It was natural to put normal contact conditions into the form of a C.P. We now want to extend and include friction if possible.

Complementarity Problems

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(1.1)

Let w and z be vectors of length n. Further, let $w(z)$ be a given function. The complementarity problem is: find z satisfying:

$$z \geq 0, \quad w(z) \geq 0, \quad w^T(z)\, z = 0$$

$$\boxed{0 \leq z \perp w(z) \geq 0}$$

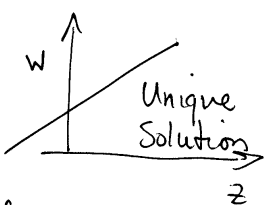
Linear Complementarity Problem (LCP)

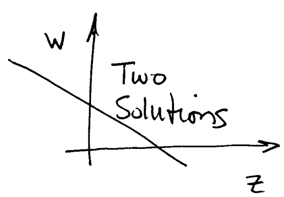
If $w(z)$ is defined as

$$w(z) = Fz + f$$

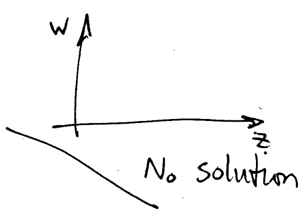
where $F \in \mathbb{R}^{n \times n}$ and $f \in \mathbb{R}^n$ are given constants

LCP of Size 1





The LCP has a unique solution if F is a P-matrix and Lemke's alg. is guaranteed to find a soln. in finite time.



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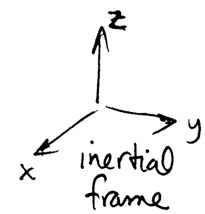
(1)

Dynamics of a Particle

Let $q = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad \nu = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \nu_x \\ \nu_y \\ \nu_z \end{bmatrix}$



Newton's Law

$$\sum \text{forces} = F = \frac{d}{dt}(m\nu)$$

$$F = m\dot{\nu} + \nu \cancel{\dot{m}}$$

Assume 0

Equations of Motion in First-Order Form

$$\boxed{\begin{aligned} \dot{\nu} &= F/m \\ \dot{q} &= \nu \end{aligned}}$$

Time Stepping

We want to approx. the solution over the time interval $[a, b]$. (Assume constant time steps.)

$t_\ell = a + h\ell$ for $\ell = 0, 1, \ldots, M$ where $h = \dfrac{b-a}{M}$

Euler's Method

Let $\dot{q} = f(t)$. Taylor expand to approx at t_{l+1}

$$\frac{1}{h}[q(t_{l+1}) - q(t_l)] \cong f(t_l)$$

$$q^{l+1} \simeq q^l + f^l h$$

Explicit if q^{l+1} = fcn of q^l
Implicit otherwise

f is normally evaluated at t_l, but this is not required.

Apply to our problem

$$\dot{\nu} = F/m$$
$$\dot{q} = \nu$$

Use Euler approximations of $\dot{\nu}$ and $\dot{q}$

$\frac{\nu^{l+1}-\nu^l}{h} = F/m$	F^l/m ? F^{l+1}/m ?
$\frac{q^{l+1}-q^l}{h} = \nu$	ν^l ? ν^{l+1} ?

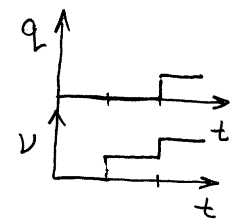
Does it matter where we evaluate F and ν ?

$$\left.\begin{aligned} \nu^{l+1} &= \nu^l + h\frac{F^l}{m} \\ q^{l+1} &= q^l + h\nu^l \end{aligned}\right\} \leftarrow \text{Explicit Method}$$

Everything on RHS is know

l	q^l	ν^l	F (constant)	m (constant)	h (constant)
0	0	0	1	1	1
1	0	1	⋮	⋮	⋮
2	1	2			
3	3	3			

delayed response

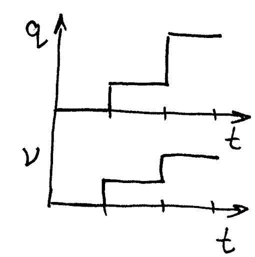


$$\nu^{l+1} = \nu^l + hF/m$$

$$q^{l+1} = q^l + h\nu^{l+1}$$

l	q^l	ν^l	Constant $F, m, h = 1$		
0	0	0	1	1	1
1	1	1			
2	3	2			
3	6	3			

extra fast response



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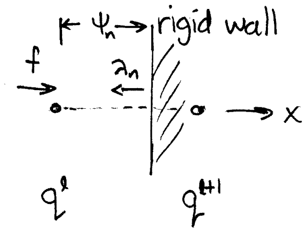
Consider Dynamics When Collision is Imminent

Assume 1-dimensional motion.

$\dot{\nu} = F/m$

$\dot{q} = \nu$



Now F = external force + wall force

$\dot{\nu} = \frac{1}{m}(f - \lambda_n)$

$\dot{q} = \nu$

Let Ψ_n = dist to wall.

$$\text{Assume} \left.\begin{array}{ll} \lambda_n = 0 & \text{if } \Psi_n > 0 \\ \lambda_n \geq 0 & \text{if } \Psi_n = 0 \\ \lambda_n > 0 & \text{only if } \Psi_n = 0 \end{array}\right\} \Rightarrow \cancel{0 \leq \lambda_n \perp} \ \boxed{\begin{array}{c} \lambda_n \geq 0 \\ \Psi_n \geq 0 \\ \lambda_n \Psi_n = 0 \end{array}}$$

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So now the dynamics are:

$$\boxed{\begin{array}{l} \dot{\nu} = \frac{1}{m}(f - \lambda_n) \\ \dot{q} = \nu \\ \text{s.t.} \quad 0 \leq \lambda_n \perp \Psi_n \geq 0 \end{array}}$$

$\Leftarrow$ complementarity constraint

Suppose velocity is high enough so that particle will bounce.

Then apply an impact model when particle reaches wall.

Newton's Hypothesis

$$\boxed{\nu(t_c^+) = -\nu(t_c^-)\, e}$$

$0 \leq e \leq 1$ (inelastic: $e=0$; elastic: $e=1$)

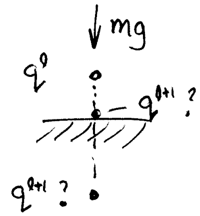
where e is known as the coeff. of rest.

Suppose collision is inelastic or particle will not bounce of by much.

$$\nu^{l+1} = \nu^l + (-g + \overset{?}{\lambda_n}/m)h$$

$$q^{l+1} = q^l + h\,\nu^{?}$$

$$0 \le \overset{?}{\lambda_n} \perp \overset{?}{\psi_n} \ge 0$$



We want to prevent penetration

Notice that ν^l is not biased by contact force, but ν^{l+1} is!

$\therefore$ To prevent q^{l+1} from penetrating, we should use ν^{l+1} in $q^{l+1} = q^l + h\nu^{l+1}$.

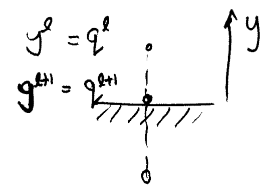
Now make complementarity constraint valid.

