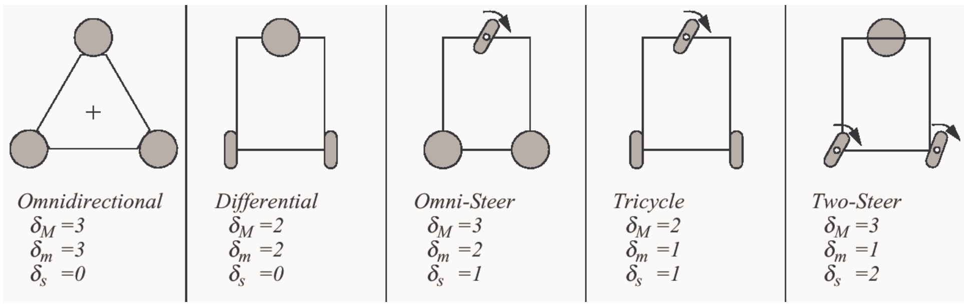
## Five Basic Types of Three-Wheel Configurations

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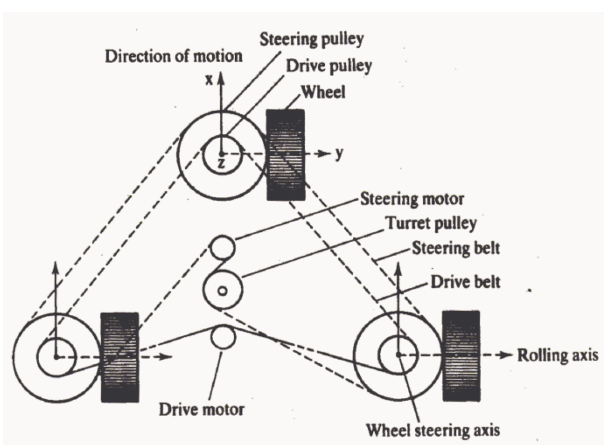
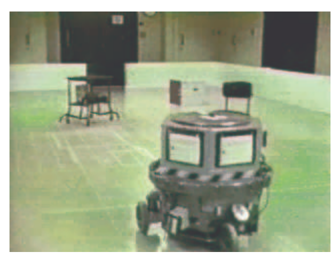
## Synchro Drive

$$\delta_M = \delta_m + \delta_s = 1 + 1 = 2$$



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## Mobile Robot Workspace: Degrees of Freedom

- Maneuverability is equivalent to the vehicle's degree of freedom (DOF)
- But what is the degree of vehicle's freedom in its environment?
  - *Car example*
- Workspace
  - *how the vehicle is able to move between different configuration in its workspace?*
- The robot's independently achievable velocities
  - = differentiable degrees of freedom (DDOF) = $\delta_m$
  - *Bicycle:* $\delta_M = \delta_m + \delta_s = 1 + 1$ *DDOF = 1; DOF=3*
  - *Omni Drive:* $\delta_M = \delta_m + \delta_s = 3 + 0$ *DDOF=3; DOF=3*

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## Mobile Robot Workspace: Degrees of Freedom, Holonomy

- DOF *degrees of freedom*:
  - *Robots ability to achieve various poses*
- DDOF *differentiable degrees of freedom*:
  - *Robots ability to achieve various path*

$$DDOF \leq \delta_m \leq DOF$$

- Holonomic Robots
  - *A holonomic kinematic constraint can be expressed a an explicit function of position variables only*
  - *A non-holonomic constraint requires a different relationship, such as the derivative of a position variable*
  - Fixed and steered standard wheels impose non-holonomic constraints

