CSCI 2200: Foundations of Computer Science

Final Exam Review Problems

These problems are in addition to the "good problems to review" in the book homework problems, and problems covered in lecture and recitation.
These questions only represent a subset of the topics that will be covered on the final exam.

1. Prove or disprove the following statement: "There exists at least one positive integer n such that $n^2 + n^3 = 100$".

   Solution:
   We look at the value with the higher power which in this case is n^3. $n^3 > 100$ when $n > 4$ So, if we show that the equation does not hold for $n = 1$ and $n = 2$ and $n = 3$ and $n = 4$, then we can say that there exists no positive integer such that $n^2 + n^3 = 100$

   When $n = 1$
   $1^2 + 1^3 = 2$
   $2 \neq 100$
   The equation does not hold.

   When $n = 2$
   $2^2 + 2^3 = 4 + 8$
   $12 \neq 100$
   The equation does not hold.

   When $n = 3$
   $3^2 + 3^3 = 9 + 27$
   $36 \neq 100$
   The equation does not hold.

   When $n = 4$
   $4^2 + 4^3 = 16 + 64$
   $80 \neq 100$
   The equation does not hold.

   Hence, we have proved that there does not exist at least one positive integer n such that $n^2+n^3 = 100$.
   QED

2. Prove that $m^2 = n^2$ if and only if $m = n$ or $m = -n$.

   Solution:
   We need to prove,

   (a) if $m = n$ or $m = -n$, then $m^2 = n^2$

   (b) if $m^2 = n^2$, then $m = n$ or $m = -n$

   First, we prove (a). Assume $m = n$ or $m = -n$. In either case, squaring both sides, we obtain, $m^2 = n^2$,

   Next, we prove (b). Assume $m^2 = n^2$, For any real number k, $\sqrt{k^2} = \pm k$. Therefore, we obtain $\pm m = \pm n$, or equivalently $m = n$ or $m = -n$.

   QED

3. Prove that if n is an integer and $n+5$ is odd, then n is even.

   Solution:
   This is proof by contraposition.
   Assume n is odd. Then, by definition, $n = 2k+1$ for some integer k. This implies:

$$\begin{aligned} n+5 &= 2k+1+5 \\ &= 2k+6 \\ &= 2(k+3). \end{aligned}$$

   Therefore, $n+5 = 2j$, where $j = k+3$ is an integer. So, $n+5$ is even.
   QED

4. Prove that that every odd integer is the difference of two squares.

   Solution:
   Let n be an odd integer. Then, by definition, $n = 2k+1$ for some integer k. Equivalently,

$$\begin{aligned} n &= 2k+1 \\ &= k^2+2k+1-k^2 \\ &= (k+1)^2-k^2. \end{aligned}$$

   Thus, n can be written as the difference of 2 squares.
   QED

5. Prove that the following three statements are equivalent whenever x is a real number:
   (a) x is rational
   (b) $x/2$ is rational
   (c) $3x-1$ is rational

6. Formulate a conjecture about the arithmetic and quadratic means of two real numbers x and y. Prove your conjecture. (The quadratic mean is defined as: $\sqrt{(x^2+y^2)/2}$.) *Hint: Is one type of mean greater than or equal to the other?*

7. Show that 37 is or is not congruent to 3 modulo 7?

8. Prove or disprove that $(p_1p_2p_3...p_n)+1$ is prime, where p_i is the i^{th} smallest prime.

9. Let $A = \{a,b,c\}$, $B = \{x,y\}$, and $C = \{0,1\}$. Find:
   (a) $A \times B \times C$
   (b) $C \times B \times A$

10. Show that $A \oplus B = (A-B) \cup (B-A)$. Illustrate your argument with a Venn diagram.

11. Determine whether $f(m,n) : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ is onto if:
    (a) $f(m,n) = |m|-|n|$
    (b) $f(m,n) = m^2-4$

12. Find $f \circ g$, where $f(x) = x^2+1$ and $g(x) = x+2$ are functions from $\mathbb{R}$ to $\mathbb{R}$.

13. Find the solution to the recurrence relation $a_n = (n+1)a_{n-1}$ with initial condition $a_0 = 2$.

14. Find the value of the sum $\sum_{j=0}^{8}(3^j - 2^j)$.

15. Prove by induction that $2 - 2\cdot 7 + 2\cdot 7^2 - ... + 2\cdot(-7)^n = (1-(-7)^{n+1})/4$ whenever n is a nonnegative integer.