Goal: make system consistent at end of time step.

$$\nu^{l+1} = \nu^l + h\left(\frac{\lambda_n^{l+1}}{m} - g^{l+1}\right)$$

$$q^{l+1} = q^l + h\nu^{l+1}$$

$$0 \le \lambda_n^{l+1} \perp q^{l+1} \ge 0$$



Substituting first two eqs into third

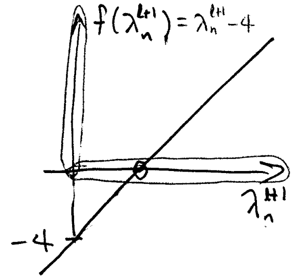
$$0 \le \lambda_n^{l+1} \perp \underbrace{\lambda_n^{l+1}\left(\frac{h^2}{m}\right) + q^l + h\nu^l - gh^2}_{f(\lambda_n^{l+1})} \ge 0$$

Find Solution $\ni$ $\boxed{\begin{matrix}\lambda_n^{l+1} = 0 \\ f(\lambda_n^{l+1}) \ge 0\end{matrix}}$ OR $\boxed{\begin{matrix}f(\lambda_n^{l+1}) = 0 \\ \lambda_n^{l+1} \ge 0\end{matrix}}$

Let $m = h = g = 1$, $\nu^l = -4$ $q^l = 1$

$$0 \le \lambda_n^{l+1} \perp \lambda_n^{l+1} + \underset{q^l}{1} - \underset{\nu^l}{4} - \underset{g}{1} \ge 0$$

$$0 \le \lambda_n^{l+1} \perp \lambda_n^{l+1} - 4 \ge 0$$



Unique Solution

$$\boxed{\lambda_n^{l+1} = 4}$$

Interpretation of solution

Substitute Back in:

$$\nu^{l+1} = -4 + 1(4-1) = -1$$

$$q^{l+1} = 1 + 1(-1) = 0$$

Enough impulse was applied to prevent penetration at end of current time step, BUT NOT ENOUGH TO REMOVE ALL APPROACH VELOCITY!

Next time step

$$0 \le \lambda_n^{l+2} \perp \lambda_n^{l+2} \overset{q^l=0}{} + \overset{\nu^l}{(-1)} - \overset{g}{1} \ge 0$$

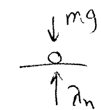
dist covered in $h=1$ at ν^{l+1}

dist that would be covered by grav. accel in $h=1$.

$$\therefore \quad \lambda_n^{l+2} = 2$$

We see it takes two time steps to fully resolve a collision.

l	q^l	ν^l	λ_n^l
l	1	−4	0
$l+1$	0	−1	4
$l+2$	0	0	2
$l+3$	0	0	1 — = mg
⋮	⋮	⋮	⋮



Why is impulse to stop particle equal to 6 & not 4?

Impulse = Δmomentum

This is because there are also 2 units of gravity impulse over the 2 time steps required to resolve the impulse.

Suppose wall is moving as $q_\omega(t)$

$$q - q_\omega(t) \geq 0$$

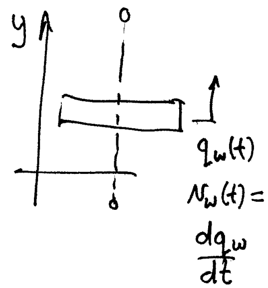
$$\psi_n(q,t) = y - q_\omega(t) \geq 0$$

Note $y = q$

Discretize

$$\frac{\psi_n^{l+1} - \psi_n^l}{h} = \frac{\partial \psi_n}{\partial q} \underbrace{\frac{q^{l+1} - q^l}{h}}_{\nu^{l+1}} + \frac{\partial \psi_n}{\partial t}$$

evaluate where?



$$\psi_n^{l+1} \simeq \psi_n^l + ((1)\nu^{l+1} - \nu_\omega(t))h$$

when negative this term acts to stabilize the constraint

if given fcn of time, then could use ν_ω^{l+1}.

ReWrite LCP

$$\nu^{l+1} = \nu^l + h\left(\frac{\lambda_n^{l+1}}{m} - g\right)$$

$$q^{l+1} = q^l + h\nu^{l+1}$$

$$0 \leq \lambda_n^{l+1} \perp \psi_n^{l+1} \geq 0$$

Substitute

$$0 \leq \lambda_n^{l+1} \perp \underbrace{q^l - q_\omega^l}_{\psi_n^l} + h(\nu^{l+1} - \nu_\omega) \geq 0$$

$$0 \leq \lambda_n^{l+1} \perp q^l - q_\omega^l + h\left[\nu^l + h\left(\frac{\lambda_n^{l+1}}{m} - g\right) - \nu_\omega\right] \geq 0$$

Let $h = m = g = 1$ $\quad q^l = 1 \quad \nu^l = -4$

$\quad q_\omega^l = 0 \quad \nu_\omega^l = 0.5$

$$0 \leq \lambda_n^{l+1} \perp 1 - 0 + 1\left[-4 + 1(\lambda_{n_1}^{l+1} - 1) - \tfrac{1}{2}\right] \geq 0$$

$$0 \leq \lambda_n^{l+1} \perp \lambda_n^{l+1} - 4.5 \geq 0$$

$$\underline{\underline{\lambda_n^{l+1} = 4.5}}$$

$\nu^{l+1} = -4 + \frac{1}{1}4.5 - 1 = \underline{\underline{-0.5 = \nu^{l+1}}}$ $\Leftarrow$ still has rel. vel. into surface

$q^{l+1} = 1 - 1 \cdot 0.5 = \underline{\underline{+0.5 = q^{l+1}}}$ $\Leftarrow$ on surface

since $q_\omega^{l+1} = q_\omega^l + h\,\nu_\omega = 0.5$

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Next time step

$$0 \le \lambda_n^{l+2} \perp q^{l+1} - q_w^{l+1} + h\left(\nu^{l+1} + \frac{h}{m}\lambda_n^{l+2} - hg - \nu_w\right) \ge 0$$

$$0 \le \lambda_n^{l+2} \perp \cancel{0.5} - \cancel{0.5} + 1\left(-0.5 + \lambda_n^{l+2} - 1 - 0.5\right) \ge 0$$

$$0 \le \lambda_n^{l+2} \perp \lambda_n^{l+2} - 2.0 \ge 0$$

$$\lambda_n^{l+2} = 2.0$$

$$\nu^{l+2} = -0.5 + 1(2.0 - 1) = 0.5 = \nu^{l+2}$$

$$q^{l+2} = 0.5 + 1 \cdot 0.5 = 1.0 = q^{l+2}$$

Future time steps will have $\lambda_n^{l+j} = 1$ for $j > 2$

l	ψ_n^l	ν^l	λ_n^l
l	1	-4	0
$l+1$	0	-0.5	4.5
$l+2$	0	0.5	2
$l+3$	0	0.5	1
⋮	⋮	⋮	⋮

~~How do we add~~

Add Friction?

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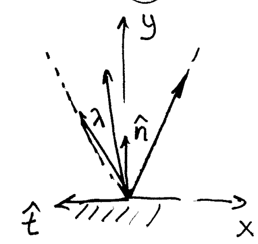
Coulomb's Law



Let velocity of particle be $\overbrace{\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix}}^{\nu} = \begin{bmatrix}-\nu_t\\ \nu_n\end{bmatrix}$

with position $\begin{bmatrix}0\\ 0\end{bmatrix}$

Let λ_n be the normal component of contact force, $\lambda_n \ge 0$

λ_t be the tangential " " " " in $\hat{t}$ directi

Coulomb's Law is given by:

$\lambda_t = -\mu\lambda_n$	if $\nu_t > 0$	$\dot{x} < 0$
$-\mu\lambda_n \le \lambda_t \le \mu\lambda_n$	if $\nu_t = 0$	$\dot{x} = 0$
$\lambda_t = \mu\lambda_n$	if $\nu_t < 0$	$\dot{x} > 0$

Let's divide friction force into positive and negative parts

$$\lambda_t = \lambda_{f_1} - \lambda_{f_2}$$

s.t. $\lambda_{f_1}, \lambda_{f_2} \ge 0$

$$\lambda_{f_1} \xleftarrow{\hat{t}} \cdot \xrightarrow{-\hat{t}} \lambda_{f_2}$$

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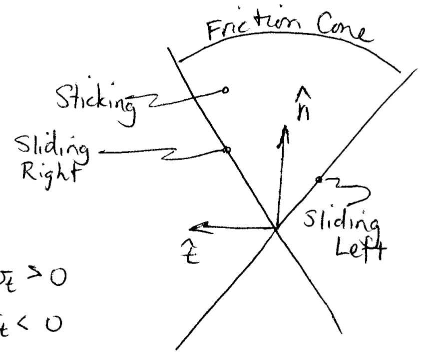
Modeling Friction in Planar Systems



There a 3 physically distinct, important cases to model:

Slide Left $\Rightarrow \lambda_t = -\mu\lambda_n, \; \nu_t > 0$

Slide Right $\Rightarrow \lambda_t = \mu\lambda_n, \; \nu_t < 0$

Stick $\Rightarrow -\mu\lambda_n \le \lambda_t \le \mu\lambda_n, \; \nu_t = 0$

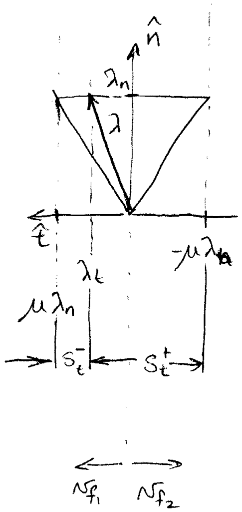
Introduce 2 nonnegative slack variables, s_t^+ and s_t^-

$$s_t^+ = \mu\lambda_n + \lambda_t$$

$$s_t^- = \mu\lambda_n - \lambda_t$$

$s_t^+ = 0 \Rightarrow$ sliding Left

$s_t^- = 0 \Rightarrow$ sliding Right



Rewrite ν_t as the sum of its nonnegative & nonpositive parts

$$\nu_t = \nu_{f_1} - \nu_{f_2}$$

$$\nu_{f_1}, \nu_{f_2} \ge 0$$

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Ideally $\nu_{f_1} \perp \nu_{f_2}$

or equivalently $|\nu_t| = \nu_{f_1} + \nu_{f_2}$

Friction Complementarity Conditions

$$\boxed{\begin{aligned} 0 \le \mu\lambda_n + \lambda_t \;&\perp\; \nu_{f_1} \ge 0 \\ 0 \le \mu\lambda_n - \lambda_t \;&\perp\; \nu_{f_2} \ge 0 \end{aligned}}$$

4 Cases

$\mu\lambda_n + \lambda_t, \; \mu\lambda_n - \lambda_t > 0 \Rightarrow \nu_{f_1} = \nu_{f_2} = \nu_t = 0$ $\boxed{\text{Sticking}}$

$\mu\lambda_n + \lambda_t, \; \nu_{f_2} > 0 \Rightarrow \lambda_t = \mu\lambda_n, \; \nu_{f_1} = 0$ $\boxed{\text{Sliding Right}}$

$\Rightarrow \nu_t = -\nu_{f_2} < 0$

$\mu\lambda_n - \lambda_t, \; \nu_{f_1} > 0 \Rightarrow \lambda_t = -\mu\lambda_n, \; \nu_{f_2} = 0$ $\boxed{\text{Sliding Left}}$

$\Rightarrow \nu_t = +\nu_{f_1} > 0$

$\nu_{f_1}, \nu_{f_2} > 0 \Rightarrow \lambda_t = -\lambda_n\mu = \mu\lambda_n$ $\boxed{\text{Degenerate Sliding}}$

$\Rightarrow \lambda_n = \lambda_t = 0$

An Alternative Formulation

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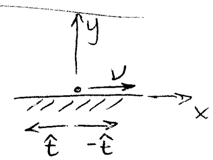
Not as efficient, but extends to 3D problems.

$$\boxed{\begin{array}{c} 0 \le \lambda_f \perp W_f^T \nu + E s \ge 0 \\ 0 \le s \perp \mu \lambda_n - E^T \lambda_f \ge 0 \end{array}}$$

$W_f^T \nu = \begin{bmatrix} \nu_{f_1} \\ \nu_{f_2} \end{bmatrix}$ = tangential velocity components

$E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Example:



$0 \le \lambda_{f_1} \perp \nu_{f_1} + s \ge 0$ (1)

$0 \le \lambda_{f_2} \perp \nu_{f_2} + s \ge 0$ (2)

$0 \le s \perp \mu\lambda_n - \lambda_{f_1} - \lambda_{f_2} \ge 0$ (3)

$W_f = [\hat{t} \;\; -\hat{t}]$

Note $\nu_{f_2} = -\hat{t}^T \nu > 0$

$\nu_{f_2} = \dot{x} > 0$

$\nu_{f_1} = -\nu_{f_2} = -\dot{x}$

Consider all 8 cases Systematically

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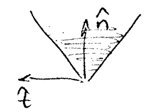
Case	Signs Left	Sign Right
1	+ + +	0 0 0
2	+ + 0	0 0 +
3	+ 0 +	0 + 0
4	+ 0 0	0 + +
5	0 + +	+ 0 0
6	0 + 0	+ 0 +
7	0 0 +	+ + 0
8	0 0 0	+ + +

Case 1 : $\boxed{\text{Inconsistent}}$

(3) ⟹ $s > 0$, but (1) & (2) ⟹ $s = -\nu_{f_1} = -\nu_{f_2}$

since $\nu_{f_1} = -\nu_{f_2}$, $s = \nu_{f_1} = 0$ q.e.d.

Case 2 : $\boxed{\text{Sticking}}$

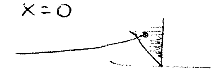


$\lambda_{f_1}, \lambda_{f_2} \ge 0$, $\nu_{f_1} = \nu_{f_2} = 0$

Case 3 : $\boxed{\text{Right Sliding}}$ $s = \nu_{f_2}$, $\lambda_{f_1} = \mu\lambda_n$

Case 4 : $\boxed{\text{Degenerate Sticking}}$ $\dot{x} = 0$

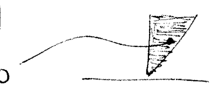
$\lambda_{f_2} = 0$ $\lambda_{f_1} \ge 0$



Case 5 : $\boxed{\text{Left Sliding}}$ $s = \nu_{f_1}$, $\lambda_{f_2} = \mu\lambda_n$

Case 6 : $\boxed{\text{Degenerate Sticking}}$

$\dot{x} = 0$, $\lambda_{f_1} = 0$ $\lambda_{f_2} \ge 0$



Case 7 : $\boxed{\text{Degenerate Sliding}}$

$\lambda_{f_1} = \lambda_{f_2} = \lambda_n = 0$, $s > 0$

Case 8 : $\boxed{\text{Degenerate Sticking}}$

$\lambda_{f_1} = \lambda_{f_2} = \lambda_n = 0$, $s = 0$, $\nu_{f_1} = \nu_{f_2} = 0$

Time Stepping Subproblem

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$$\left.\begin{aligned}
&v^{l+1} = v^l + Fh/m \\
&q^{l+1} = q^l + h v^{l+1} \\
&0 \le \begin{bmatrix}\lambda_n^{l+1}\\ \lambda_f^{l+1}\\ s^{l+1}\end{bmatrix} \perp \begin{bmatrix}\Psi_n(q^{l+1}, t_{l+1})\\ W_f^T v^{l+1} + E s^{l+1}\\ \mu\lambda_n^{l+1} - E^T\lambda_f^{l+1}\end{bmatrix} \ge 0
\end{aligned}\right\}\ \text{Mixed Nonlinear Complement. Problem}$$

Where are the nonlinearities?

F(t) could be nonlinear. Could integrate if easy enough

If $\Psi_n(q,t)$ is nonlinear, eg. circular obstacle

If $\hat{t}$ changes over time step, $W_f^T = \begin{bmatrix}\hat{t}\\ -\hat{t}\end{bmatrix}$

LCP's are much easier to solve (use PATH solver),

So linearize

$$\Psi_n^{l+1} = \Psi_n^l + \frac{\partial\Psi_n^l}{\partial q}\Delta q + \frac{\partial\Psi_n^l}{\partial t}\Delta t + \cancel{H.O.T.}$$

ignore

$$\Delta q = q^{l+1} - q^l = h v^{l+1}$$

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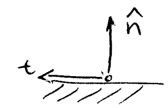
$$p_n^{l+1} = \frac{\Psi^{l+1}}{h} \simeq \frac{\Psi_n^l}{h} + W_n^T v^{l+1} + \frac{\partial\Psi_n}{\partial t}$$

where $W_n^T = \hat{n}^T$

Write F in terms of external and contact forces

$$F = \underbrace{W_n\lambda_n}_{\text{normal force}} + \underbrace{W_f\lambda_f}_{\text{friction force}} + \underbrace{g_{ext}}_{\text{gravity, wind resistance, etc.}}$$



Organize Eqs.:

Let: $M = \begin{bmatrix} m & \\ & m\end{bmatrix}$, $U = \mu$

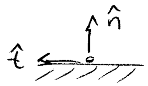
$h g_{ext} = p_{ext}$, $h\lambda = p$

$$\underbrace{\begin{bmatrix}0\\ \hline p_n^{l+1}\\ p_f^{l+1}\\ \sigma^{l+1}\end{bmatrix}}_{\text{just definitions}} = \begin{bmatrix} M & -W_n & -W_f & 0\\ \hline W_n^T & 0 & 0 & 0\\ W_f^T & 0 & 0 & E\\ 0 & U & -E^T & 0\end{bmatrix}\underbrace{\begin{bmatrix}v^{l+1}\\ p_n^{l+1}\\ p_f^{l+1}\\ s^{l+1}\end{bmatrix}}_{\text{unknowns}} + \begin{bmatrix}-v^l - p_{ext}^l\\ \hline \frac{\Psi_n^l}{h} + \frac{\partial\Psi_n}{\partial t}\\ 0\\ 0\end{bmatrix}$$

$$q^{l+1} = q^l + h v^{l+1}$$

Define: $\rho_n^{\ell+1} = \dfrac{\Psi_n^{\ell}}{h} + W_n^T \nu^{\ell+1} + \dfrac{\partial \Psi_n}{\partial t}$

note that $W_n^T = \hat{n}^T$



ReWrite $F = W_n \lambda_n + W_f \lambda_f + g_{ext}$

external forces

friction force

normal force

Let $M = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$, $U = \mu$, $p_{ext} = h g_{ext}$, $p_\alpha = h\lambda_\alpha$

Time Stepping SubProblem – Mixed LCP

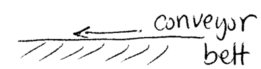
$$\begin{bmatrix} 0 \\ \hline \rho_n^{\ell+1} \\ \rho_f^{\ell+1} \\ \sigma^{\ell+1} \end{bmatrix} = \begin{bmatrix} M & -W_n & -W_f & 0 \\ \hline W_n^T & 0 & 0 & 0 \\ W_f^T & 0 & 0 & E \\ 0 & U & -E^T & 0 \end{bmatrix} \begin{bmatrix} \nu^{\ell+1} \\ \lambda_n^{\ell+1} \\ \lambda_f^{\ell+1} \\ s^{\ell+1} \end{bmatrix} + \begin{bmatrix} -M\nu^{\ell} - p_{ext}^{\ell} \\ \hline \dfrac{\Psi_n^{\ell}}{h} + \dfrac{\partial \Psi_n}{\partial t} \\ 0 \\ 0 \end{bmatrix}$$

$$0 \le \begin{bmatrix} \rho_n^{\ell+1} \\ \rho_f^{\ell+1} \\ \sigma^{\ell+1} \end{bmatrix} \perp \begin{bmatrix} \lambda_n^{\ell+1} \\ \lambda_f^{\ell+1} \\ s^{\ell+1} \end{bmatrix} \ge 0$$

$$q^{\ell+1} = q^{\ell} + h\nu^{\ell+1}$$

Another variation. Suppose the contact surface is moving in tangential direction.

Define Ψ_f analogous to $\Psi_n \ge 0$



$\Psi_f(q,t)$ only velocity matters

$$\Psi_f^{\ell+1} \simeq \Psi_f^{\ell} + \frac{\partial \Psi_f}{\partial q}\Delta q + \frac{\partial \Psi_f}{\partial t}\Delta t$$

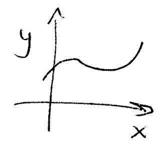
rel. tang. velocity $= \cancel{\partial t}\; W_f^T \nu^{\ell+1} + \dfrac{\partial \Psi_f}{\partial t}$

Change to LCP is in only the constant vector.

It becomes
$$\begin{bmatrix} -M\nu^{\ell} - p_{ext}^{\ell} \\ \dfrac{\Psi_n^{\ell}}{h} + \dfrac{\partial \Psi_n}{\partial t} \\ \dfrac{\partial \Psi_f}{\partial t} \\ 0 \end{bmatrix}$$

One more variation: Equality Constraints

$\Theta(q,t) = 0$ e.g. particle moves on a wire



$$\Theta^{\ell+1} \simeq \Theta^{\ell} + \frac{\partial \Theta}{\partial q}\Delta q + \frac{\partial \Theta}{\partial t}\Delta t$$

$$\frac{\Theta^{\ell}}{h} + W_b^T \nu^{\ell+1} + \frac{\partial \Theta}{\partial t} \cong 0$$

New Matrix & Vector of Mixed LCP

$$\begin{bmatrix} 0 \\ 0 \\ \hline \rho_n^{\ell+1} \\ \rho_f^{\ell+1} \\ \sigma^{\ell+1} \end{bmatrix} = \begin{bmatrix} M & -W_b & -W_n & -W_f & 0 \\ W_b^T & 0 & & & 0 \\ \hline W_n^T & & & & 0 \\ W_f^T & & & & E \\ 0 & 0 & U & -E^T & 0 \end{bmatrix} \begin{bmatrix} \nu^{\ell+1} \\ p_b^{\ell+1} \\ p_n^{\ell+1} \\ p_f^{\ell+1} \\ s^{\ell+1} \end{bmatrix} + \begin{bmatrix} -M\nu^{\ell} - p_{ext}^{\ell} \\ \frac{\Theta^{\ell}}{h} + \frac{\partial \Theta^{\ell}}{\partial t} \\ \hline \frac{\Psi_n^{\ell}}{h} + \frac{\partial \Psi_n}{\partial t} \\ \frac{\partial \Psi_f}{\partial t} \\ 0 \end{bmatrix}$$

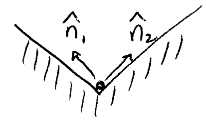
$$0 \leq \begin{bmatrix} \rho_n^{\ell+1} \\ \rho_f^{\ell+1} \\ \sigma^{\ell+1} \end{bmatrix} \perp \begin{bmatrix} p_n^{\ell+1} \\ p_f^{\ell+1} \\ s^{\ell+1} \end{bmatrix} \geq 0$$

$$q^{\ell+1} = q^{\ell} + h\nu^{\ell+1}$$

What chagges for multiple unilateral contacts

$\Psi_n = [\Psi_{1n} \quad \Psi_{2n} \cdots]^T$

$W_n = \begin{bmatrix} \hat{n}_1 & \hat{n}_2 & \cdots \end{bmatrix}$



$W_f = \begin{bmatrix} \hat{t}_1 & -\hat{t}_1 & \hat{t}_2 & -\hat{t}_2 & \cdots \end{bmatrix}$

$E = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ & & \ddots \end{bmatrix}$ block diagonal

$U = \begin{bmatrix} \mu_1 & \\ & \mu_2 \\ & & \ddots \end{bmatrix}$

$\frac{\partial \Psi_n}{\partial t}$ is $(n_c \times 1)$ $\quad$ $\frac{\partial \Psi_f}{\partial t}$ is $(2n_c \times 1)$

What about more equality constraints?

$\Theta = \begin{bmatrix} \Theta_1 \\ \vdots \\ \Theta \end{bmatrix}$

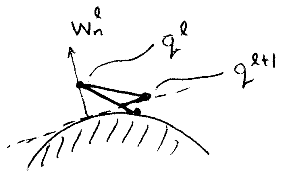
Solution existence

Can Prove soln existence (or not) by eliminating ν^{l+1} & p_o^{l+1}

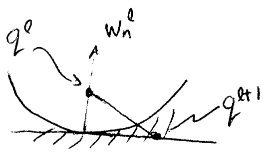
If can eliminate, and $\frac{\psi_n^l}{h} \geq 0$, then solution exists and can be found in finite time by Lemke's algorithm.

