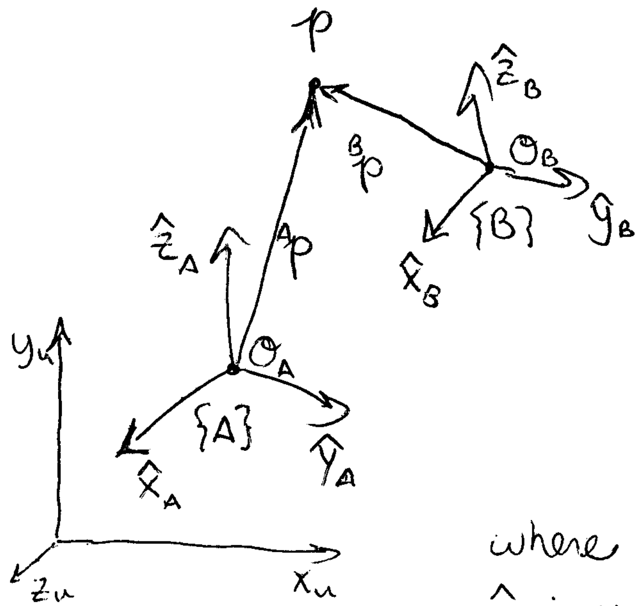
Summary of Displacement Representations as 4x4 Homogenous Transformations



Position representation

$$p = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \qquad {}^Ap = \begin{bmatrix} {}^Ap_x \\ {}^Ap_y \\ {}^Ap_z \end{bmatrix}$$

where $\hat{x}$ is unit vector.

$^Ap = (p - O_A)$ expressed in $\{A\}$

$$^Ap = \begin{bmatrix} \hat{x}_A^T (p - O_A) \\ \hat{y}_A^T (p - O_A) \\ \hat{z}_A^T (p - O_A) \end{bmatrix}$$

⇐ each of these ~~vectors~~ rows must be ~~represented in~~ expressed in a common frame!

$$^Ap = \begin{bmatrix} \hat{x}_A^T \\ y_A^T \\ \hat{z}_A^T \end{bmatrix} (p - O_A)$$

~~Assume ^Bp is~~

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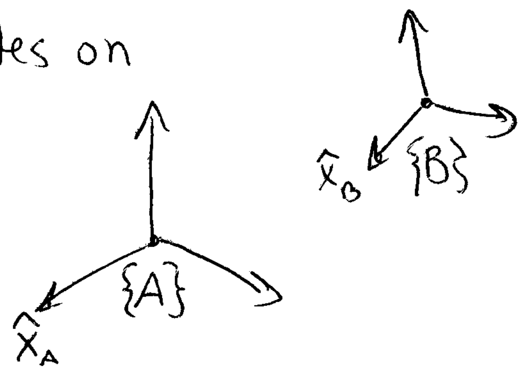
Orientation Representation

How do we do this?

We ~~represent~~ express the axis directions of one frame in coordinates on another frame.



Let ${}^B_A R$ be the matrix whose columns are ${}^B\hat{x}_A, {}^B\hat{y}_A, {}^B\hat{z}_A$, but expressed in $\{B\}$.

$$
{}^B_A R = \begin{bmatrix} \hat{x}_A \cdot \hat{x}_B & \hat{y}_A \cdot \hat{x}_B & \hat{z}_A \cdot \hat{x}_B \\ \hat{x}_A \cdot \hat{y}_B & \hat{y}_A \cdot \hat{y}_B & \hat{z}_A \cdot \hat{y}_B \\ \hat{x}_A \cdot \hat{z}_B & \hat{y}_A \cdot \hat{z}_B & \hat{z}_A \cdot \hat{z}_B \end{bmatrix}
$$

$$
\underbrace{\quad}_{{}^B\hat{x}_A} \qquad \underbrace{\quad}_{{}^B\hat{y}_A} \qquad \underbrace{\quad}_{{}^B\hat{z}_A}
$$

