26. Dynamics and frictional contact

Mechanics of Manipulation

Matt Mason
matt.mason@cs.cmu.edu

http://www.cs.cmu.edu/~mason

Carnegie Mellon

tact.

Outline.

Review of frictional contact (static and quasistatic)

Planar dynamic single contact

Inconsistency and indeterminacy

Graphical methods (acceleration center)

Multiple contact

Review of frictional contact

Static problems

- Friction cones in wrench space
- Polyhedral convex cones in 2D and 3D
- Graphical methods in 2D
- Applications to grasp and fixture design

Quasistatic problems

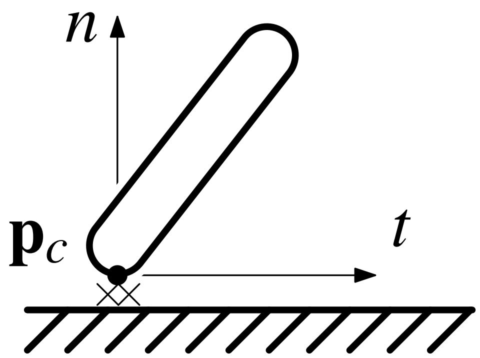
- Planar sliding
- Maximum power inequality
- The Limit Surface
- Application to stable pushing and parts orienting

Today: planar dynamic. Applications to all the above.

Recall sliding rod

From Lecture 16:

$\dot{p}_{cn}$	$\ddot{p}_{cn}$	$\dot{p}_{ct}$	$\ddot{p}_{ct}$		
< 0					impact
> 0					separation
$= 0$	< 0				impact
$= 0$	> 0				separation
$= 0$	$= 0$	< 0		$f_t = \mu f_n$	left sliding
$= 0$	$= 0$	> 0		$f_t = -\mu f_n$	right sliding
$= 0$	$= 0$	$= 0$	< 0	$f_t = \mu f_n$	left sliding
$= 0$	$= 0$	$= 0$	> 0	$f_t = -\mu f_n$	right sliding
$= 0$	$= 0$	$= 0$	$= 0$	$\|f_t\| \leq \|\mu f_n\|$	fixed



Basic approach: case analysis of contact modes

Look at table of previous slide. Force is not expressed as function of state! Contact mode is function of acceleration, not just position and velocity. Solving for acceleration leads to circularity?

But if contact mode is given, friction is not so weird—no circularity. So . . .

1. Enumerate the contact modes.
2. Solve the mechanics problem associated with each contact mode to obtain velocities and accelerations at each contact point.
3. Discard the solution if the contact velocities and accelerations are not consistent with the hypothesized contact mode.

Inconsistency and nondetermism

Solve a Newtonian mechanics problem for each contact mode? How strange! Presumably only one of them gives a solution consistent with the assumed contact mode. Wrong!

> Sometimes two or more modes are valid. Nondeterminism.
>
> Sometimes there are *no* valid solutions. Inconsistency.

(Except, when there are no valid solutions, maybe treating it as impact will save the day.)

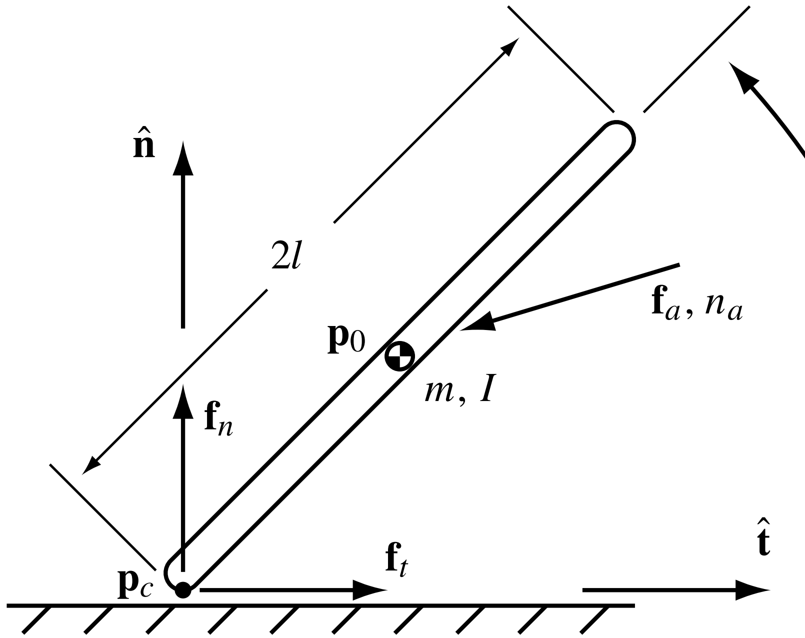
Example: the sliding rod

To write equations of motion using contact point $\mathbf{p}_c$, start with kinematics:

$$\mathbf{p}_c = \mathbf{p}_0 - l \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$\dot{\mathbf{p}}_c = \dot{\mathbf{p}}_0 - l \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$\ddot{\mathbf{p}}_c = \ddot{\mathbf{p}}_0 + l \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \dot{\theta}^2 - l \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} \ddot{\theta}$$