Errors

ψ_n nonlinear

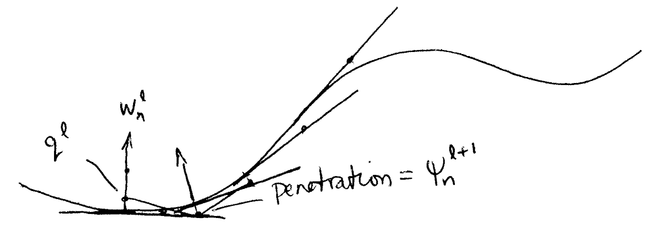


contact force will exist



penetration exists

Constraint Stabilization



$\frac{\psi_n^l}{h}$ = outward normal component of velocity needed to eliminate penetration

contains

$p_n \perp p_n$

$\therefore \frac{\psi_n^l}{h}$ requires p_n^{l+1} to be large enough to eliminate penetration.

if

Not physically realistic impulse.

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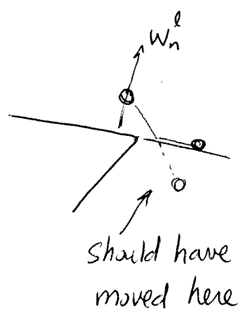
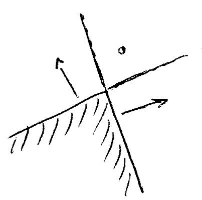
Error due to Polygonalization (and explicit integration method)



If we used $W_n^{\ell+1}$ we could avoid this.

Alternative: exact representation of polygonal free space.



$$\Psi_{in}^{\ell+1} \cong \Psi_{in}^{\ell} + W_{in}^{T} \nu^{\ell+1} + \frac{\partial \Psi_{in}}{\partial t} \geq 0$$

OR

$$\Psi_{2n}^{\ell+1} \cong \Psi_{2n}^{\ell} + W_{2n}^{T} \nu^{\ell+1} + \frac{\partial \Psi_{2n}}{\partial t} \geq 0$$

$\boxed{\text{Egan, Berard, Trinkle, Tech report}}$

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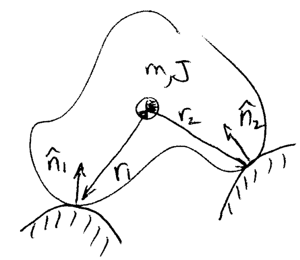
Extension to Planar Rigid Bodies

$$M\dot{\nu} = W_n \lambda_n + W_f \lambda_f + g_{ext}$$



$$M = \text{diag}(m, m, J)_{(3\times3)}$$

$$W_n = \begin{bmatrix} \hat{n}_1 & \hat{n}_2 & \cdots \\ (r_1 \times \hat{n}_1)_z & (r_2 \times \hat{n}_2)_z & \cdots \end{bmatrix}_{(3\times n_c)}$$

$$W_f = \begin{bmatrix} \hat{t}_1 & -\hat{t}_1 & \cdots \\ (r_1 \times \hat{t}_1)_z & -(r_1 \times \hat{t}_1)_z & \cdots \end{bmatrix}_{(3\times 2n_c)}$$

$\underset{(3\times1)}{p_{ext}} = h g_{ext}$ includes moment component

$$E = \underset{\text{diag}}{\text{block}} \left(\cdots \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdots \right)$$

$$U = \text{diag}(\mu_1, \mu_2, \dots)$$

$\Psi_n \quad (n_c \times 1)$

$\Psi_f \quad (2n_c \times 1)$

Multiple Planar Rigid Bodies

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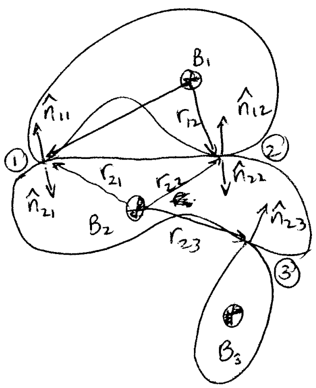
$$M = \text{blockdiag}\left(\begin{bmatrix} m_1 & & \\ & m_1 & \\ & & J_1 \end{bmatrix} \cdots \right)$$

$$W_n = \begin{array}{c} \left[\begin{array}{c|c|c} \hat{n}_{11} & \hat{n}_{12} & 0 \\ r_{11} \times \hat{n}_{11} & r_{12} \times \hat{n}_{12} & \\ \hline \hat{n}_{21} & \hat{n}_{22} & \hat{n}_{23} \\ r_{21} \times \hat{n}_{21} & r_{22} \times \hat{n}_{22} & r_{23} \times \hat{n}_{23} \\ \hline 0 & 0 & \hat{n}_{33} \\ & & r_{33} \times \hat{n}_{33} \end{array}\right] \\ \text{Cont 1} \quad\quad \text{Cont 2} \quad\quad \text{Cont 3} \end{array} \begin{array}{l} \} B_1 \\ \\ \} B_2 \\ \\ \} B_3 \\ \\ \end{array}$$



W_f analogous

p_{ext} $(3n_b \times 1)$

An application that's not just simulation.

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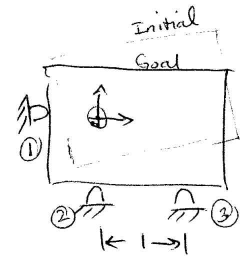
Frictionless Parts Seating

Determine impulse to apply to cause contact at all three points.



$$\begin{bmatrix} 0 \\ \rho_n^{\ell+1} \end{bmatrix} = \begin{bmatrix} M & -W_n \\ W_n^T & 0 \end{bmatrix} \begin{bmatrix} \nu^{\ell+1} \\ p_n^{\ell+1} \end{bmatrix} + \begin{bmatrix} -M\nu^\ell - p_{ext}^\ell \\ \frac{\psi_n^\ell}{h} + \frac{\partial \psi_n}{\partial t} \end{bmatrix}$$

Eliminate $\nu^{\ell+1} = \nu^\ell$

$$\rho_n^{\ell+1} = W_n^T M^{-1} W_n \, p_n^{\ell+1} + W^T M^{-1} (\nu^\ell + p_{ext}^\ell) + \frac{\psi_n^\ell}{h} + \frac{\partial \psi_n}{\partial t}$$

Assume $M = I$, $\nu^\ell = 0$, $\frac{\partial \psi_n}{\partial t} = 0$, $h = 1$

Note that $W_n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad W_n^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ $\quad \det(W_n) = 1$

Close contacts $\Rightarrow$ $\underline{\underline{\rho_n^{\ell+1} = 0}}$

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Simplify

$$p_n^{\ell+1} = \boxed{-W_n^{-1} p_{ext} - W_n^{-1} W_n^{-T} \psi_n^\ell \geq 0}$$

Assume sensor can measure gaps.

Then all is known except p_{ext}

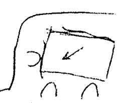
Inequality represents a polytope in p_{ext} space.

Multiply by

$$p_{ext} \leq \begin{bmatrix} -\psi_{1n}^\ell \\ -\psi_{2n}^\ell \\ \psi_{2n}^\ell - \psi_{3n}^\ell \end{bmatrix}$$

Since $\psi_n^\ell > 0$, p_{ext} has strictly negative x & y components

if $\psi_{2n}^\ell > \psi_{3n}^\ell$, $(p_{ext})_3$ is $\leq$ positive number



force helps twist block

$\psi_{3n}^\ell > \psi_{2n}^\ell$, (p_{ext}) ~~has~~ is strictly ~~neg~~ ~~≥ component~~

What if ψ_{1n}^ℓ not known accurately?

What about pt into corner problem?

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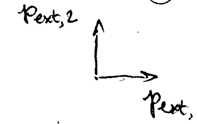
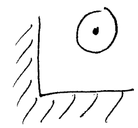
$$W_n = M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

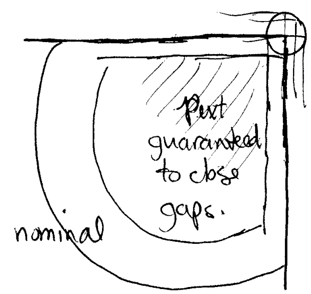
then $p_{ext} \leq \begin{bmatrix} -\psi_{1n}^\ell / h \\ -\psi_{2n}^\ell / h \end{bmatrix}$



Uncertainty.