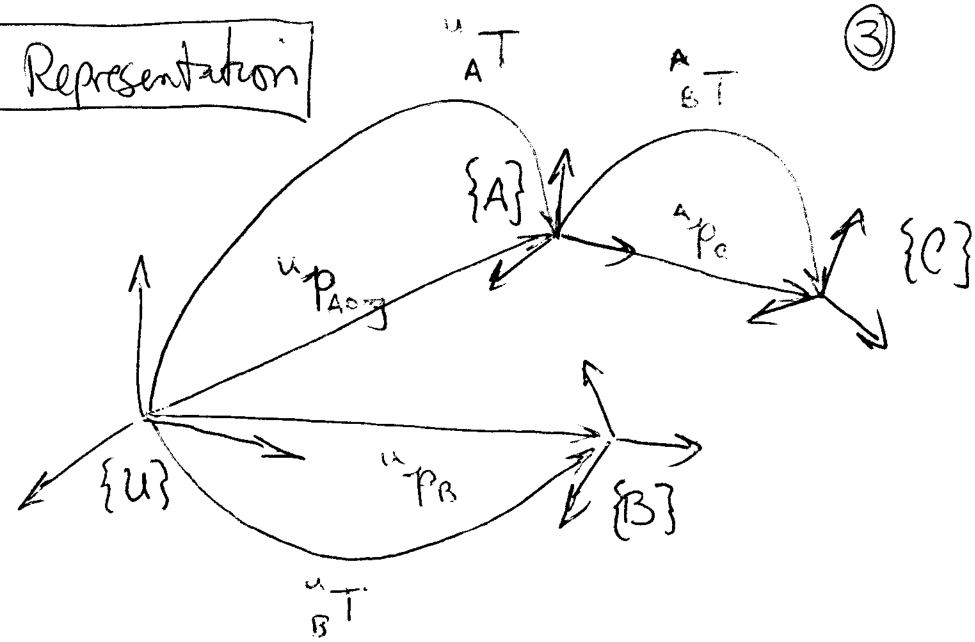
Note that inverse operation is to express $\hat{x}_B, \hat{y}_B, \hat{z}_B$ in frame $\{A\}$. $\Rightarrow$ $\boxed{{}^B_A R^{-1} = {}^A_B R}$

Note that from structure, we see

$$\boxed{{}^B_A R^{-1} = {}^B_A R^T} \Rightarrow \cancel{\hat{x}_A \cdot \hat{x}_B = 0}$$

~~Note~~ $\cancel{{}^A_C R = {}^A_B R \, {}^B_C R}$

Displacement Representation



$$ {}^{U}_{A}T = \left[\begin{array}{c|c} {}^{U}_{A}R & {}^{U}p_{A} \\ \hline 0 & 1 \end{array}\right] \qquad {}^{A}_{B}T = \left[\begin{array}{c|c} {}^{A}_{B}R & {}^{A}p_{B} \\ \hline 0 & 1 \end{array}\right] $$

Go back to idea of projective space.–

In homogenous coordinates $\left[\begin{array}{c} p \\ \hline 1 \end{array}\right]$ is a point

and each column of R is a direction!

R as an Operator

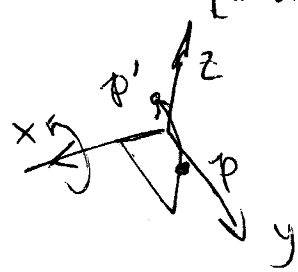
$$rot_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_\theta & -S_\theta \\ 0 & S_\theta & C_\theta \end{bmatrix} \qquad rot_y(\theta) = \begin{bmatrix} C_\theta & 0 & S_\theta \\ 0 & 1 & 0 \\ -S_\theta & 0 & C_\theta \end{bmatrix}$$

$$rot_z(\theta) = \begin{bmatrix} C_\theta & -S_\theta & 0 \\ S_\theta & C_\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Rotate point about origin by $\left(rot_{(\cdot)}(\theta)\right) p = p'$

Rotate $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ about x-axis by $30^\circ \Rightarrow p' \simeq \begin{bmatrix} 1 \\ 1.2 \\ 1.4 \end{bmatrix}$

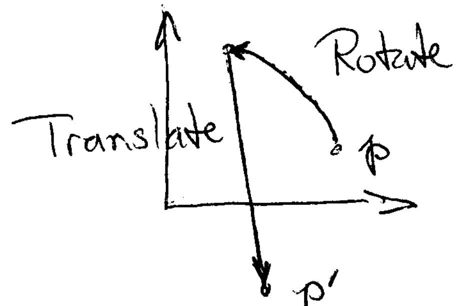
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \sqrt{3}/2 & -1/2 \\ 0 & 1/2 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.2 \\ 1.4 \end{bmatrix}$$



T as an Operator

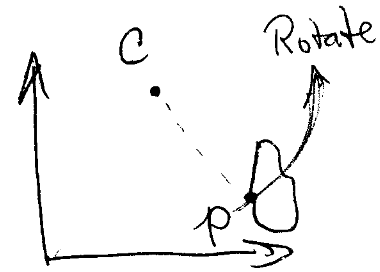
$$T = \left[\begin{array}{c|c} I & p \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} R & 0 \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} R & p \\ \hline 0 & 1 \end{array}\right]$$

Rotate about origin (of frame of expression) then translate



How do we effect rotation about a specific point?