Sliding rod equation of motion

$$\mathbf{f}_n + \mathbf{f}_t + \mathbf{f}_a = m\ddot{\mathbf{p}}_0$$

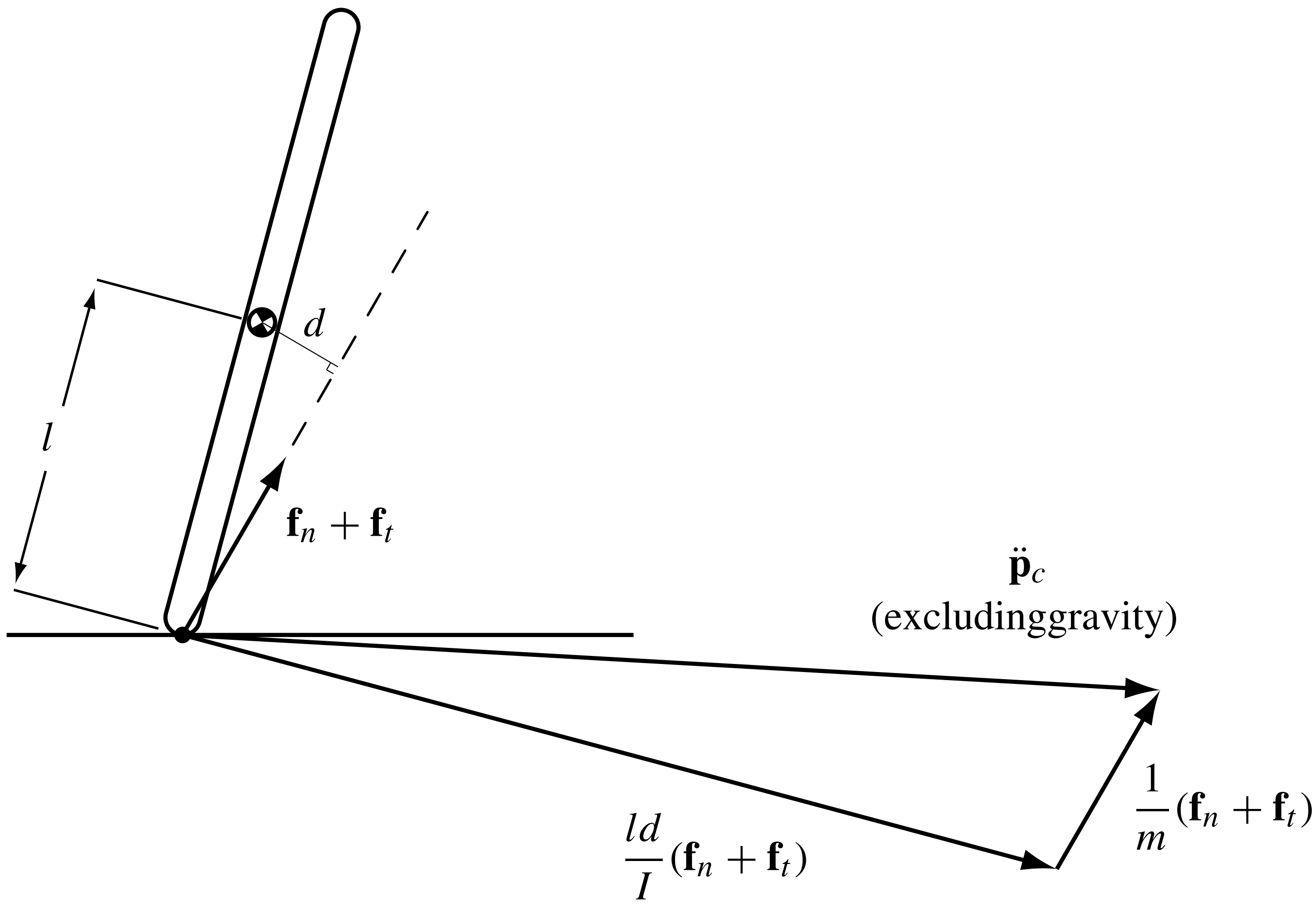
$$(\mathbf{p}_c - \mathbf{p}_0) \times (\mathbf{f}_n + \mathbf{f}_t) + n_a = I\ddot{\theta}$$