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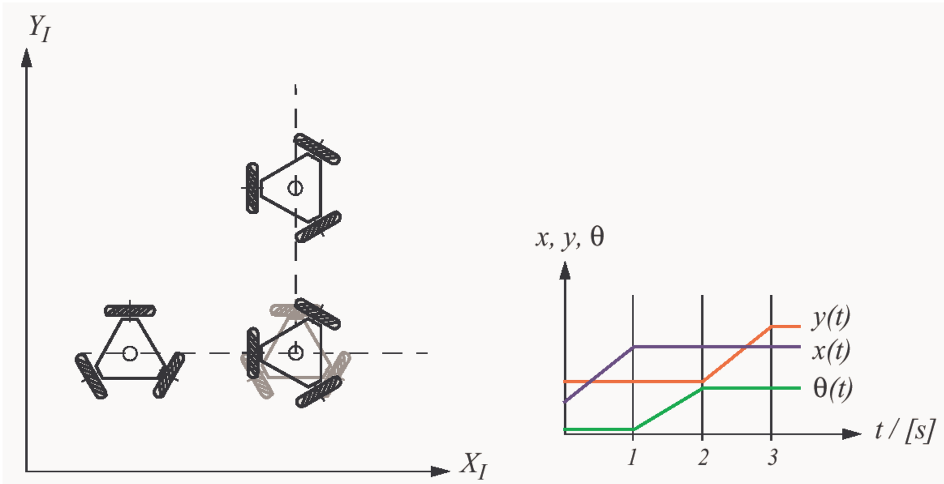
## Path / Trajectory Considerations: Omnidirectional Drive

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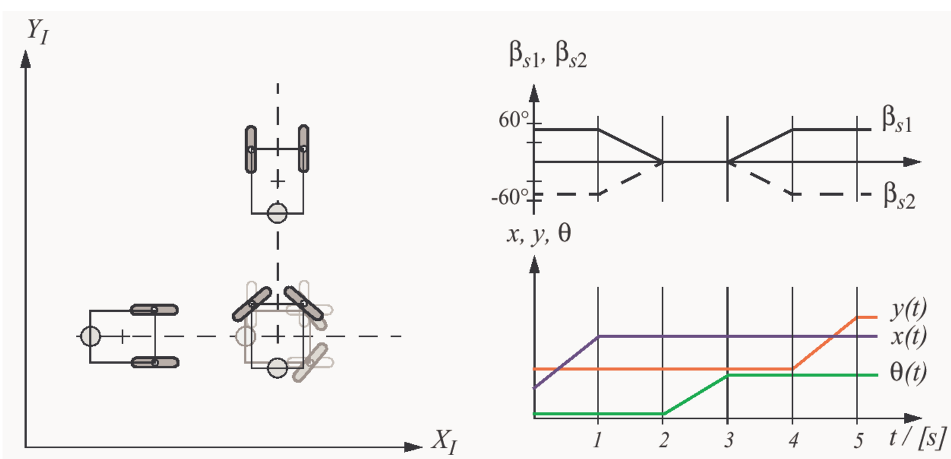
## Path / Trajectory Considerations: Two-Steer

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## An Aside Into Motion Control (kinematic control)

- The objective of a kinematic controller is to follow a trajectory described by its position and/or velocity profiles as function of time.
- Motion control is not straight forward because mobile robots are non-holonomic systems.
- However, it has been studied by various research groups and some adequate solutions for (kinematic) motion control of a mobile robot system are available.
- Most controllers are not considering the dynamics of the system

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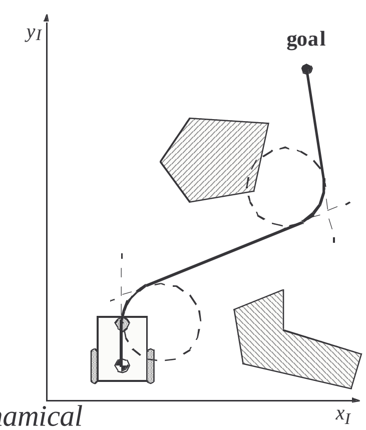
## Motion Control: Open Loop Control

- trajectory (path) divided in motion segments of clearly defined shape:
  - *straight lines and segments of a circle.*
- control problem:
  - *pre-compute a smooth trajectory based on line and circle segments*
- Disadvantages:
  - *It is not at all an easy task to pre-compute a feasible trajectory*
  - *limitations and constraints of the robots velocities and accelerations*
  - *does not adapt or correct the trajectory if dynamical changes of the environment occur.*
  - *The resulting trajectories are usually not smooth*

