Planar Sliding

Coulomb frict. at a pt. is well defined.

What if we have a contact patch?

We would like a law like Coulomb's that is "easy" to apply.

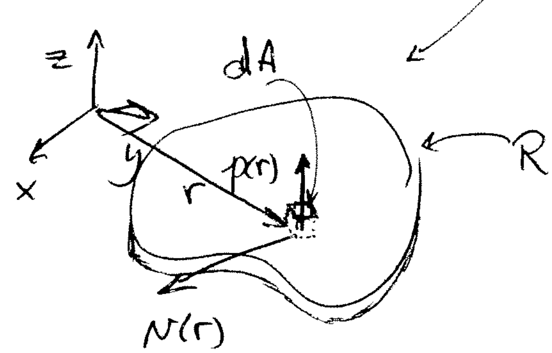
Let μ = coeff. of frict
assume constant

R = contact region

$p(r)$ = pressure

$v(r)$ = velocity

Planar Slider



Differential Forces

Normal $= p(r)\, dA$

~~Tangential~~ Friction $= -\mu \frac{v(r)}{\|v(r)\|} p(r)\, dA \quad , \quad v(r) \neq 0$

Differential Moments

Normal $= r \times p(r)\, dA$

Friction $= -\mu r \times \frac{v(r)}{\|v(r)\|} p(r)\, dA \quad , \quad v(r) \neq 0$

Integrate over R to obtain total forces & moments.

$$f_0 = f_n = \int_R p(r)\, dA$$

$$n_n = -\cancel{\mu} \int_R r \times \cancel{\frac{v(r)}{\|v(r)\|}}\, p(r)\, dA$$

$v(r) \neq 0$ in finite patch

$$f_f = -\mu \int_R \frac{v(r)}{\|v(r)\|}\, p(r)\, dA$$

"

$$n_f = -\mu \int r \times \frac{v(r)}{\|v(r)\|}\, p(r)\, dA$$

"

Removing one point from integral where $p(r)\, dA$ is finite does not change $\int_R$.

In general, one needs to know $p(r)$, $v(r)$, & R to integrate.

Consider two cases:

1.) Translation

2.) Rotation about instantant center.

First define centroid o

Skip to pages 4,5,6

$$\cancel{r_0 = \frac{\int_R r\, p(r)\, dA}{\int_R p(r)\, dA}} = \ldots \, p(r)\, dA$$

Case 1: Pure Translational Sliding

$$v(r) = \text{constant}$$

$$f_f = -\mu \frac{v(r)}{\|v(r)\|} \int p(r)\, dA = \boxed{-\mu \frac{v(r)}{\|v(r)\|} f_n = f_f}$$

$$n_f = -\mu \int_R r\, p(r)\, dA \times \frac{v(r)}{\|v(r)\|}$$

$$= \int r\, p(r)\, dA \times -\mu \frac{v(r)}{\|v(r)\|} \cdot \frac{\int_R p(r)\, dA}{\int_R p(r)\, dA}$$

$$= \frac{\int r\, p(r)\, dA}{\int p(r)\, dA} \times -\mu \frac{v(r)}{\|v(r)\|} \int p(r)\, dA$$

$$\boxed{n_f = r_0 \times f_f}$$

Skip back to page 6: rotation

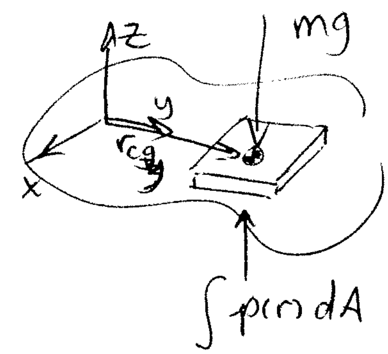
Center of Friction

Center of gravity is place where one can lump gravity force, mg.

Would like an analogous point for friction force.