Solve for $\ddot{\mathbf{p}}_0$ and $\ddot{\theta}$ and substitute:

$$\ddot{\mathbf{p}}_c = \frac{1}{m}(\mathbf{f}_n + \mathbf{f}_t + \mathbf{f}_a) + l\dot{\theta}^2 \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} - \frac{l}{I} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} [(\mathbf{p}_c - \mathbf{p}_0) \times (\mathbf{f}_n + \mathbf{f}_t) + n_a]$$

We use this equation and choose different parameters to illustrate inconsistency and nondeterminism.

Inconsistency



Inconsistency (the math)

We give the object a leftward velocity on a horizontal support with a gravitational field:

$$\dot{\mathbf{p}}_0 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$\dot{\theta} = 0$$

$$\mathbf{f}_a = \begin{pmatrix} 0 \\ -mg \end{pmatrix}$$

$$n_a = 0$$

Left sliding? Maybe, we'll check it on next slide.

Separation? Nope—with no contact force rod would accelerate downward.

Inconsistency example: try left sliding

$$\mathbf{f}_t = \mu \mathbf{f}_n$$

or

$$\mathbf{f}_t + \mathbf{f}_n = f_n \begin{pmatrix} \mu \\ 1 \end{pmatrix}$$

The moment is

$$\begin{aligned} &(\mathbf{p}_c - \mathbf{p}_0) \times (\mathbf{f}_n + \mathbf{f}_t) \\ &= -l \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \times f_n \begin{pmatrix} \mu \\ 1 \end{pmatrix} \\ &= l f_n (\mu \sin\theta - \cos\theta) \\ &= l f_n \frac{1}{\cos\alpha} (-\cos(\alpha + \theta)) \end{aligned}$$

where $\alpha = \tan^{-1}\mu$.

Left sliding analysis continued

Substituting into equation of motion

$$\ddot{\mathbf{p}}_c = \frac{1}{m} f_n \begin{pmatrix} \mu \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ -g \end{pmatrix} + \mathbf{0} + \frac{l^2 f_n}{I} \begin{pmatrix} -\sin\theta \\ cos\theta \end{pmatrix} \frac{\cos(\alpha+\theta)}{\cos\alpha}$$

Now the left sliding mode specifies that $\ddot{p}_{cn} = 0$:

$$\frac{f_n}{m} - g + \frac{l^2 f_n}{I} \frac{\cos\theta}{\cos\alpha} \cos(\alpha+\theta) = 0 \tag{1}$$

or

$$f_n\left(\frac{1}{m} + \frac{l^2}{I} \frac{\cos\theta}{\cos\alpha} \cos(\alpha+\theta)\right) = g \tag{2}$$

Inconsistency, left sliding mode, continued

Now g is positive, and f_n is positive, so all values of m, l, I, α, and θ must satisfy

$$(\frac{1}{m} + \frac{l^2}{I}\frac{\cos\theta}{\cos\alpha}\cos(\alpha+\theta)) > 0$$

It turns out that for some θ this condition is violated by large values of μ, or small values of I. For example, if $m = I = 1$, $l = 4$, $\alpha = 30$, and $\theta = 75$, then the equation is violated. There is no contact mode that works.

Frictional indeterminacy example

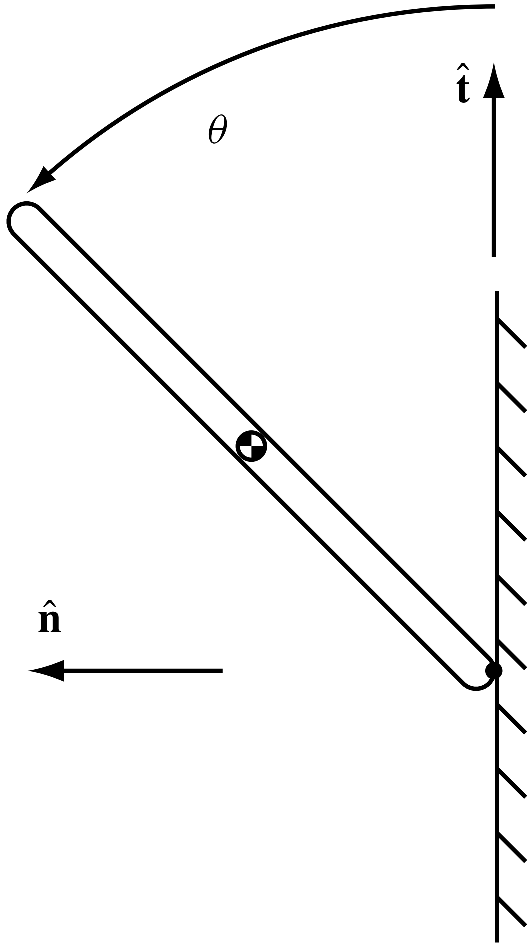
Change gravity, start with motionless rod:

$$\mathbf{f}_a = \begin{pmatrix} -mg \\ 0 \end{pmatrix}$$

$$n_a = 0$$

$$\dot{\mathbf{p}}_0 = \mathbf{0}$$

$$\dot{\theta} = 0$$



Three modes to worry about. Separation won't work: $\mathbf{f}_t + \mathbf{f}_n = \mathbf{0}$, hence $\ddot{p}_{cn} = 0$, i.e. there is no normal acceleration. That leaves *left-sliding*, and *fixed*.

Indeterminacy: left sliding

For left sliding, there is a solution—if $\mathbf{f}_t = \mu \mathbf{f}_n = \mathbf{0}$, then we get $\ddot{p}_{ct} = -g$, i.e. the rod falls straight down, skimming the surface but with zero contact force.

Indeterminacy: fixed

Let $s = f_t/f_n$. (Coulomb's law: $-\mu \leq s \leq \mu$.)

$$\mathbf{f}_t + \mathbf{f}_n = f_n \begin{pmatrix} s \\ 1 \end{pmatrix}$$

Substituting into equation of motion with $\ddot{\mathbf{p}}_c = \mathbf{0}$,

$$\mathbf{0} = \frac{1}{m}\left(f_n \begin{pmatrix} s \\ 1 \end{pmatrix} + \begin{pmatrix} -mg \\ 0 \end{pmatrix} \right) + \frac{l}{I} \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} l \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \times f_n \begin{pmatrix} s \\ 1 \end{pmatrix}$$

Given the physical parameters m, I, g, l, and the initial angle θ, this yields two equations in F_n and s. μ might have to be large, but solutions exist.

Graphical methods for planar dynamics

Force dual transform works for planar rigid body dynamics problems!

- Assume body is initially at rest, accelerating with twist $\mathbf{a}$ due to total applied wrench $\mathbf{f}$.
- Let origin be the center of mass of planar body.
- Let unit of length be the radius of gyration.
- Then Newton's second law gives:

$$\mathbf{a} = \begin{pmatrix} a_x \\ a_y \\ \alpha_z \end{pmatrix} = \frac{1}{m} \begin{pmatrix} f_x \\ f_y \\ n_z \end{pmatrix}$$

Recall force dual is projection of wrench $\mathbf{f}$. Acceleration center is projection of twist $\mathbf{a}$. They're the same!

Graphical method for frictional contact

Let applied force $\mathbf{f}$ be sum of contact wrench $\mathbf{c}$ and some additional applied wrench $\mathbf{w}$ to be determined.

$$\mathbf{a} = \frac{1}{m}(\mathbf{w} + \mathbf{c})$$

where $\mathbf{c}$ is the positive weighted sum of those contact normals or friction cone edges consistent with the contact mode

$$\mathbf{c} \in \mathrm{pos}(\{\mathbf{c}_i\}).$$

Define a relation: wrench $\mathbf{w}$ and twist $\mathbf{a}$ are related iff the first equation holds for some choice of contact forces for the assumed contact mode.

Extend relation to rays

The graphical methods work with *rays* in wrench and twist space, so extend the relation as follows: the pair $\mathbf{w}, \mathbf{a}$ are related if and only if

$$\exists_{s \geq 0, s_i \geq 0} \; \mathbf{w} + \sum s_i \mathbf{c}_i = s m \mathbf{a}$$

which is equivalent to

$$\mathbf{w} \in \mathrm{pos}(\{\mathbf{a}\} \cup \{-\mathbf{c}_i\}).$$

Force dual for dynamic contact problem

For a given contact mode,

- Let $\{-\mathbf{c}_i\}$ be the negated contact normals or friction cone edges consistent with the contact mode;
- Let $\mathbf{a}$ be any acceleration twist consistent with the contact mode;
- Let $\mathbf{a}'$ and $-\mathbf{c}_i'$ be the corresponding force duals;
- Then the possible applied wrenches must have force duals satisfying:

$$\mathbf{w}' \in \mathrm{conv}(\{\mathbf{a}'\} \cup \{-\mathbf{c}_i'\})$$

Multiple contact problems!!!

Here's the most amazing part: the method for works for multiple contacts.