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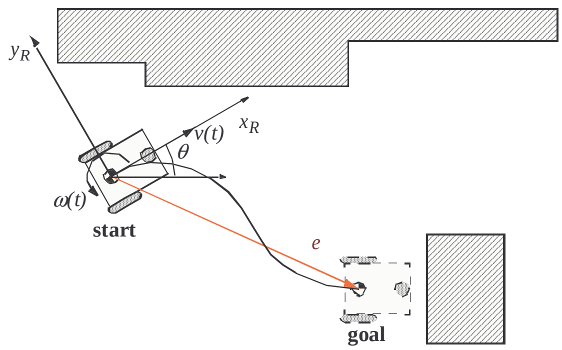
## Motion Control: Feedback Control, Problem Statement




- Find a control matrix $K$, if exists

$$K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \end{bmatrix}$$

with $k_{ij}=k(t,e)$

- such that the control of $v(t)$ and $\omega(t)$

$$\begin{bmatrix} v(t) \\ \omega(t) \end{bmatrix} = K \cdot e = K \cdot {}^{R}\begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

- drives the error e to zero.

$$\lim_{t \to \infty} e(t) = 0$$

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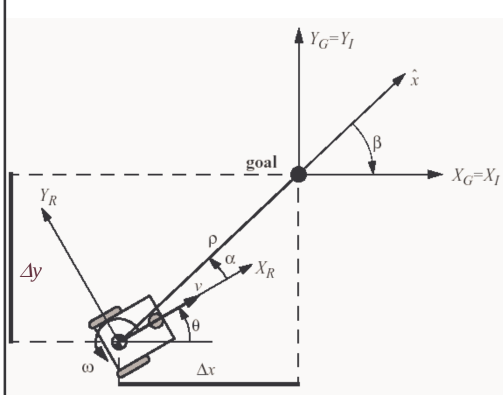
**Motion Control:**

## Kinematic Position Control




The kinematic of a differential drive mobile robot described in the initial frame $\{x_I, y_I, \theta\}$ is given by,

$${}^{I}\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 \\ \sin\theta & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

where and are the linear velocities in the direction of the $x_I$ and $y_I$ of the initial frame.

Let $\alpha$ denote the angle between the $x_R$ axis of the robots reference frame and the vector connecting the center of the axle of the wheels with the final position.

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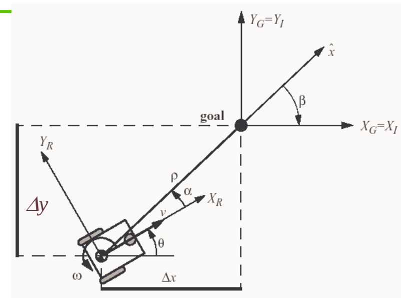
## Kinematic Position Control: Coordinates Transformation

**Coordinates transformation into polar coordinates with its origin at goal position:**

$$\rho = \sqrt{\Delta x^2 + \Delta y^2}$$

$$\alpha = -\theta + \operatorname{atan2}(\Delta y, \Delta x)$$

$$\beta = -\theta - \alpha$$




**System description, in the new polar coordinates**

$$\begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -\cos\alpha & 0 \\ \dfrac{\sin\alpha}{\rho} & -1 \\ -\dfrac{\sin\alpha}{\rho} & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

*for* $I_1 = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\begin{bmatrix} \dot{\rho} \\ \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} \cos\alpha & 0 \\ -\dfrac{\sin\alpha}{\rho} & 1 \\ \dfrac{\sin\alpha}{\rho} & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}$$

*for* $I_2 = (-\pi, -\pi/2] \cup (\pi/2, \pi]$

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## Kinematic Position Control: Remarks

- The coordinates transformation is not defined at $x = y = 0$; as in such a point the determinant of the Jacobian matrix of the transformation is not defined, i.e. it is unbounded
- For $\alpha \in I_1$ the forward direction of the robot points toward the goal, for $\alpha \in I_2$ it is the backward direction.
- By properly defining the forward direction of the robot at its initial configuration, it is always possible to have $\alpha \in I_1$ at $t$=0. However this does not mean that α remains in $I_1$ for all time $t$.

