5. Nonholonomic constraint

Mechanics of Manipulation

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Outline.

- An example: the unicycle.
- Integrable and nonintegrable constraints
- Vector fields and distributions
- Frobenius's theorem

Holonomic does not mean unconstrained!!!

- Holonomic means the constraints can be written as equations independent of $\dot{q}$

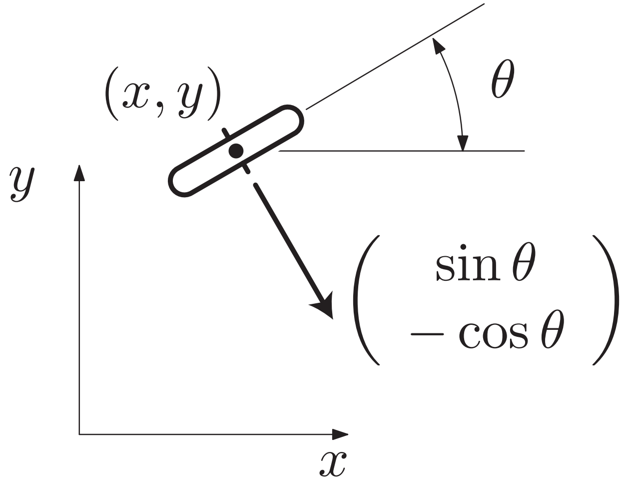
$$f(q, t) = 0$$

- A mobile robot with no constraints is holonomic.
- A mobile robot capable of arbitrary planar velocities is holonomic.
- A mobile robot capable of only translations is holonomic.

Unicycle constraint

The unicycle cannot move sideways.
Let

$$\dot{\mathbf{q}} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix}$$



and let

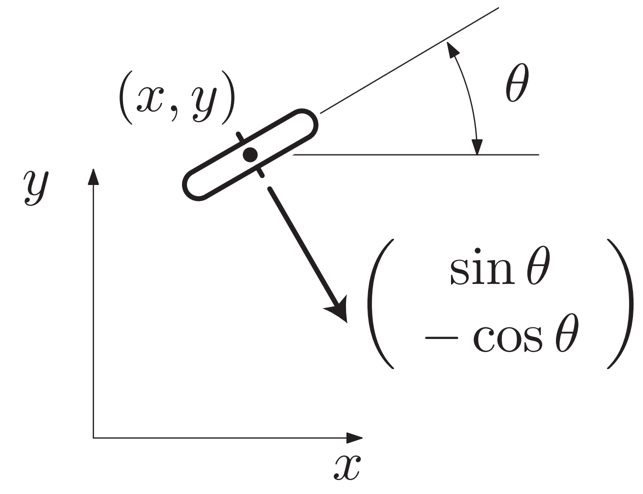
$$\mathbf{w}_1 = (\sin\theta, -\cos\theta, 0)$$

so the constraint is written $\mathbf{w}_1 \dot{\mathbf{q}} = 0$.

Unicycle freedom

The unicycle can move in two directions, expressed by defining

$$\mathbf{g}_1(\mathbf{q}) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{g}_2(\mathbf{q}) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$



and noting that the unicycle's motion is (missing from book)

$$\dot{\mathbf{q}} = u_1\mathbf{g}_1 + u_2\mathbf{g}_2$$

where u_1 and u_2 are arbitrary reals. They are the *controls*.

So, how many DOFs does the unicycle have?

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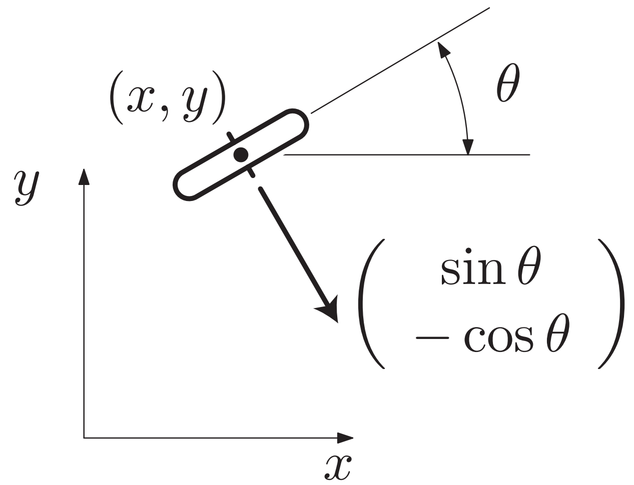