First we need a way of representing contact modes. For the ith contact we will abbreviate the mode m_i as follows:

p	kinematically infeasible ("penetration")
s	separation
l	left sliding
r	right sliding
f	fixed

Then the contact mode of the body can be written as a string

$$m_1 m_2 \dots m_n$$

How many contact modes for n contacts?

5^n?

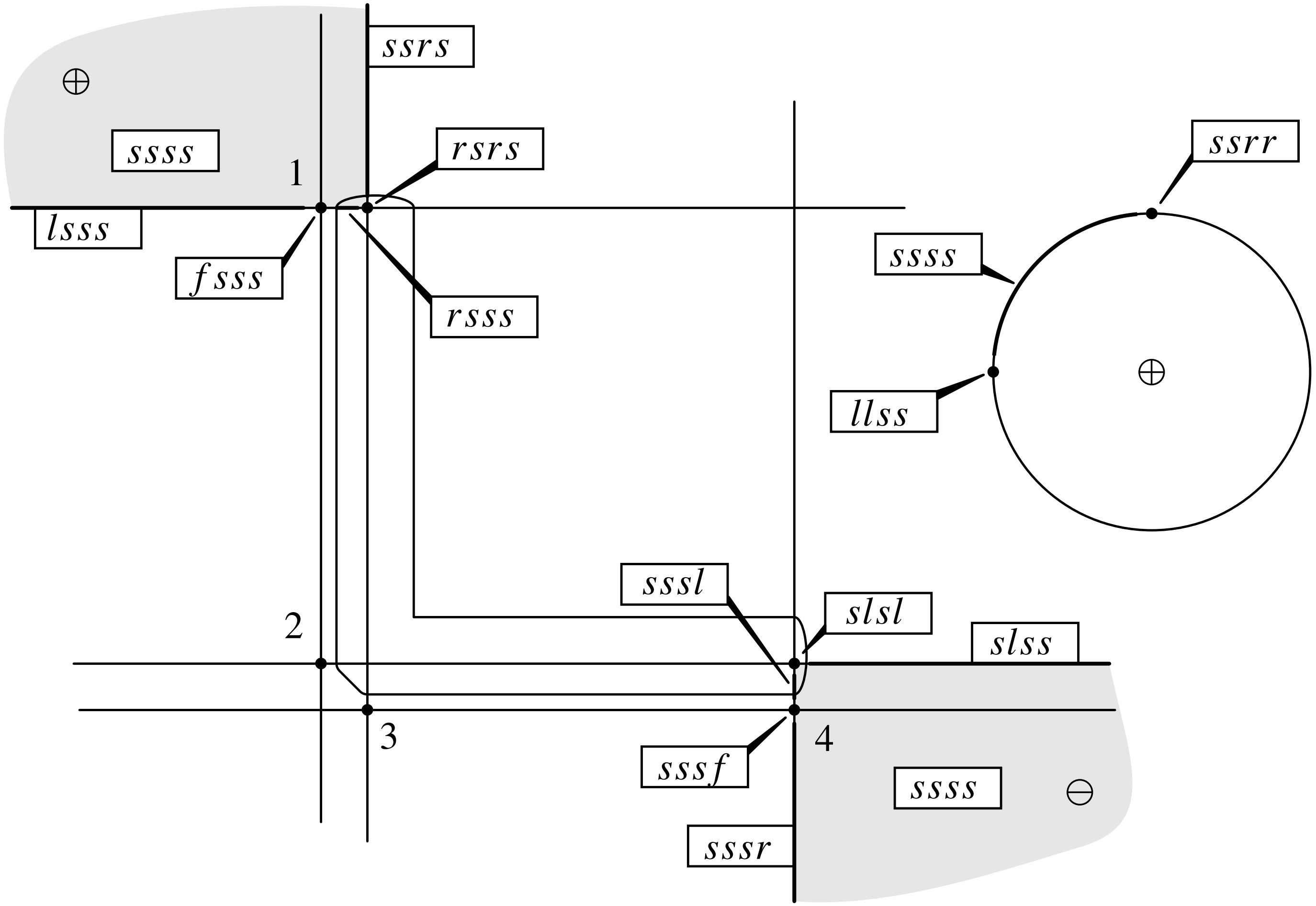
Use variation on Reuleaux.

The method

Then, for each contact mode:

1. identify the set of acceleration centers;
2. identify the set of possible contact forces $\mathbf{c}_i$, negate them, and map them to their force duals;
3. form the convex hull of the acceleration centers and negated contact force duals, to obtain the set of applied wrenches represented as force duals.

Analysis of mode rsrs



Contact modes for wrench in tray