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## Kinematic Position Control: The Control Law

- It can be shown, that with

$$v = k_\rho \rho \qquad \omega = k_\alpha \alpha + k_\beta \beta$$

the feedback controlled system

$$\begin{bmatrix} \dot\rho \\ \dot\alpha \\ \dot\beta \end{bmatrix} = \begin{bmatrix} -k_\rho \rho \cos\alpha \\ k_\rho \sin\alpha - k_\alpha \alpha - k_\beta \beta \\ -k_\rho \sin\alpha \end{bmatrix}$$

- will drive the robot to $(\rho, \alpha, \beta) = (0,0,0)$
- The control signal $v$ has always constant sign,
  - *the direction of movement is kept positive or negative during movement*
  - *parking maneuver is performed always in the most natural way and without ever inverting its motion.*

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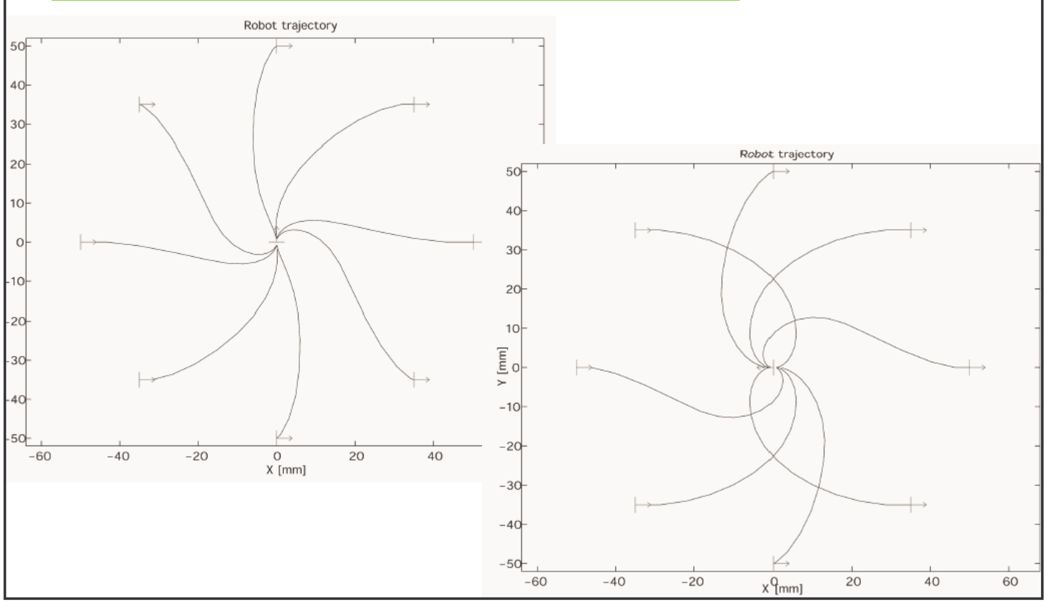
Autonomous Mobile Robots, Chapter 3 3.6.2

## Kinematic Position Control: Resulting Path



## Kinematic Position Control: Stability Issue

- It can further be shown, that the closed loop control system is locally exponentially stable if

$$k_\rho > 0 \;\; ; \;\; k_\beta < 0 \;\; ; \;\; k_\alpha - k_\rho > 0$$

- Proof:
for small $x$ –> cos$x$ = 1, sin$x$ = $x$

$$\begin{bmatrix}\dot{\rho}\\ \dot{\alpha}\\ \dot{\beta}\end{bmatrix} = \begin{bmatrix}-k_\rho & 0 & 0\\ 0 & -(k_\alpha - k_\rho) & -k_\beta\\ 0 & -k_\rho & 0\end{bmatrix}\begin{bmatrix}\rho\\ \alpha\\ \beta\end{bmatrix} \qquad A = \begin{bmatrix}-k_\rho & 0 & 0\\ 0 & -(k_\alpha - k_\rho) & -k_\beta\\ 0 & -k_\rho & 0\end{bmatrix}$$

and the characteristic polynomial of the matrix $A$ of all roots

$$(\lambda + k_\rho)(\lambda^2 + \lambda(k_\alpha - k_\rho) - k_\rho k_\beta)$$

have negative real parts.

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## Summary

- Using a fairly strong set of assumptions, we were able to develop kinematic models for robot motion and control
  - *No sliding*
  - *Single contact point*
  - *These will be sufficient for our purposes*
- Does not address kid steering vehicles (*e.g.* caterpillar tracks) that rely on sliding to turn
- Requires study of vehicle dynamics (outside the scope of this course)
- Feedback control for robot systems is itself a major research area, in case you are interested