First translate space so C is at origin, then rotate about origin, then rotate back

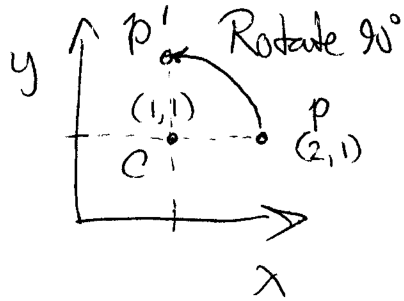


maybe p is pt. on body.

$$\left[\begin{array}{c|c} I & C \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} R & 0 \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c|c} I & -C \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c} p \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} p' \\ \hline 1 \end{array}\right]$$

$$\left[\begin{array}{c|c} R & -RC + C \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c} p \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} p' \\ \hline 1 \end{array}\right]$$

Example: Rotate p $\pi/2$ rad about C.

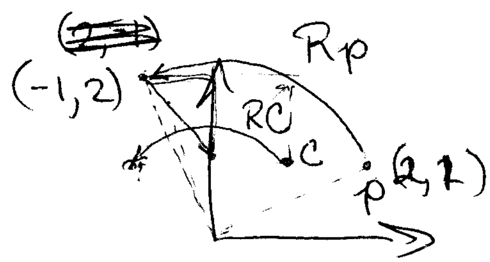


$$\left[\begin{array}{c|c} R & -RC + C \\ \hline 0 & 1 \end{array}\right] \left[\begin{array}{c} p \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} p' \\ \hline 1 \end{array}\right]$$

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \qquad p = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$p' = Rp - RC + C$$

$$p' = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

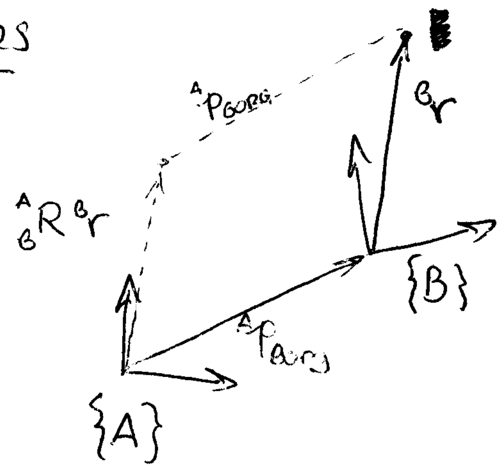


$$p' = \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Homogeneous Transformations as a Mapping Between Frames

$$\begin{bmatrix} {}^A r \\ \hline 1 \end{bmatrix} = {}^A_B T \begin{bmatrix} {}^B r \\ \hline 1 \end{bmatrix}$$

where ${}^A r$ and ${}^B r$

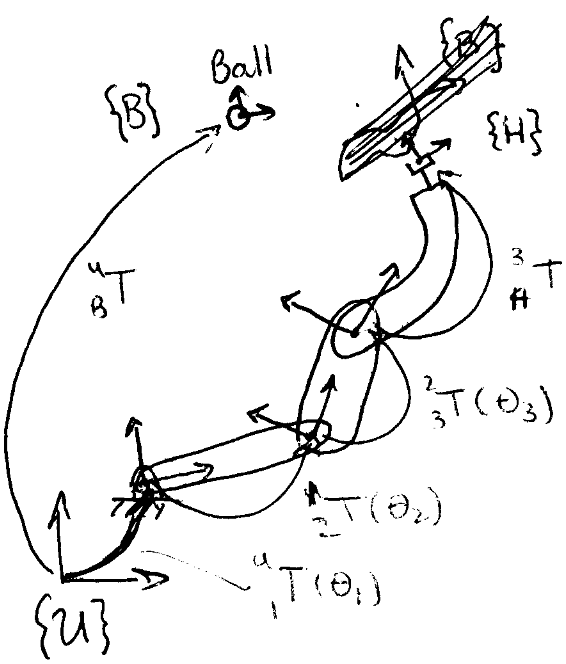


$$\begin{bmatrix} {}^A r \\ \hline 1 \end{bmatrix} = \left[\begin{array}{c|c} {}^A_B R & {}^A P_{Borg} \\ \hline 0 & 1 \end{array}\right] \begin{bmatrix} {}^B r \\ \hline 1 \end{bmatrix}$$

$${}^A r = {}^A_B R\, {}^B r + {}^A P_{BORG}$$

$\Leftarrow$ All quantities must be in same coordinate frame!