If q^ℓ uncertain, then ψ_n^ℓ is uncertain

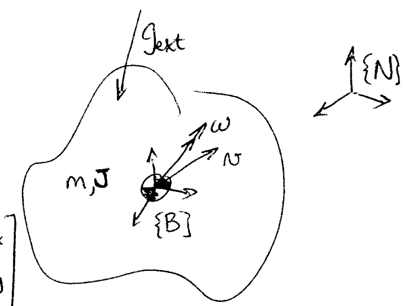




Generalize to Spatial Case

Significant Changes:

- Rotation Kinematics
- Nonlinear Friction Constraint
- New term in dynamics
- Matrix dimensions



Generalized velocity

$$\nu = \begin{bmatrix} v \\ \omega \end{bmatrix} \qquad v = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} \qquad \omega = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

Configuration

$$q = \begin{bmatrix} x \\ y \\ z \\ e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

e_0, e_1, e_2, e_3: unit quaternion a.k.a. Euler parameters

$$e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$$

Rotational Kinematics

$$\dot{q} = G\nu$$

$$G = \begin{bmatrix} I_{(3\times 3)} & \\ & {}^{B}B(q)_{(4\times 3)} \end{bmatrix}_{(7\times 6)}$$

$I_{(3\times 3)}$ = 3×3 identity matrix $\Rightarrow$ $\begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$

$${}^{B}B = \frac{1}{2}\begin{bmatrix} -e_1 & -e_2 & -e_3 \\ e_0 & -e_3 & e_2 \\ e_1 & e_0 & -e_1 \\ -e_2 & e_3 & e_0 \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{e}_0 \\ \dot{e}_1 \\ \dot{e}_2 \\ \dot{e}_3 \end{bmatrix} = {}^{B}B(q) \, {}^{B}\!\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

very important

Properties: $G^T G = I_{(6\times 6)}$

$$G G^T \dot{q} = \dot{q}$$

Also need rotation matrix

$${}^{N}_{B}R(q) = \begin{bmatrix} \underbrace{\begin{matrix} 1-2(e_2^2+e_3^2) \\ 2(e_1e_2+e_0e_3) \\ 2(e_1e_3-e_0e_2) \end{matrix}}_{{}^{N}x_B} & \begin{matrix} 2(e_1e_2-e_0e_3) \\ 1-2(e_1^2+e_3^2) \\ 2(e_2e_3+e_0e_1) \end{matrix} & \begin{matrix} 2(e_1e_3+e_0e_2) \\ 2(e_2e_3-e_0e_1) \\ 1-2(e_2^2+e_1^2) \end{matrix} \end{bmatrix}$$

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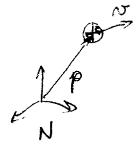
Change in Dynamic Eqs.

Sum of forces $\sum_i f_i = F$

Sum of Moments $\sum_i r_i \times f_i + n_i = N$

Newton : $F = \frac{d}{dt}(m v) \xrightarrow{m=\text{const}} \boxed{F = m\dot{v}}$



Euler : $N = \frac{d}{dt}(J\omega)$

because $J\omega$ is the angular momentum of a rotating body, its derivative has two parts

$$\frac{d}{dt}(J\omega) = \boxed{J\dot{\omega} + \omega \times J\omega = N}$$

Prepare for integration / simulation – put in first-order form.

$$\dot{v} = F/m$$

$$\dot{\omega} = J^{-1}(N - \omega \times J\omega)$$

$$\dot{x} = v$$

$$\dot{e} = B\omega$$

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J = Inertia Matrix

J is 3×3, P.D., & symmetric.

represents mass distribution

In frame B, J is constant $= {}^BJ$

Frames of representation of dynamic eqs.

$${}^N\dot{v} = {}^NF/m$$

$${}^N\dot{\omega} = {}^NJ^{-1}({}^NN - {}^N\omega \times {}^NJ\,{}^N\omega)$$

$${}^N\dot{x} = {}^Nv$$

$${}^N\dot{e} = {}^NB\,{}^N\omega$$

~~Define ${}^NJ$ from ${}^BJ$.~~

~~${}^B\omega = {}^NR\,{}^N\dot{\omega}$~~

Define ${}^N J$ from ${}^B J$

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$${}^N_B R \left({}^B J\, {}^B\dot{\omega} = {}^B N - {}^B\omega \times {}^B J\, {}^B\omega \right)$$

$$\underbrace{{}^N_B R\, {}^B J\, {}^B_N R}_{{}^N J}\, {}^N\dot{\omega} = \underbrace{{}^N_B R\, {}^B N}_{{}^N N} - \underbrace{{}^N_B R\, {}^B\omega}_{{}^N\omega} \times \underbrace{{}^N_B R\, {}^B J\, {}^B_N R}_{{}^N J}\, {}^N\omega$$

$${}^N J = {}^N_B R\, {}^B J\, {}^B_N R$$

${}^N J$ is P.D. & Symmetric.

$$g_{ext} = \begin{bmatrix} \text{gravity,} \\ \text{drag,} \\ \text{etc} \end{bmatrix} + \begin{bmatrix} 0 \\ -\omega \times J\omega \end{bmatrix}$$

Complementarity Conditions and the following

$$\dot{\nu} = M^{-1}\left(W_n \lambda_n + W_f \lambda_f + g_{ext} \right) + W_b \lambda_b$$

$$\dot{q} = G\nu \qquad , \qquad e_0^2 + e_1^2 + e_2^2 + e_3^2 - 1 = 0 = \Theta(q)$$

where $M = \left[\begin{array}{c|c} mI & 0 \\ \hline 0 & J \end{array}\right]$ $\qquad G = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & B \end{array}\right]$

3D Dynamics – Contact Friction

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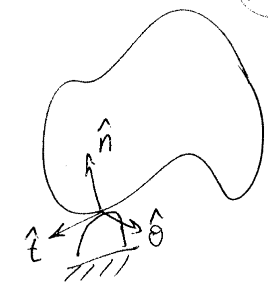
$$f = \hat{n}\lambda_n + \hat{t}\lambda_t + \hat{o}\lambda_o$$

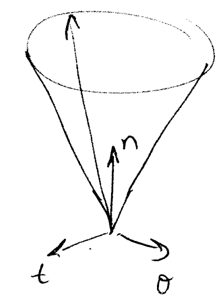
$$r \times f = r \times \hat{n}\lambda_n + r \times \hat{t}\lambda_t + r \times \hat{o}\lambda_o$$



Friction Model – ~~When Slid~~

Friction acts to maximize rate at which energy is dissipated

Friction force lies with a cone.



Sliding or Rolling → $\lambda_t^2 + \lambda_o^2 \le \mu^2 \lambda_n^2$

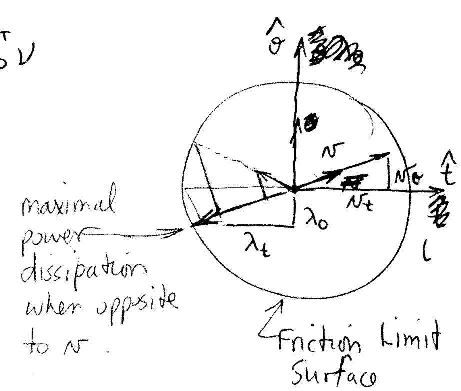
Sliding $(\nu_t, \nu_o) \neq 0$ $\quad (\lambda_t, \lambda_o) \in \arg\max\left\{ -\nu_t \lambda_t' - \nu_o \lambda_o' : \lambda_t'^2 + \lambda_o'^2 \le \mu^2 \lambda_n^2 \right\}$

where $\nu_t = W_t^T \nu$, $\quad \nu_o = W_o^T \nu$

~~$(\lambda_t, \lambda_o) \in \arg\max\{-W_t$~~

~~$(\lambda_t, \lambda_o) \in \arg\max\{-\nu(W_t + W_o)$~~

$$(\lambda_t, \lambda_o) \in \arg\max\left\{ -\nu^T \left[W_t \;\; W_o \right] \begin{bmatrix} \lambda_t' \\ \lambda_o' \end{bmatrix} : (\lambda_t', \lambda_o') \in \mathcal{F} \right\}$$



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When sliding we can solve for λ_t, λ_o :

$$\lambda_t = \frac{-\mu \lambda_n \nu_t}{\sqrt{\nu_t^2 + \nu_o^2}} \qquad \lambda_o = \frac{-\mu \lambda_n \nu_o}{\sqrt{\nu_t^2 + \nu_o^2}}$$

NONLINEAR CONSTRAINTS

If we knew the approximate sliding direction, then we could linearize with Taylor series

But we don't!