Consider slider at rest on plane.

$$m\ddot{z} = \int p(r)\,dA - mg = 0$$



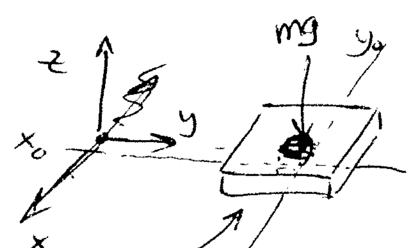
$$\boxed{\int p(r)\,dA = mg}$$

Centroid of Pressure Distribution is point about which moments applied by pressure distrib are zero.

Find (x_0, y_0) w/ no moment

$$\cancel{\int_R (y - y_0)\, p(r)\, dA \pm mg\, x_{cg} = 0}$$

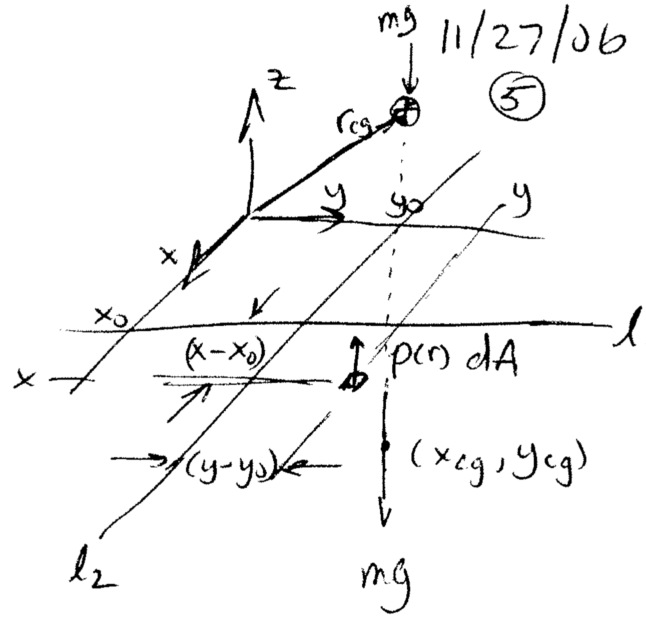
$$\cancel{\int_R (x - x_0)\, p(r)\, dA}$$



(See next page)

let $r_{cg} = \begin{bmatrix} x_{cg} \\ y_{cg} \end{bmatrix}$

Sum of moments about lines ∥ to x-axis & y-axis equal zero



Mom. about $l_1 = 0$

$$(x_{cg}-x_0)mg + \int_R -(x-x_0)\,p(r)\,dA = 0$$

$$x_{cg}mg - x_0 mg - \int_R x\,p(r)\,dA + \underbrace{\overbrace{\int_R x_0\,p(r)\,dA}^{x_0 \text{ is constant}}}_{x_0 mg} = 0$$

$$\frac{x_{cg}\,mg}{mg} = \frac{\int_R x\,p(r)\,dA}{mg = f_n}$$

$$\therefore \quad x_{cg} = x_0 \quad \left(\text{Note: } \frac{\int_R x\,p(r)\,dA}{\int p(r)\,dA} = x_0 \text{ by def.}\right)$$

$\therefore$ Under non-accelerating planar motion

$$\boxed{\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} x_{cg} \\ y_{cg} \end{bmatrix} \quad \therefore \text{ Center of friction is projection of center of mass to plane of sliding.}}$$

Center of Friction allows other forces as long as they are in the support plane, since then they don't alter the preceding moment sums.

Skip back to page 3

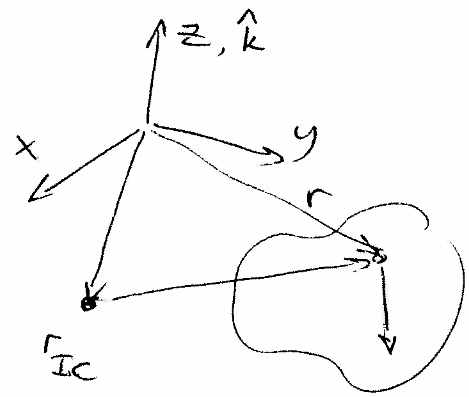
Case 1:

Case 2: Rotation

Suppose body is rotating about a point r_{Ic}

$$v(r) = \omega \times (r - r_{Ic})$$

$$= \hat{k}\,\dot{\theta} \times (r - r_{Ic})$$



sliding direction at r is ...

