12. Foundations of Statics

Mechanics of Manipulation

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Outline.

Summary:

- According to Newton, particles interact through forces.
- Rigid bodies interact through *wrenches*.
- Wrenches and twists are dual.

Outline:

1. Foundations.
2. Equivalence theorems.
3. Line of action.
4. Poinsot's theorem.
5. Wrenches.

What is force?

You cannot measure force, only its effects: deformation of structures, acceleration.

We could start from Newton's laws, but instead we hypothesize:

> A force applied to a particle is a vector.
>
> Motion is determined by vector sum.
>
> A particle remains at rest only if total force acting on it is zero.

Moment of force

Moment of force about a line:

Let l be line through origin with direction $\hat{\mathbf{l}}$,

Let $\mathbf{f}$ act at $\mathbf{x}$.

moment of force or **torque** of f about l:

$$n_l = \hat{\mathbf{l}} \cdot (\mathbf{x} \times \mathbf{f})$$

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Moment of force about a point:

moment of force or **torque** of f about O:

$$\mathbf{n}_O = (\mathbf{x} - \mathbf{O}) \times \mathbf{f}$$

If the origin is $\mathbf{O}$ this reduces to $\mathbf{n} = \mathbf{x} \times \mathbf{f}$.

If $\mathbf{n}$ is moment about the origin, and n_l is moment about l through the origin,

$$n_l = \hat{\mathbf{l}} \cdot \mathbf{n}$$

Rigid body force

Consider a rigid body, and a set of forces $\{\mathbf{f}_i\}$ acting at $\{\mathbf{x}_i\}$ resp.

Total force $\mathbf{F}$: sum of all external forces.

Total moment $\mathbf{N}$: sum of all corresponding moments.

$$\mathbf{F} = \sum \mathbf{f}_i$$

$$\mathbf{N} = \sum \mathbf{x}_i \times \mathbf{f}_i$$

Two systems of forces are **equivalent** if they have equal $\mathbf{F}$ and $\mathbf{N}$.

(Equivalent, specifically, because they would have the same effect on a rigid body, according to Newton.

Resultant

Resultant of a system of forces: a system comprising a single force, equivalent to the given system.

Line of action

Consider a force $\mathbf{f}$ applied at some point $\mathbf{x}_1$.

Total force: $\mathbf{F} = \mathbf{f}$

Total moment: $\mathbf{N} = \mathbf{x}_1 \times \mathbf{f}$.

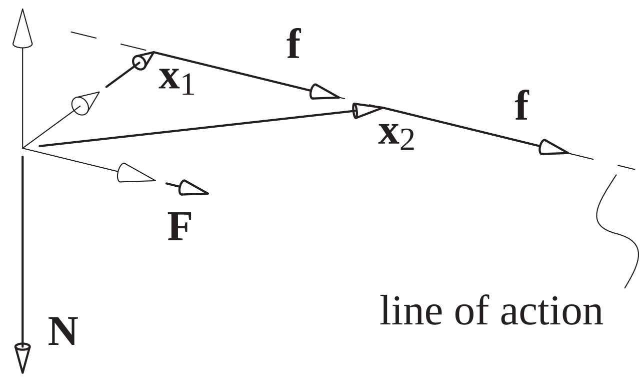
Consider line parallel to $\mathbf{f}$ through $\mathbf{x}_1$, and a second point $\mathbf{x}_2$ on the line.

Force $\mathbf{f}$ through $\mathbf{x}_2$ is *equivalent* to force $\mathbf{f}$ through $\mathbf{x}_1$.

So *point* of application is more than you need to know . . .

Line of action of a force: line through point of application parallel to force.

Bound vector, **free vector**, **line vector**, **point vector**.

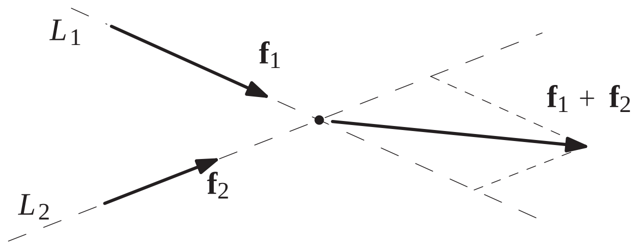


Resultant of two forces

Let $\mathbf{f}_1$ and $\mathbf{f}_2$ act along L_1 and L_2 respectively.

