Foundations of Computer Science— CSCI 2200
Exam 2 - Selected Solutions

Question 3 (12 points). Prove by mathematical induction that, for all positive integers n, $\sum_{j=1}^{n} j2^j < n2^{n+1}$.

Solution:
We will prove this by mathematical induction. Let $P(n)$ denote $\sum_{j=1}^{n} j2^j < n2^{n+1}$.

Basis Step:
$\sum_{j=1}^{1} j2^j = 1 \cdot 2^1$ and $n2^{n+1} = 1 \cdot 2^2 = 4$.
Since, $2 < 4$, P(1) is true.

Inductive Step:
We assume that $P(k)$ is true, i.e., for $k \geq 1$ $\sum_{j=1}^{k} j2^j < k2^{k+1}$.
We will show this implies $P(k+1)$ is true, i.e., $\sum_{j=1}^{k+1} j2^j < (k+1)2^{k+2}$.

$$\begin{aligned}
\sum_{j=1}^{k+1} j2^j &= \sum_{j=1}^{k} j2^j + (k+1)2^{k+1} \\
&< k2^{k+1} + (k+1)2^{k+1} \quad \text{(by the inductive hypothesis)} \\
&= (k+k+1)2^{k+1} \\
&= (2k+1)2^{k+1} \\
&\leq 2(k+1)2^{k+1} \quad \text{(since } k \geq 1\text{)} \\
&= (k+1)2^{k+1+1} \\
&= (k+1)2^{k+2}.
\end{aligned}$$

Therefore, $P(k)$ implies $P(k+1)$, and by mathematical induction $\sum_{j=1}^{n} j2^j < n2^{n+1}$ for all integers $n \geq 1$.

QED

Question 4 (12 points). Consider the sequence:
$a_1 = 1$
$a_2 = 5$
$a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 3$
Give a proof by strong induction that $a_n = 3^n - 2^n$ for all positive integers n.

Solution:
This is a proof by strong induction.

Basis Step:
$n = 1 \quad a_1 = 1 \quad 3^1 - 2^1 = 1$
$n = 2 \quad a_2 = 5 \quad 3^2 - 2^2 = 5$

Inductive Step:
Assume $a_j = 3^j - 2^j$ for $1 \leq j \leq k, k \geq 2$.
We will show this implies $a_{k+1} = 3^{k+1} - 2^{k+1}$.

$a_{k+1} = 5a_k - 6a_{k-1}$ by definition

$= 5(3^k - 2^k) - 6(3^{k-1} - 2^{k-1})$ by the inductive hypothesis

$= (5 \cdot 3 - 6)3^{k-1} - (5 \cdot 2 - 6)2^{k-1}$

$= (15 - 6)3^{k-1} - (10 - 6)2^{k-1}$

$= 9 \cdot 3^{k-1} - 4 \cdot 2^{k-1}$

$= 3^{k+1} - 2^{k+1}$

QED

Question 5 (17 = 2+5+10 points). Recall the recursive definition of a **full binary tree**:
Basis step: A single vertex r is a full binary tree.
Recursive step: If T_1 and T_2 are disjoint full binary trees, then there is a full binary tree $T = T_1 \circ T_2$ consisting of a root node r, with edges connecting to the roots of T_1 and T_2.

(a) Give the recursive definition of the height $H(T)$ of a full binary tree.

Solution:
Basis step: For a full binary tree T consisting of a single node, $H(T) = 0$.
Recursive step: For a full binary tree $T = T_1 \circ T_2$, where T_1 and T_2 are disjoint full binary trees, $H(T) = 1 + \max(H(T_1), H(T_2))$.

(b) An internal node of a full binary tree is any node that is not a leaf node. Give a recursive definition of the number of internal nodes $\mathcal{I}(T)$ in a full binary tree T.

Solution:
Basis step: For a full binary tree T consisting of a single node, $\mathcal{I}(T) = 0$
Recursive step: For a full binary tree $T = T_1 \circ T_2$, where T_1 and T_2 are disjoint full binary trees, $\mathcal{I}(T) = 1 + \mathcal{I}(T_1) + \mathcal{I}(T_2)$

(c) Prove that for any full binary tree T, $\mathcal{I}(T) \leq 2^{H(T)} - 1$.

Solution:

This is a proof by structural induction.

Basis step:
For T consisting of a single node, we have $\mathcal{I}(T) = 0$, and $2^{H(T)} - 1 = 2^0 - 1 = 1 - 1 = 0$.
Thus, $\mathcal{I}(T) \leq 2^{H(T)} - 1$.

Inductive step:. Assume that for disjoint full binary trees T_1 and T_2,

$$\mathcal{I}(T_1) \leq 2^{H(T_1)} - 1, \qquad \mathcal{I}(T_2) \leq 2^{H(T)_2} - 1.$$

We will show this implies that for $T = T_1 \circ T_2$,

$$\mathcal{I}(T) \leq 2^{H(T)} - 1.$$

$$\begin{aligned}
\mathcal{I}(T) &= 1 + \mathcal{I}(T_1) + \mathcal{I}(T_2) \quad \text{by definition of } \mathcal{I}(T) \\
&\leq 1 + (2^{H(T_1)} - 1) + (2^{H(T_2)} - 1) \quad \text{by the inductive hypothesis} \\
&= 2^{H(T_1)} + 2^{H(T_2)} + 1 - 1 - 1 \\
&= 2^{H(T_1)} + 2^{H(T_2)} - 1 \\
&\leq 2^{\max(H(T_1), H(T_2))} + 2^{\max(H(T_1), H(T_2))} - 1 \\
&= 2 \cdot (2^{\max(H(T_1), H(T_2))}) - 1 \\
&= 2^{\max(H(T_1), H(T_2)) + 1} - 1 \\
&= 2^{H(T)} - 1
\end{aligned}$$

QED