$\therefore$ ${}^A_B R\, {}^B r$ gives coords of ${}^B r$ in frame {A}



$\boxed{\text{To grasp}}$

$${}^U_1 T(\theta_1)\; {}^1_2 T(\theta_2)\; {}^2_3 T(\theta_3)\; {}^3_H T = {}^U_B T$$

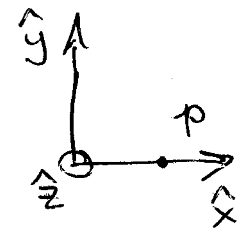
$\boxed{\text{Where is center of gripper?}}$

$${}^U_H T(\theta_1, \theta_2, \theta_3) \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Equivalence Between Operator and Mapping Interpretations

Rotation Operator -

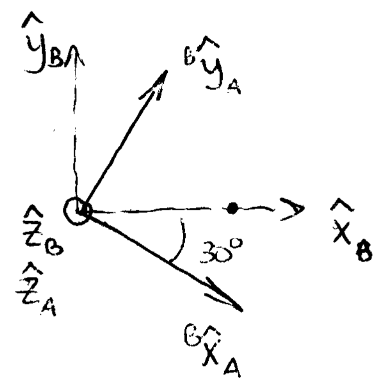
$$\mathrm{rot}_z(30°) \cdot p$$



$$\begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

Interpret $\mathrm{rot}_z(30°)$ as a frame

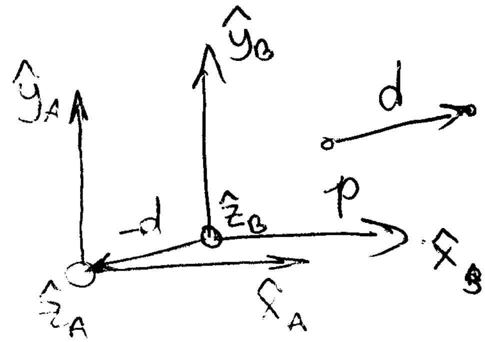
$$\begin{bmatrix} {}^Ap_x \\ {}^Ap_y \\ {}^Ap_z \end{bmatrix} = \begin{array}{r} {}^B\hat{x}_A \\ {}^B\hat{y}_A \\ {}^B\hat{z}_A \end{array} \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} {}^Bp_x \\ {}^Bp_y \\ {}^Bp_z \end{bmatrix}$$



One sees that to write p in frame $\{A\}$, one dots p with $\hat{x}_A$, $\hat{y}_A$, $\hat{z}_A$.

Translation Operator

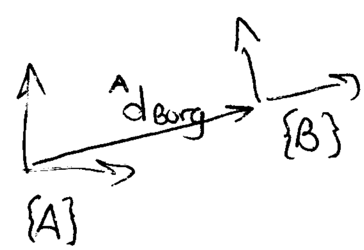
$$\left[\begin{array}{c|c} I & d \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c} p \\ \hline 1 \end{array}\right] = \left[\begin{array}{c} d+p \\ \hline 1 \end{array}\right]$$



Now think of p as fixed in $\{B\}$.

What is the location of $\{A\}$ that is consistent?

$$\left[\begin{array}{c} {}^{A}p \\ \hline 1 \end{array}\right] = \left[\begin{array}{c|c} {}^{A}_{B}R & {}^{A}d_{Borg} \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c} {}^{B}p \\ \hline 1 \end{array}\right]$$

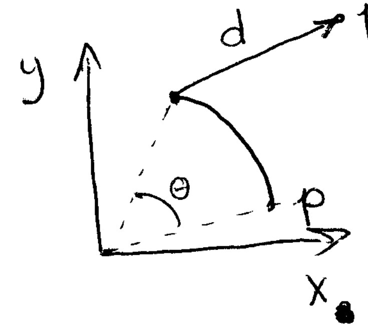


$$\therefore \quad {}^{B}d_{AORG} = {}^{B}_{A}R(-{}^{A}d_{BORG})$$

Displacement Operator

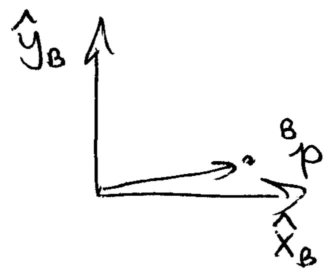
Rotate the Translate

$$\left[\begin{array}{c|c} R & d \\ \hline 0 & 1 \end{array}\right] \longrightarrow$$



Now view p from a different frame so that it "looks" like p'

$$\begin{bmatrix} {}^{A}p \\ \hline 1 \end{bmatrix} = \overbrace{\left[\begin{array}{c|c} {}^{A}_{B}R & {}^{A}d \\ \hline 0 & 1 \end{array}\right]}^{{}^{A}_{B}T} \begin{bmatrix} {}^{B}p \\ \hline 1 \end{bmatrix}$$



To see how to locate & orient frame $\{A\}$ from $\{B\}$, invert matrix to yield ${}^{B}_{A}T$

$${}^{A}_{B}T^{-1} = {}^{B}_{A}T = \left[\begin{array}{c|c} {}^{A}_{B}R^{T} & -{}^{A}_{B}R^{T}\,{}^{A}d \\ \hline 0 & 1 \end{array}\right]$$

$$= \left[\begin{array}{c|c} {}^{B}_{A}R & -{}^{B}_{A}R\,{}^{A}d \\ \hline 0 & 1 \end{array}\right]$$