$$\frac{v(r)}{\|v(r)\|} = \operatorname{sgn}(\dot{\theta})\,\hat{k} \times \frac{r - r_{Ic}}{\|r - r_{Ic}\|}$$

Substituting into defs of page 2 gives

$$\boxed{f_f = -\mu \operatorname{sgn}(\dot{\theta})\,\hat{k} \times \int_R \frac{r - r_{Ic}}{\|r - r_{Ic}\|}\, p(r)\, dA}$$

$$\boxed{(n_f)_z = -\mu \operatorname{sgn}(\dot{\theta}) \int r \cdot \frac{r - r_{Ic}}{\|r - r_{Ic}\|}\, p(r)\, dA}$$

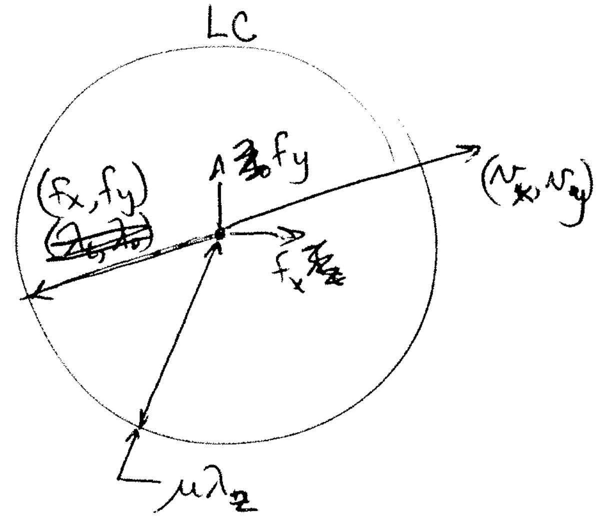
Limit Surfaces & Curves

Consider a point slider in the plane

For a given normal force, f_z

the limit curve, LC,

contains all possible friction forces



Friction force is one that dissipates maximum power

$$(f_x, f_y) = \underset{(f_x, f_y) \in LC}{\operatorname{argmax}} \left((f_x, f_y) \cdot (N_x, N_y)\right)$$

Equivalently, (f_x, f_y) satisfies:

$$\left[(f_x, f_y) - (f_x, f_y)^*\right] \cdot (N_x, N_y) \geq 0$$

~~$\forall (\lambda_x, \lambda_y)^* \in LC \cup \ldots$ $\ldots$ Int LC~~

$\forall (f_x, f_y)^* \in \mathcal{F}$

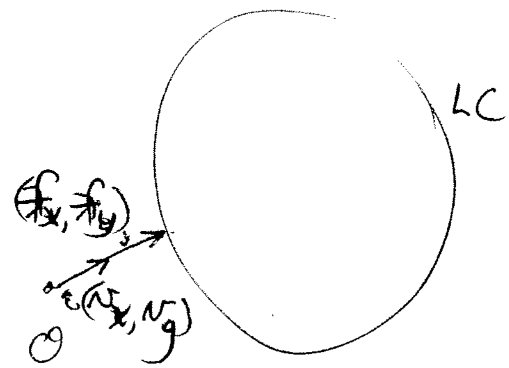
where $\mathcal{F} = LC \cup \text{Int}(LC)$

Notice that:

$(f_x, f_y) \perp LC$ when $(v_x, v_y) \neq 0$!

$(f_x, f_y) = (0, 0)$ is interior to LC

otherwise friction could generate energy



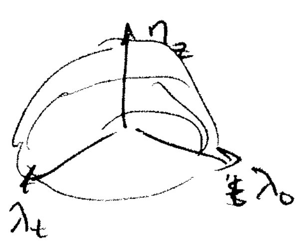
We want more graphical techniques - for planar sliding.

We would like to extend this idea to contact regions.

would like LS (limit surface) to contain all possible friction wrenches.

Would like to retain property that $(v_t, v_o) \perp LS$!

Goyal Cornell Phd 1989 did this extension



Here are the properties of LS's.