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So, how many DOFs does the unicycle have? *THREE!!!*

Unsteered cart constraint and freedom

The unsteered cart cannot turn, and cannot move sideways. Let

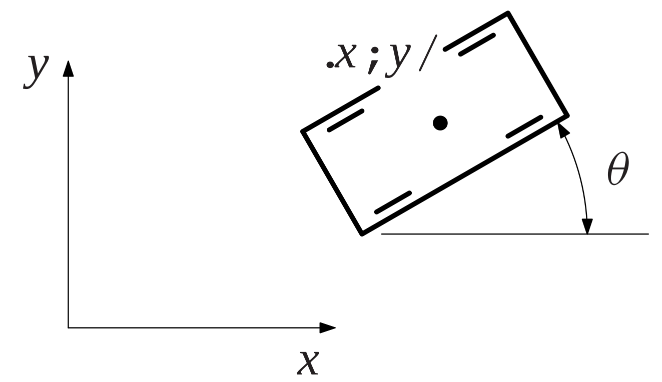
$$\mathbf{w}_1 = (\sin\theta, -\cos\theta, 0)\,, \mathbf{w}_2 = (0, 0, 1)$$

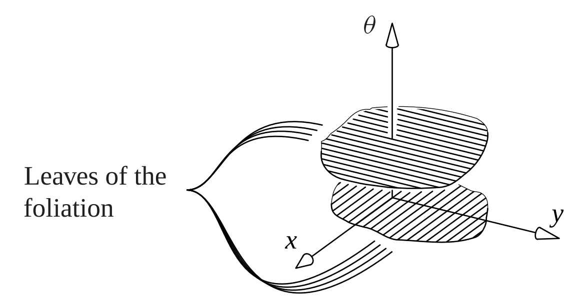
so the two constraints are written $\mathbf{w}_1\dot{\mathbf{q}} = 0$, $\mathbf{w}_2\dot{\mathbf{q}} = 0$. Expanding the products:

$$\dot{x}\sin\theta - \dot{y}\cos\theta = 0$$
$$\dot{\theta} = 0$$

These can be integrated:

$$\theta = \theta_0$$
$$(x - x_0)\sin\theta_0 - (y - y_0)\cos\theta_0 = 0$$

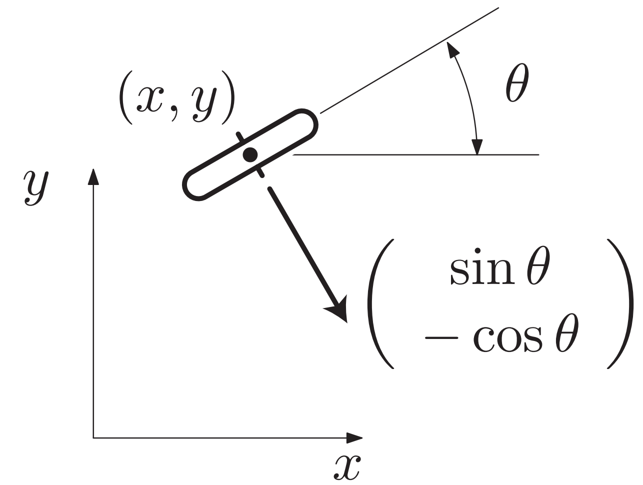


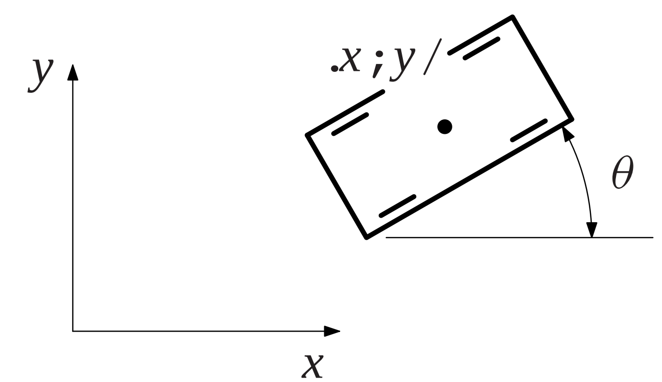


Unicycle versus cart

- Unicycle.
  - One velocity constraint.
  - Three freedoms.
- Unsteered cart
  - Two velocity constraints.
  - *Integrable*. Equivalent to two configuration constraints.
  - One freedom.

System is nonholonomic if the constraint *cannot* be written in the form $f(q,t) = 0$.