And $\sqrt{\nu_t^2 + \nu_o^2}$ can go to zero!

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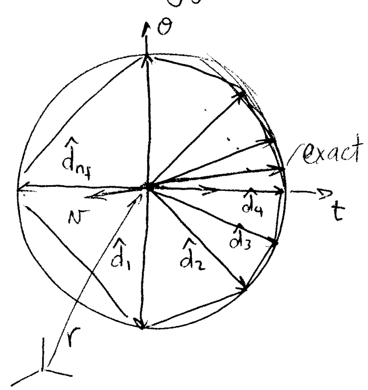
Approximate Friction Limit Surface as a Polygon



Friction force :

$$\hat{d}_1 \lambda_{1f} + \hat{d}_2 \lambda_{2f} + \cdots + \hat{d}_{n_f} \lambda_{n_f f}$$

$$\lambda_{if} \geq 0 \quad \forall \ i$$

Friction moment :

$$r \times \hat{d}_1 \lambda_{1f} + \cdots + r \times \hat{d}_{n_f} \lambda_{n_f f}$$

Friction Wrench

$$W_f \lambda_f \qquad W_f = \underbrace{\begin{bmatrix} \hat{d}_1 & \cdots & \hat{d}_{n_f} \\ r_1 \times \hat{d}_1 & & r \times \hat{d}_{n_f} \end{bmatrix}}_{(6 \times n_f)} \qquad \lambda_f = \underbrace{\begin{bmatrix} \lambda_{1f} \\ \lambda_{2f} \\ \vdots \\ \lambda_{n_f f} \end{bmatrix}}_{(n_f \times 1)} \geq 0$$

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7.1

Instantaneous Dynamics

$$\dot{\nu} = M^{-1}(W_n \lambda_n + W_t \lambda_t + W_o \lambda_o + W_b \lambda_b + g_{ext})$$

$$\dot{q} = G \nu$$

$$\Theta = 0 \quad , \quad \lambda_b \text{ free}$$

$$0 \leq \lambda_n \perp \Psi_n \geq 0$$

$$(\lambda_t, \lambda_o) \in \text{argmax} \left\{ -\nu^T [W_t \ W_o] \begin{bmatrix} \lambda'_t \\ \lambda'_o \end{bmatrix} : (\lambda'_t, \lambda'_o) \in \mathcal{F} \right\}$$

$$\text{where } \mathcal{F} = \mathcal{F}_1 \times \mathcal{F}_2 \times \cdots \times \mathcal{F}_{n_c}$$

Sum all contact forces

$$W_n \lambda_n + W_f \lambda_f$$

$$\lambda_n, \lambda_f \geq 0$$

$$\lambda_n = \begin{bmatrix} \lambda_{1n} \\ \lambda_{2n} \\ \vdots \\ \lambda_{n_c n} \end{bmatrix} \qquad \lambda_f = \begin{bmatrix} \lambda_{1f} \\ \lambda_{2f} \\ \vdots \\ \lambda_{n_c f} \end{bmatrix}$$

where $\lambda_{jf} = [\lambda_{jf_1} \ \cdots \ \lambda_{jf_{n_d}}]$

Over small time step

$$W_n p_n + W_f p_f$$

$$p_n, p_f \geq 0 \quad , \quad p_\alpha = h \lambda_\alpha$$

How do we write constraints to pick best friction force?

$$0 \leq p_f^{\ell+1} \perp W_f^T \nu^{\ell+1} + E s^{\ell+1} \geq 0$$

$$0 \leq s^{\ell+1} \perp U p_n^{\ell+1} - E^T p_f^{\ell+1} \geq 0$$

$$U = \text{diag}(\mu_1, \cdots, \mu_{n_c}) \qquad E = \text{BlkDiag}(e_1, e_2, \ldots e_{n_c})$$

$$\text{where } e = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}_{(n_d \times 1)}$$

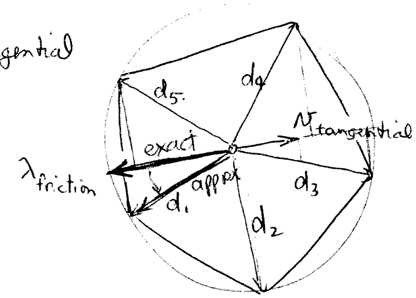
Nondegenerate Solutions of the LCP

Assume $(W_t + W_o)\nu = \nu_{tangential}$

Note that

$$\nu_{tang} \cdot d_1$$

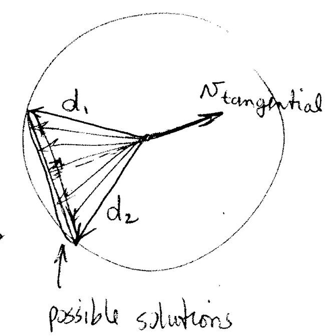
dissipates most energy.



$$\therefore \ \lambda_{f_1} \geq 0, \quad \lambda_{f_2} = \lambda_{f_3} = \lambda_{f_4} = \lambda_{f_5} = 0$$

Some solutions find $\nu_{tangential}$ such that

$$W_f^T \nu = \begin{bmatrix} \text{min} \\ \text{min} \\ \text{larger} \\ \vdots \\ \text{larger} \end{bmatrix}$$



Then $\lambda_{f_1}, \lambda_{f_2} \geq 0 \ , \ \lambda_{f_j} = 0 \quad \forall j \neq 1, 2$

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10

Time Stepping LCP

Same as page 22 11/9/06

Note that for every body we have

$$\Theta_i = e_{i0}^2 + e_{i1}^2 + e_{i2}^2 + e_{i3}^2 - 1 = 0$$

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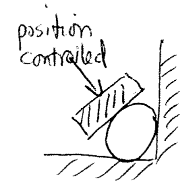
11

LCP Solution non-existence



$\psi_n^{l+1} \geq 0$ is infeasible

i.e. block moves toward corner by some finite amount over timestep h.

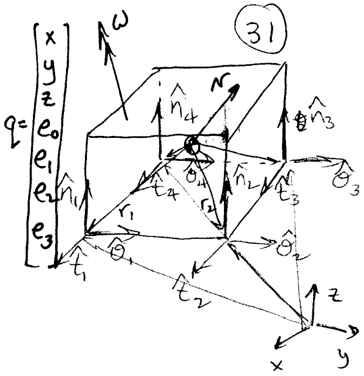
Disk is larger than ~~gap~~ space between

$\psi_n^{l+1} \not\geq 0$.

An Example: Box on Floor

Assume Body-Fixed Frame is principal axes

$$\nu = \begin{bmatrix} v_x \\ v_y \\ v_z \\ \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} \qquad q = \begin{bmatrix} x \\ y \\ z \\ e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix}$$



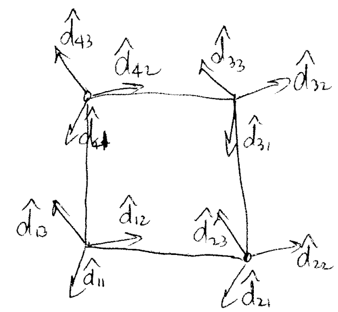
$${}^BJ = \text{diag}(1, 2, 3)$$

$$m = 1$$

r's from c.g. to contact point.