If $p(r)$ is finite everywhere (no concentrated points of force), the LS is strictly convex and ~~each~~ mapping between (f_x, f_y) & (N_x, N_y) is 1-1 & onto if $(N_x, N_y) \neq 0$. ~~and~~

~~(λ_t, λ_0)~~ ~~(N_t, N_0)~~

If $p(r)$ contains discrete points of support, then LS contains flat facets.

~~If p(r) is~~

~~If~~ Where LS ~~has~~ a well defined normal, ~~(λ_t, λ_0) is~~ the friction force is on LS such that the normal to LS is $\parallel E[(N_t, N_0)]$.

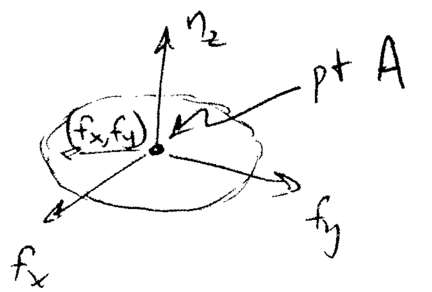
The projection of the LS onto the (λ_t, λ_0) plane is the circle of radius

LS contains the origin
LS is symmetric wrt the (f_x, f_y)-plane.

Example: 1 point of Support

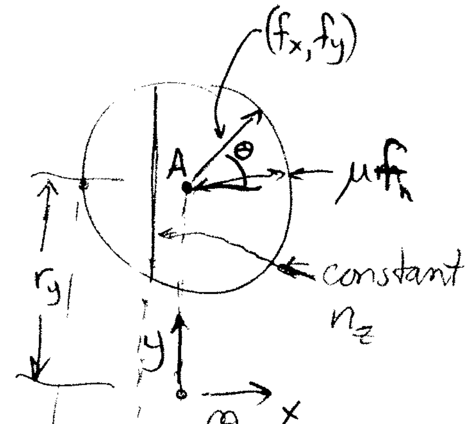
~~First~~. what is LS for a single point.

Origin at contact point. $n_z = 0$



$$\text{wrench} \triangleq w = \begin{bmatrix} f_x \\ f_y \\ n_z \end{bmatrix}$$

Origin at ~~cont~~ point not equal to contact point



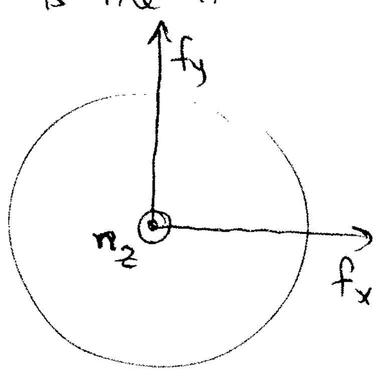
$$n_z = (r \times f)_z$$

$$r = \begin{bmatrix} 0 \\ r_y \\ 0 \end{bmatrix} \qquad f = \begin{bmatrix} \mu f_n \cos\theta \\ \mu f_n \sin\theta \\ 0 \end{bmatrix}$$

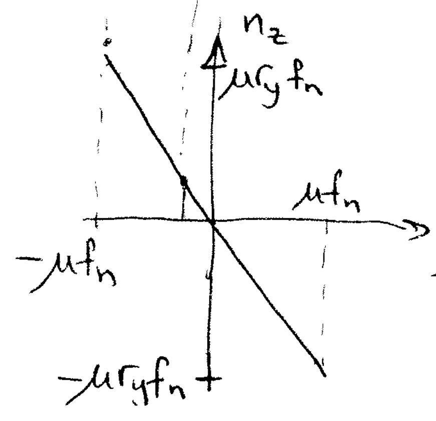
$$\begin{vmatrix} i & j & k \\ 0 & r_y & 0 \\ f_x & f_y & 0 \end{vmatrix}$$

$n_z = -r_y f_x$ $\Longleftarrow$ moment is linear in f_x

What is the limit surface?