Slide $\mathbf{f}_1$ and $\mathbf{f}_2$ along their respective lines of action to the intersection (if any)

Resultant: the vector sum $\mathbf{f}_1 + \mathbf{f}_2$, acting at the intersection.



Change of reference

Using reference Q or R, a system is described by

$$\mathbf{F}_Q = \sum f_i \qquad \mathbf{N}_Q = \sum (\mathbf{x}_i - \mathbf{Q}) \times \mathbf{f}_i$$
$$\mathbf{F}_R = \sum f_i \qquad \mathbf{N}_R = \sum (\mathbf{x}_i - \mathbf{R}) \times \mathbf{f}_i$$

From which it follows

$$\mathbf{F}_R = \mathbf{F}_Q$$
$$\mathbf{N}_R - \mathbf{N}_Q = \sum (\mathbf{Q} - \mathbf{R}) \times \mathbf{f}_i$$

which gives

$$\mathbf{N}_R = \mathbf{N}_Q + (\mathbf{Q} - \mathbf{R}) \times \mathbf{F}$$

Couple

Is a moment like a force? Can you apply a moment? Does it have a line of action?

A **couple** is a system of forces whose total force $\mathbf{F} = \sum \mathbf{f}_i$ is zero.

I.e. a pure moment.

Notice that the moment $\mathbf{N}$ of a couple is independent of reference point.

For an arbitrary couple, can you construct an equivalent system of just two forces?

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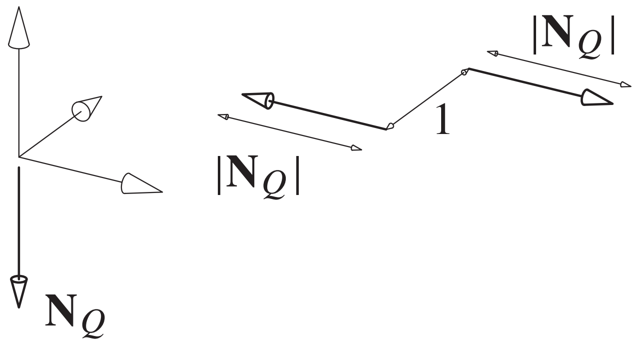


Equivalence theorems

Our goal: defining *wrench*, and showing every system of forces is equivalent to some wrench.

Analogous to the program for kinematics, resulting in definition of *screw*.

Theorem: For any reference point Q, any system of forces is equivalent to a single force through Q, plus a couple.
Proof:

> Let $\mathbf{F}$ be the total force;
>
> let $\mathbf{N}_Q$ be the total moment about Q.
>
> Apply $\mathbf{F}$ at Q;
>
> construct a couple with moment $\mathbf{N}_Q$.

Two forces are sufficient

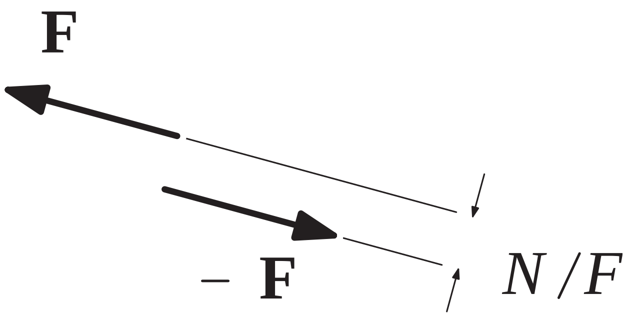
Theorem: Every system of forces is equivalent to a system of just two forces.

Proof:

> Given arbitrary $\mathbf{F}$ and $\mathbf{N}$, construct equivalent force and couple, comprising three forces in total.
>
> Move couple so that one of its forces acts at same point as $\mathbf{F}$.
>
> Replace those two forces with their resultant.