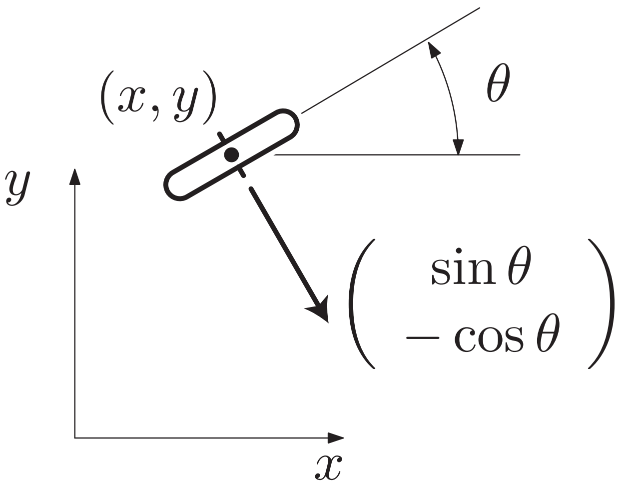


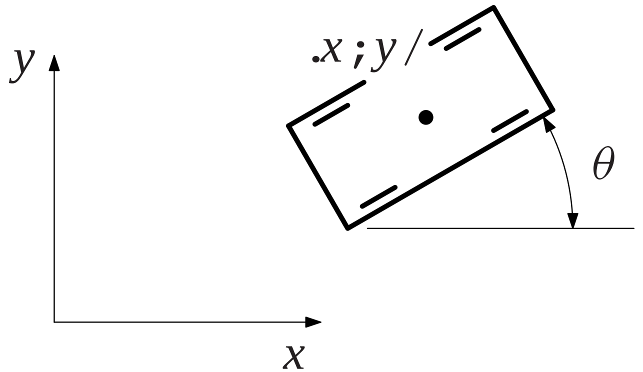


How can you tell?

How can you tell whether a velocity constraint is integrable?

1. Try to integrate it for a while.
2. Determine whether the DOFs were reduced.
3. Lie brackets!!! (Frobenius's theorem)





Pfaffian constraints

A set of k **Pfaffian constraints** are of the form

$$\mathbf{w}_i(\mathbf{q})\dot{\mathbf{q}} = 0, i = 1 \ldots k$$

where the $\mathbf{w}_i$ are linearly independent row vectors, and $\dot{\mathbf{q}}$ is a column vector.

All the velocity constraints we have considered for the unicycle and the cart are Pfaffian.

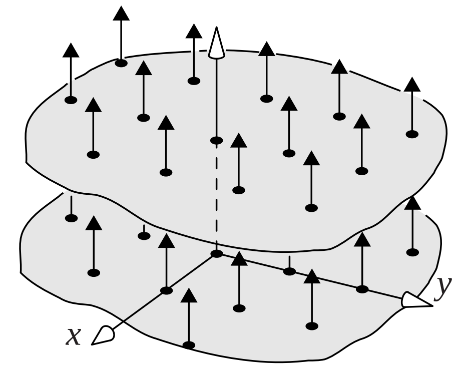
Vector fields

A **vector field** is a smooth map

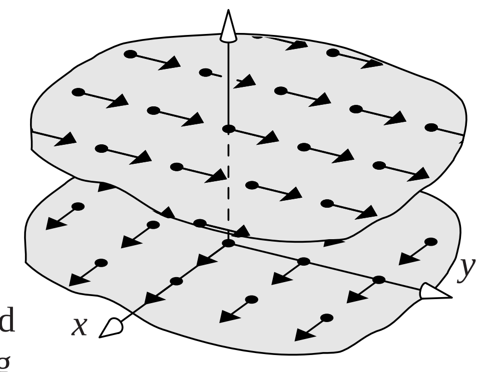
$$f(\mathbf{q}) : \mathbf{C} \mapsto \mathbf{T_q C}$$

from configurations $\mathbf{q}$ to velocity vectors $\dot{\mathbf{q}}$.

Note: In differential geometry "vector" sometimes means specifically "velocity vector".



$\mathbf{g}_1$: turning



$\mathbf{g}_2$: forward rolling

Distributions

A **distribution** is a smooth map assigning a linear subspace of $\mathbf{T_q C}$ to each configuration $\mathbf{q}$ of $\mathbf{C}$.