Tilted ellipse



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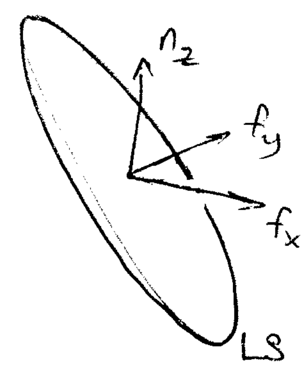
Since projection to (f_x, f_y) plane is a circle

And since n_z varies linearly with f_x,

The Limit Surface is an elliptical disk containing the origin. & the f_y axis



Recall that twists & wrenches map forces, torques, velocities, & ang vels. to the origin!

Where is a twist, frict force pair?
That maximizes dissipation?

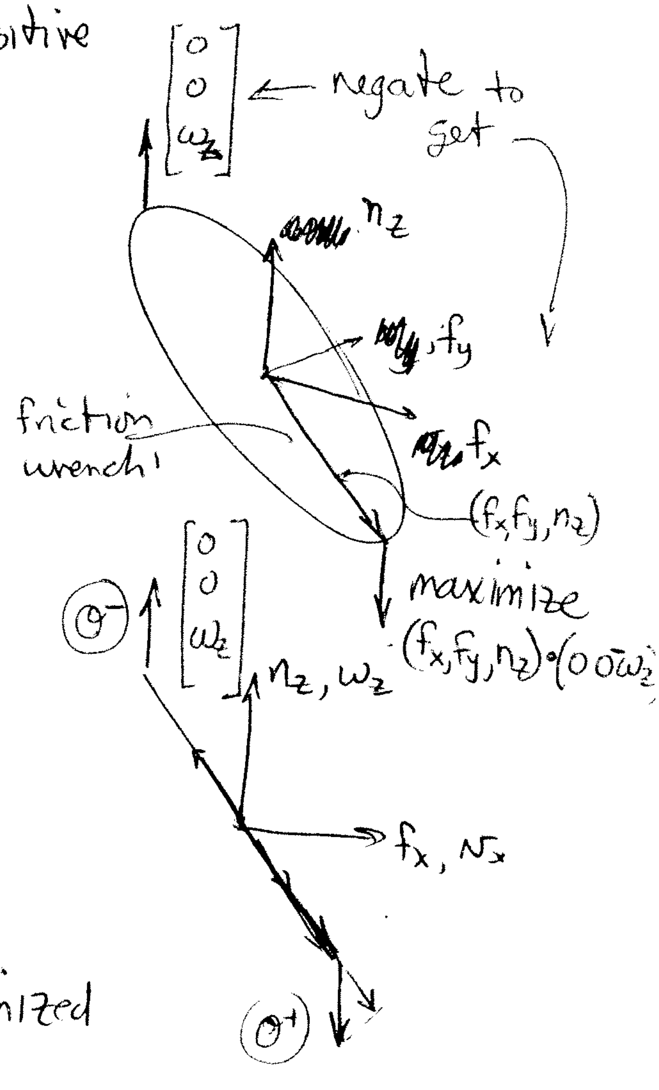
Let twist be rotation in positive sense about origin

$$W^T \nu = \begin{bmatrix} \nu_x \\ \nu_y \\ \omega_z \end{bmatrix} \quad \nu_x = \nu_y = 0 \quad \omega_z > 0$$

ν_x f_x O ω_z

maximize power dissipation.

$$\max -(f_x, f_y, n_z) \cdot (\nu_x, \nu_y, \omega_z) = -\nu^T W w$$



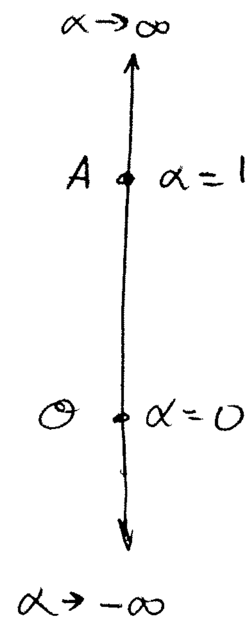
Project negative twist into the space of the Limit Surface. Then find location where