$${}^NW_n = \begin{bmatrix} {}^N\hat{n}_1 & \cdots & {}^N\hat{n}_4 \\ {}^Nr_1 \times {}^N\hat{n}_1 & \cdots & {}^Nr_4 \times {}^N\hat{n}_4 \end{bmatrix}$$

$$\cancel{{}^NW_f = \begin{bmatrix} {}^NW_t & -W_t \end{bmatrix}}$$



$${}^NW_f = \begin{bmatrix} {}^N\hat{d}_{11} & \hat{d}_{12} & \hat{d}_{13} & \hat{d}_{21} & \cdots \\ {}^Nr_1 \times {}^N\hat{d}_{11} & r_1 \times \hat{d}_{12} & r_1 \times \hat{d}_{13} & r_2 \times \hat{d}_{21} & \cdots \end{bmatrix}$$

$$M = \left[\begin{array}{ccc|c} m & & 0 & \\ & m & & 0 \\ 0 & & m & \\ \hline & 0 & & {}^N I \end{array}\right] \qquad p_{ext} = \left[\begin{array}{c} 0 \\ 0 \\ -mg \\ \hline h({}^N\omega \times {}^N I\, {}^N\omega) \end{array}\right]$$

$\cancel{W_E}$ $\Theta^l = e_0^2 + e_1^2 + e_2^2 + e_3^2 - 1 = 0$ $\quad [0 \;\; 0 \;\; 0 \;\; 2e_0 \;\; 2e_1 \;\; 2e_2 \;\; 2e_3]$

$$W_E^T = \frac{\partial\Theta}{\partial q} G(q) \qquad W_E = G^T(q)\left(\frac{\partial\Theta}{\partial q}\right)^T \quad \text{where} \left(\frac{\partial\Theta}{\partial q}\right)^T = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2e_0 \\ 2e_1 \\ 2e_2 \\ 2e_3 \end{bmatrix}$$

$$U = \begin{bmatrix} \mu_1 & & & 0 \\ & \mu_2 & & \\ & & \mu_3 & \\ 0 & & & \mu_4 \end{bmatrix}_{(n_c \times n_c)}$$

$$E^T = \begin{bmatrix} 1\;1\;1 & & & \\ & 1\;1\;1 & & \\ & & 1\;1\;1 & \\ & & & 1\;1\;1 \end{bmatrix}_{(n_c \times (n_c \cdot n_f))}$$

$$\frac{\partial\Theta}{\partial t} = 0$$

$\Psi_n^l / h \iff \Psi_n^l$ = gap at current time

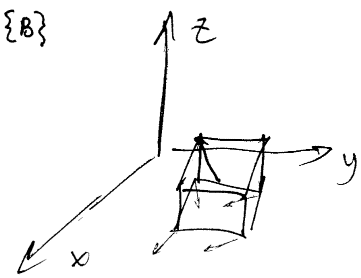
$$\Psi_n^l = \begin{bmatrix} 0.001 \\ -0.0014 \\ -0.010 \\ -0.0004 \end{bmatrix}$$

e.g.

Assume ~~unit cube.~~ Look at nonpenetration for one corner.

$$\Psi_n = \left[\begin{array}{c|c} {}^N_B R(q) & \begin{matrix} q_1 \\ q_2 \\ q_3 \end{matrix} \\ \hline 0\;0\;0 & 1 \end{array}\right]_3 \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1 \end{bmatrix} \geq 0$$

corner in {B}

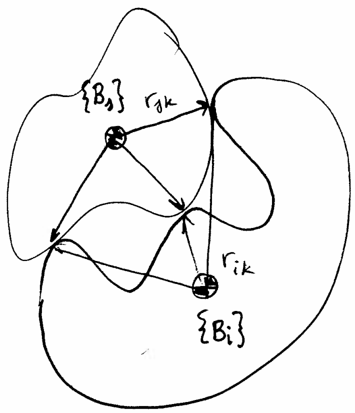


It is ok to Keep all 8 constraints active at all times.

More Bodies

$$q = \begin{bmatrix} q_1 \\ \vdots \\ q_{nb} \end{bmatrix} \quad \nu = \begin{bmatrix} \nu_1 \\ \vdots \\ \nu_{nb} \end{bmatrix} \quad p_{ext} = \begin{bmatrix} p_{1,ext} \\ p_{2,ext} \\ \vdots \\ p_{n_b,ext} \end{bmatrix}$$

$(7nb\times 1)$ $\quad (6n_b\times 1)$ $\quad (6nb\times 1)$



$$M = \mathrm{diag}(M_1, \dots, M_{nb})$$

$$W_n = \begin{array}{c} \\ \\ \end{array}\left[\begin{array}{ccc} & W_{ik} & \\ & W_{jk} & \end{array}\right]\begin{array}{l} \}\text{ body } i \\ \}\text{ body } j \end{array}$$

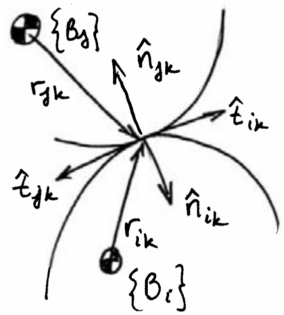
$(6n_b \times n_c)$

$1\ 2 \cdots k \cdots n_c$ $\quad (6nb\times n_c)$

$\longleftarrow$ contacts $\longrightarrow$

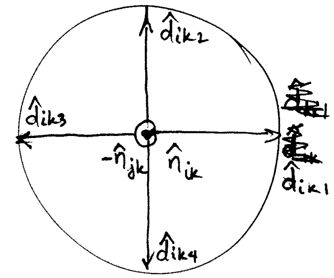
Where $${}^N W_{ik} = \begin{bmatrix} {}^N\hat{n}_{ik} \\ {}^N r_{ik} \times {}^N\hat{n}_{ik} \end{bmatrix} \Rightarrow$$

$${}^N W_{jk} = \begin{bmatrix} -{}^N\hat{n}_{ik} \\ {}^N r_{jk} \times (-{}^N\hat{n}_{ik}) \end{bmatrix} = \begin{bmatrix} {}^N\hat{n}_{jk} \\ {}^N r_{jk} \times {}^N\hat{n}_{jk} \end{bmatrix}$$



$$W_f = \left[\begin{array}{ccc} & \text{Contact } k & \\ & W_{ik,f} & \\ & W_{jk,f} & \end{array}\right]\begin{array}{l} \\ \text{Body } i \\ \text{Body } j \end{array}$$

Contact k



$$W_{ik,f} = \begin{bmatrix} \hat{d}_{ik,1} & \cdots & \hat{d}_{ik,n_f} \\ r_{ik}\times\hat{d}_{ik,1} & \cdots & r_{ik}\times\hat{d}_{ik,n_f} \end{bmatrix}$$

Similar for $W_{jk,f}$.

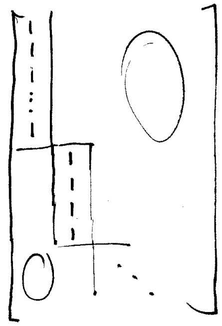
Convenient to let ${}^N\hat{d}_{ik,1} = -{}^N\hat{d}_{jk,1}$, etc.

Then it is easy to use one set of contact ~~force~~ impulse parameters $p_{kn}, p_{kf1}, p_{kf2}, \dots$

$$\Theta(q) = \begin{bmatrix} e_{10}^2 + e_{11}^2 + e_{12}^2 + e_{13}^2 - 1 \\ e_{20}^2 + e_{21}^2 + e_{22}^2 + e_{23}^2 - 1 \\ \vdots \end{bmatrix}$$

$$\frac{\partial\Theta}{\partial q} = \begin{bmatrix} 0\ 0\ 0\ 2e_{10}\ 2e_{11}\ 2e_{12}\ 2e_{13} & & \\ 0\ 0\ 0 & 0\ 0\ 0\ 2e_{20}\ 2e_{21}\ 2e_{22}\ 2e_{23} & \cdots \\ & 0 & \ddots \end{bmatrix}$$

$E =$
$(n_c \times (n_c \cdot n_f))$



$\longleftarrow$ contacts $\longrightarrow$

$$p_f = \begin{bmatrix} \vdots \\ p_{kf1} \\ p_{kf2} \\ \vdots \\ p_{(k+1)f1} \\ p_{(k+1)f2} \\ p_{(k+1)f3} \\ \vdots \end{bmatrix} \updownarrow \text{contacts}$$

Final Size of LCP $(7n_b + n_c(2+n_f))$

Could eliminate $7n_b$ variables to make problem smaller, but then the LCP matrix becomes dense and solver converges more slowly.