16. Prove by induction that $\sum_{j=0}^{n}\left(-\frac{1}{2}\right)^j = \frac{2^{n+1}+(-1)^n}{3\cdot 2^n}$ whenever n is a nonnegative integer.

17. Determine all gift amounts that can be formed by (nonnegative) combinations of \$25 and \$40 gift certificates. Prove your result using strong induction.

18. What is more likely: rolling a total of eight when two dice are rolled or rolling a total of eight when three dice are rolled? Prove your answer.

    Solution:
    When two dice are rolled, the sample space is the space of ordered pairs (d_1, d_2) of the faces of the dice (assume each die is 6-sided). There are 36 such pairs, 5 of which sum to 8. These 5 pairs are (2,6) , (3,5) ,(4,4) , (5,3), and (6,2). The probability of getting a sum of 8 on two dice is $5/36 \approx 0.1389$.

    When three dice are rolled, the sample space consists of ordered triples (d_1, d_2, d_3). There are $6^3 = 216$ such triples. We want $d_1 + d_2 + d_3 = 8$. We define $\delta_i = d_i - 1$, for $i = 1, 2, 3$. Then, we require $\delta_1 + \delta_2 + \delta_3 = 5$, where each $\delta_i \in \{0, 1, 2, 3, 4, 5\}$. Thus the number of triples (d_1, d_2, d_3) with $d_1 + d_2 + d_3 = 8$ is equivalent to the number of non-negative integer solutions to $\delta_1 + \delta_2 + \delta_3 = 5$. The number of solutions is $\binom{5+3-1}{5} = \binom{7}{5} = 21$. Thus, the probability of rolling an 8 on three dice is $21/216 \approx 0.09722$.

    Therefore, it is more likely to roll an 8 on two dice than on three.

19. Answer the questions below regarding discrete probability. For parts (a), (b), and (c), assume that births are independent events (i.e. no twins) and that a birth occurs with equal probability for each day of the week (even though doctors typically schedule C-sections on weekdays). [Probability of a birth each day is 1/7]

    (a) What is the probability that two people chosen at random were born on the same day of the week?

    Solution: The sample space is $S = \{1, 2, 3, 4, 5, 6, 7\} \times \{1, 2, 3, 4, 5, 6, 7\}$, and the size of the sample space is $|S| = 49$. The event that two people chosen at random were born on the same day of the week has 7 outcomes. Therefore, the probability is $p = 7/49 = 1/7$.

    (b) What is the probability that in a group of n people chosen at random, there are at least two born on the same day of the week?

    Solution: If $n > 7$, the probability is $p = 1$ (by the pigeonhole principle). If $n \le 7$, the probability is $p = 1 - \frac{P(7,n)}{7^n}$.

    (c) How many people chosen at random are needed to make the probability that there are at least two people born on the same day of the week exceed 1/2?

    Solution: From (b), to make $1 - \frac{P(7,n)}{7^n} > 1/2$, we need $n \ge 4$.

    (d) What is the probability of q preceding t when we randomly select a permutation of the set $\{q, r, s, t\}$?

    Solution: There are $4! = 24$ ways to arrange the 4 letters, i.e., the size of the sample space is 24. There are 12 outcomes in the event set E with q preceding t. Therefore, $p(E) = 12/24 = 1/2$.

    (e) What is the probability of q preceding r and s preceding t when we randomly select a permutation of the set $\{q, r, s, t\}$?

Solution: The probability of q preceding r and the s preceding t are both 1/2. These two events are independent. Therefore, $p = 1/2 * 1/2 = 1/4$.

20. Consider a biased coin, where the probability of heads is $\frac{2}{3}$. Suppose you flip the coin 5 times. What is the probability that the coin comes up heads at least 3 times?