$\boxed{-\text{twist} \cdot \text{frict wrench}}$ is maximized

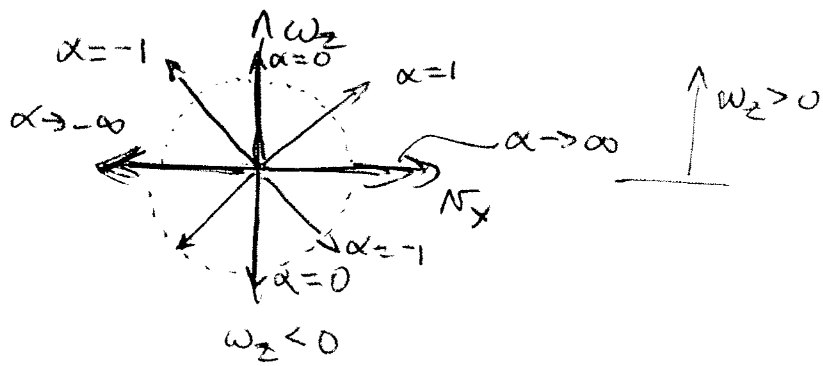
Consider all IC's along line   11/30/06

containing A & O.   11.1

$$IC = \alpha A + (1-\alpha)\cancelto{0}{O} \; ; \; -\infty < \alpha < \infty$$

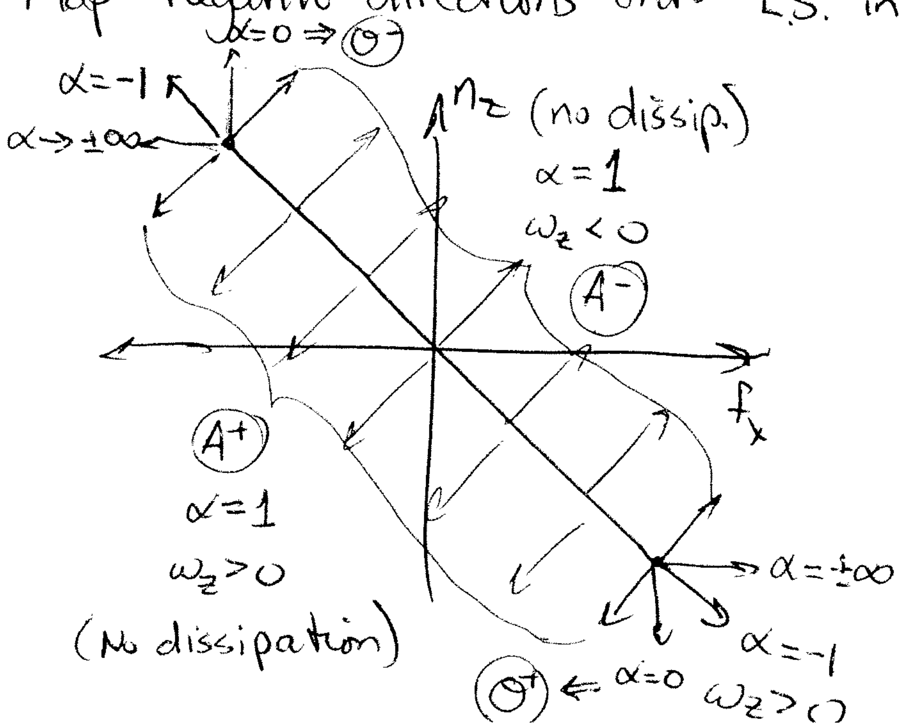


Plot all possible twists for these IC's



All directions in (v_x, ω_z) plane are possible.