Planar system with nonzero F has a resultant

Theorem: A system consisting of a single non-zero force plus a couple in the same plane, i.e. a torque vector perpendicular to the force, has a resultant.



Proof:

Let $\mathbf{F}$ be the force, acting at P.

Let $\mathbf{N}$ be the moment of the couple.

Construct an equivalent couple as in the figure.

Translate the couple so $-\mathbf{F}$ is applied at P.

Poinsot's theorem

Every system of forces is equivalent to a single force, plus a couple with moment parallel to the force.

Proof:

> Let $\mathbf{F}$ and $\mathbf{N}$ be the given force and moment.
>
> Decompose the moment: $\mathbf{N}_{||}$ parallel to $\mathbf{F}$, and $\mathbf{N}_{\perp}$ perpendicular to $\mathbf{F}$.
>
> Since planar system with nonzero force has a resultant, replace $\mathbf{F}$ and $\mathbf{N}_{\perp}$ by a single force $\mathbf{F}'$ parallel to $\mathbf{F}$.
>
> Now we construct a couple with moment $\mathbf{N}_{||}$ to obtain the desired result: a force and a couple with moment parallel to the force.

Wrench

Wrench: a screw plus a scalar magnitude, giving a force along the screw axis plus a moment about the screw axis.

The force magnitude is the wrench magnitude, and the moment is the twist magnitude times the pitch.

Thus the pitch is the ratio of moment to force.

Poinsot's theorem is succinctly stated: every system of rigid body forces reduces to a wrench along some screw.

Screw coordinates for wrenches

Let f be the magnitude of the force acting along a line l,

Let n be the magnitude of the moment about l.

The magnitude of the wrench is f.

Recall definition in terms of Plücker coordinates:

$$\mathbf{w} = f\mathbf{q}$$
$$\mathbf{w}_0 = f\mathbf{q}_0 + fp\mathbf{q}$$

where $(\mathbf{q}, \mathbf{q}_0)$ are the normalized Plücker coordinates of the wrench axis l, and p is the pitch, which is defined to be

$$p = n/f$$

Screw coordinates for wrenches demystified

Let $\mathbf{r}$ be some point on the wrench axis

$$\mathbf{q}_0 = \mathbf{r} \times \mathbf{q}$$

With some substitutions ...

$$\mathbf{w} = \mathbf{f}$$
$$\mathbf{w}_0 = \mathbf{r} \times \mathbf{f} + \mathbf{n}$$

which can be written:

$$\mathbf{w} = \mathbf{f}$$
$$\mathbf{w}_0 = \mathbf{n}_0$$

where $\mathbf{n}_0$ is just the moment of force at the origin.

Screw coordinates of a wrench are actually a familiar representation $(\mathbf{f}, \mathbf{n}_0)$.

Reciprocal product of twist and wrench

Reciprocal product:

$$(\omega, \mathbf{v}_0) * (\mathbf{f}, \mathbf{n}_0) = \mathbf{f} \cdot \mathbf{v}_0 + \mathbf{n}_0 \cdot \omega$$

The power produced by the wrench $(\mathbf{f}, \mathbf{n}_0)$ and differential twist $(\omega, \mathbf{v}_0)$.

A differential twist is reciprocal to a wrench if and only if no power would be produced.

Repelling if and only if positive power.

Contrary if and only if negative power.

Force versus motion

Wrench coordinates and twist coordinates seem to use different conventions.

Rotation or translation first? Pitch is translation over rotation, or its inverse?

Not just a peculiar convention. Roles of translation and rotation are reversed!

Example: rotation axis versus line of action.

Comparing motion and force

Motion

A zero-pitch twist is a pure rotation.

For a pure translation, the direction of the axis is determined, but the location is not.

A differential translation is equivalent to a rotation about an axis at infinity.

In the plane, any motion can be described as a rotation about some point, possibly at infinity.

Force

A zero-pitch wrench is a pure force.

For a pure moment, the direction of the axis is determined, but the location is not.

A couple is equivalent to a force along a line at infinity.

In the plane, any system of forces reduces to a single force, possibly at infinity.