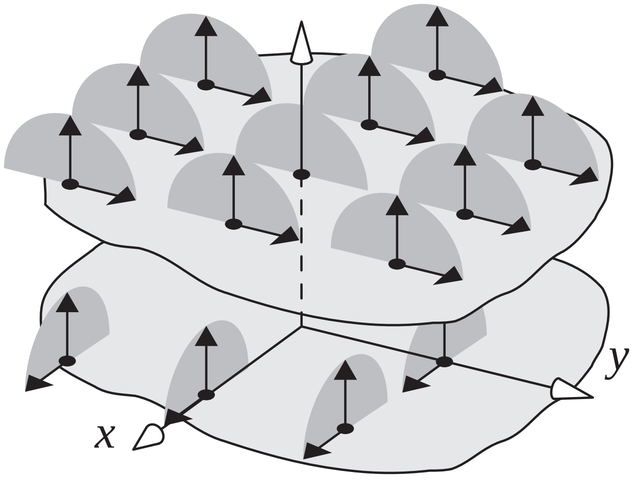
Example: The linear span of $\mathbf{g}_1$ and $\mathbf{g}_2$.

Recall that for the unicycle

$$\mathbf{q} = u_1\mathbf{g}_1 + u_2\mathbf{g}_2$$

for $u_1, u_2 \in \mathbf{R}$. So the figure shows the feasible velocities for every $\mathbf{q}$.

(Well, it only shows a circular patch where it should show a whole plane at every $\mathbf{q}$.)



Regular distributions and Lie brackets

A distribution is **regular** if its dimension is constant over the configuration space.

Let $\mathbf{f}$, $\mathbf{g}$ be two vector fields on $\mathbf{C}$. Define the **Lie bracket** $[\mathbf{f}, \mathbf{g}]$ to be the vector field

$$\frac{\partial \mathbf{g}}{\partial \mathbf{q}}\mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{q}}\mathbf{g}$$

What is this thing written $\frac{\partial \mathbf{g}}{\partial \mathbf{q}}$ or $\frac{\partial \mathbf{f}}{\partial \mathbf{q}}$? Matrix. Each column is partial of

velocity w.r.t. configuration variable.

Lie brackets, example.

Let's take the Lie bracket $[\mathbf{g}_1, \mathbf{g}_2]$.

$$\frac{\partial \mathbf{g}_1}{\partial \mathbf{q}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\frac{\partial \mathbf{g}_2}{\partial \mathbf{q}} = \begin{pmatrix} 0 & 0 & -\sin\theta \\ 0 & 0 & \cos\theta \\ 0 & 0 & 0 \end{pmatrix}$$

For the new vector field defined by the Lie bracket we obtain

$$\mathbf{g}_3 = [\mathbf{g}_1, \mathbf{g}_2] = \frac{\partial \mathbf{g}_2}{\partial \mathbf{q}}\mathbf{g}_1 - \frac{\partial \mathbf{g}_1}{\partial \mathbf{q}}\mathbf{g}_2$$

$$= \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

Lie brackets example continued

$$\mathbf{g}_3 = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$$

Physically, $\mathbf{g}_3$ moves sideways. It is linearly independent of $\mathbf{g}_1$ and $\mathbf{g}_2$, and it violates the constraint $\mathbf{w}_1$.

What is its physical significance? Given two vector fields $\mathbf{f}$ and $\mathbf{g}$,

1. Follow $\mathbf{f}$ for some time ϵ;
2. Follow $\mathbf{g}$ for ϵ;
3. Follow $-\mathbf{f}$ for ϵ;
4. Follow $-\mathbf{g}$ for ϵ.

In the limit as ϵ approaches zero, the result of the above motion approaches the Lie bracket $[\mathbf{f}, \mathbf{g}]$. The Lie bracket could have been called "parallel parking product".

Involutive distribution

- A distribution is **involutive** if it is closed under Lie bracket operations.
- The **involutive closure** of a distribution Δ is the closure $\overline{\Delta}$ of the distribution under Lie bracketing.

Frobenius's theorem

Theorem 2.8 (Frobenius's theorem):
A regular distribution is integrable if and only if it is involutive.
Proof:

To prove that an integrable distribution is involutive, take the Taylor series expansion of the parallel parking maneuver as a function of ϵ. The second order terms are Lie brackets! If the distribution is involutive, the Lie brackets must also be contained in the distribution.

To prove that involutive distributions are integrable . . . $\square$

nonholonomic $\leftrightarrow$ parallel parking helps

Rotatio

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