Map negative directions onto L.S. in wrench space



$\underline{\underline{\alpha = 1}}$

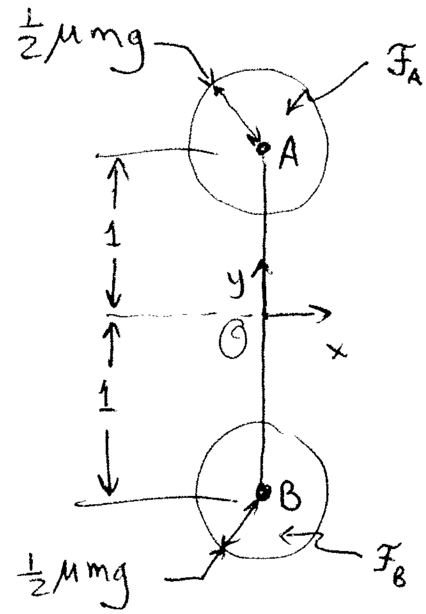
Many wrenches for 1 twist

$\underline{\alpha \neq 1}$

Many twists give 1 wrench

Example: Two Points of Support

Assume weight is evenly distributed



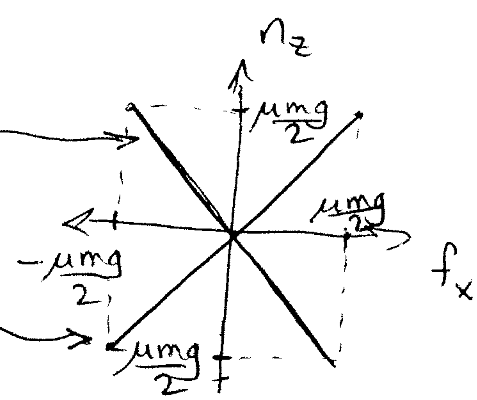
The L.S. contains all possible wrenches.

What does it look like?

Edge view

Ellipse from contact A

Ellipse from contact B



The L.S. contains all wrenches from ~~A~~ $\bigcirc$A ~~+~~ $\bigcirc$B,

i.e. the ~~a~~ Minkowski sum of wrenches from the two ellipses.

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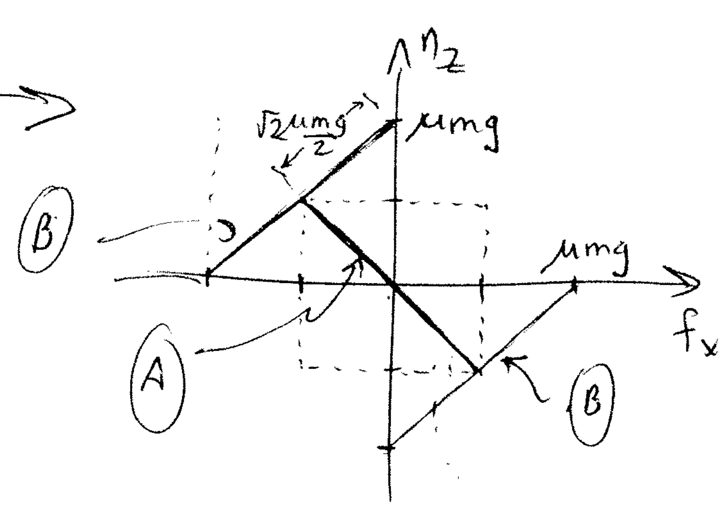
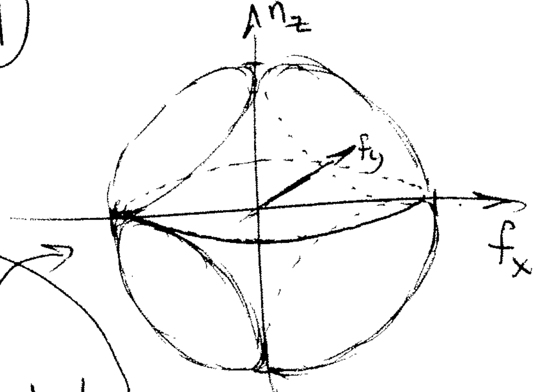
Fix A.