21. Suppose that the probability that a spam email contains the word 'lottery' is 97%, and the probability that a non-spam email contains the word 'lottery' is 5%. Assume that every email I receive has equal probability of being spam or not spam. For each question below, give your answer and explain how you derived it.

    (a) What is the probability that an email I receive is spam given it contains the word 'lottery'?

    (b) What is the probability that an email I receive is not spam given it does not contain the word 'lottery'?

    Solution:
    Let S be the event that the email is spam. We are told that every email I receive has equal probability of being spam or not spam, i.e., $p(S) = 0.5$ and $p(\bar{S}) = 0.5$. Let L be the event that the email contains the word 'lottery'. We are told that $p(L|S) = 0.97$ and that $p(L|\bar{S}) = 0.05$. From these, we can conclude the values of $p(\bar{L}|S) = 1 - 0.97 = 0.03$ and $p(\bar{L}|\bar{S}) = 1 - 0.05 = 0.95$

    (a) We are asked for $p(S|L)$. We use Bayes' theorem:

    $$p(S|L) = \frac{p(L|S)p(S)}{p(L|S)p(S)+p(L|\bar{S})p(\bar{S})} = \frac{(0.97)(0.5)}{(0.97)(0.5)+(0.06)(0.5)} \approx 0.95$$

    (b) We are asked for $p(\bar{S}|\bar{L})$. We use Bayes' theorem:

    $$p(\bar{S}|\bar{L}) = \frac{p(\bar{L}|\bar{S})p(\bar{S})}{p(\bar{L}|\bar{S})p(\bar{S})+p(\bar{L}|S)p(S)} = \frac{(0.95)(0.5)}{(0.95)(0.5)+(0.003)(0.5)} \approx 0.97$$

22. Suppose that there is an equal probability of having a boy or a girl. What is the expected number of children a family has until it has both a boy and a girl.

23. Given alphabet $\{x, y, z\}$, construct DFAs that recognize the languages below:

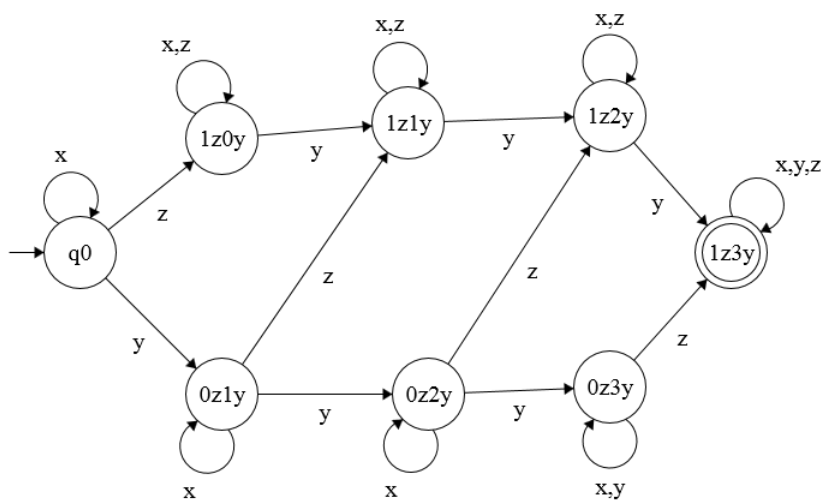
    (a) $\{w \mid w$ contains at least one z and three y's$\}$

    (b) $\{w \mid w$ ends in $yyz\}$

    (c) $\{w \mid w$ is any string except those that begin and end with $y\}$

    (d) $\{w \mid w$ does not contain the substring $xx\}$

    Solution:



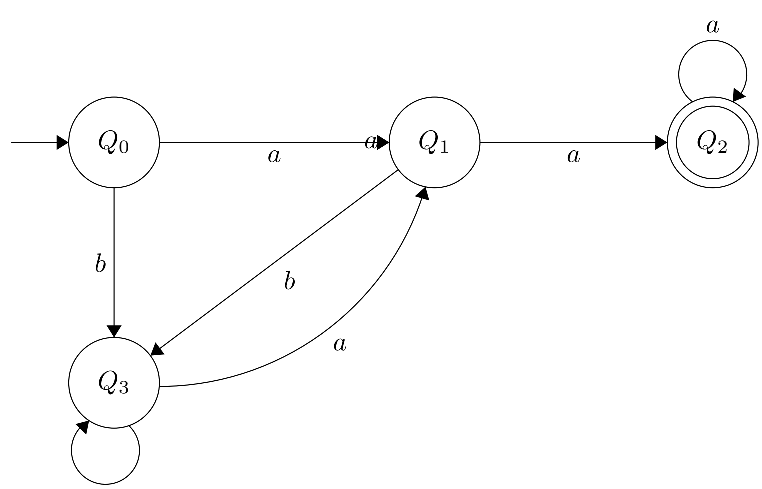
(a)

24. Construct NFAs with the specified number of states that recognize the languages below:

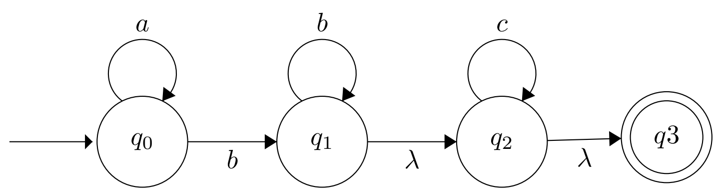
    (a) $\{w \mid w \text{ ends with } aa\}$ with four states, where the alphabet is $\Sigma = \{a, b\}$.

    (b) $a^*b^+c^*$ with four states

Solution:



(a)



(b)

25. Consider the following context-free grammar:

$$
\begin{aligned}
S &\rightarrow A1B \\
A &\rightarrow 0A \mid \lambda \\
B &\rightarrow 0B \mid 1B \mid \lambda
\end{aligned}
$$

Give a leftmost derivation of the string 00101.

Solution:
$S \to A1B \to 0A1B \to 00A1B \to 001B \to 0010B \to 00101B \to 00101$

26. Determine whether the word *cbab* belongs to the language generated by the context-free grammar, and if so, give its derivation.

$$\begin{aligned} S &\to AB \\ A &\to Ca \\ B &\to Ba \\ B &\to Cb \\ B &\to b \\ C &\to cb \\ C &\to b \end{aligned}$$

27. Let the set of terminals be $\{a, b\}$. For each of the following context-free grammars, describe what language it generates.

(a)

$$\begin{aligned} S &\to AB \\ A &\to ab \\ B &\to bb \end{aligned}$$

(b)

$$\begin{aligned} S &\to AB \\ S &\to aA \\ A &\to a \\ B &\to ba \end{aligned}$$

Solution:

(a) This CFG generates only the string *abbb*.
(b) This CFG generates the language $\{aa, aba\}$.

28. Give a complete description of a Turing Machine that accepts the language (over the alphabet $\{0, 1\}$) described by the regular expression $(0 \cup 1)1(0 \cup 1)^*$.

29. Give a complete description of a Turing machine with input tape symbols 0 and 1 that, when given a bit string as input, replaces the first 0 with a 1 and does not change any of the other symbols on the tape. The TM accepts all strings.

30. Answer the following questions, and justify your answers using the formal definition of a Turing machine.

(a) Can the tape alphabet Γ be the same as the input alphabet Σ?
(b) Can a Turing machine's head ever be in the same location in two successive steps?
(c) Can a Turing machine contain just a single state?
(d) Can a Turing machine's computation ever fill up the entire tape?

31. For each of the following statements determine whether they are **true** or **false**.

    (a) We cannot design an unambiguous context-free grammar for every context-free language.
    (b) The language $L = \{a^n b^n,\ 0 \leq n \leq 1000\}$ is regular.
    (c) Deterministic Finite Automata accept a proper subset of the languages accepted by Non-Deterministic Finite Automata.
    (d) Let $\Sigma = \{0, 1\}$. The set of all languages over the alphabet Σ is uncountably infinite.
    (e) If a polynomial time algorithm exists for any problem in NP, then $P = NP$.

    Solution:

    (a) True
    (b) True
    (c) False
    (d) True
    (e) False

32. Which of the following are decision problems:

    (a) What is the smallest prime number greater than n?
    (b) Given a set of strings, is there a finite automaton that recognizes this set of strings?