Slide B along



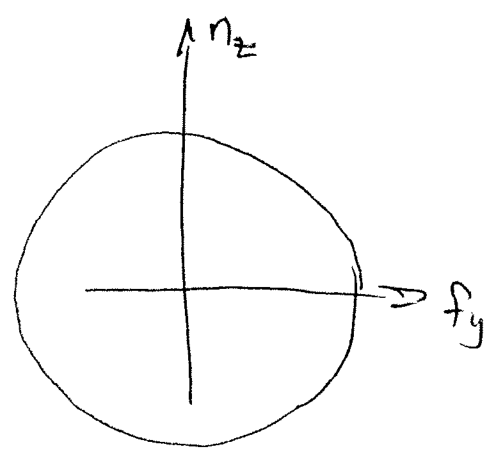
Fix B

Slide A

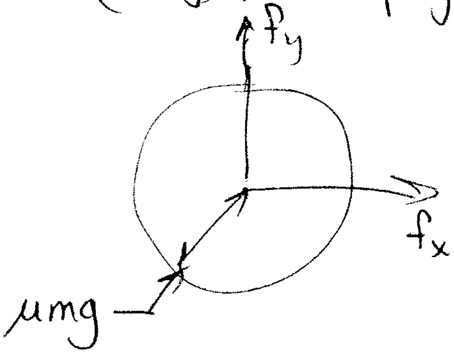


See figure 6.15 on pg 138 of text

(f_y, n_z) plane projection



(f_x, f_y) plane projection



Test claims that cross-section of L.S. in (f_y, n_z) plane is a circle. I don't see it!

Must be non-circular cross-sections as you rotate about n_z

Then LS is a sphere minus 4 flat ~~so~~ solid chords! ?

No. Then elliptical facets would be circular!

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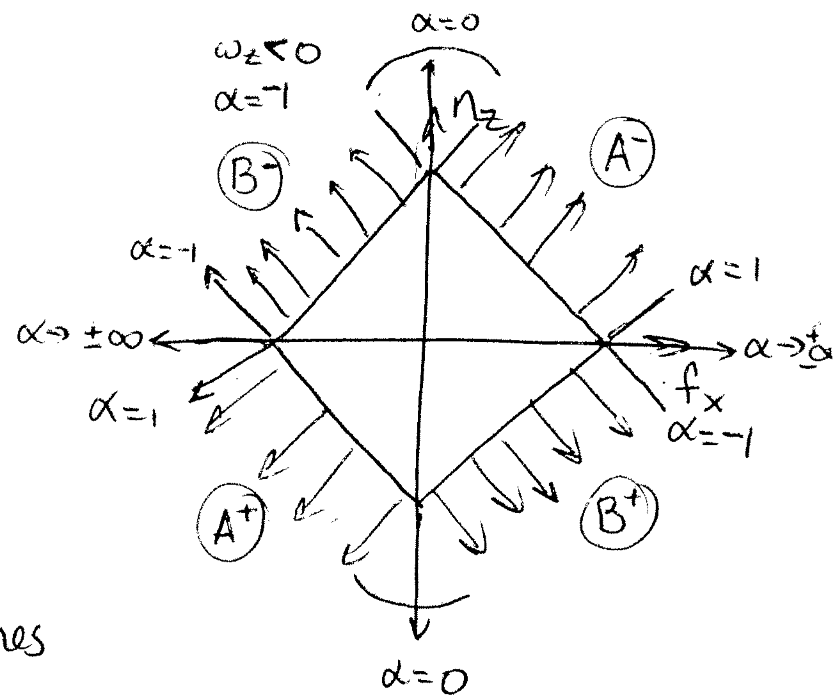
Look at Space of possible rotations about line thru O & A

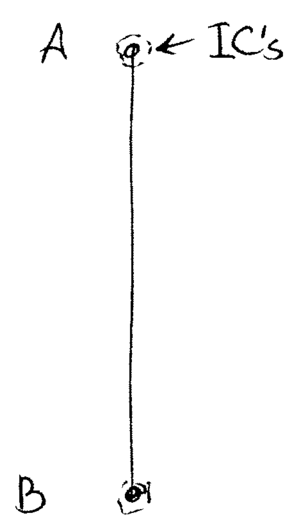
Same twist plot as before.

Map onto L.S.

As expected Rotation about A or B gives many possible wrenches



Edges of Elliptical Facets correspond to IC's infinitesimally close to B or A.



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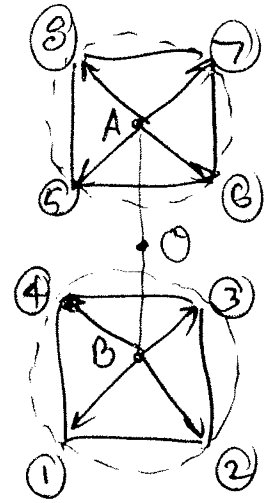
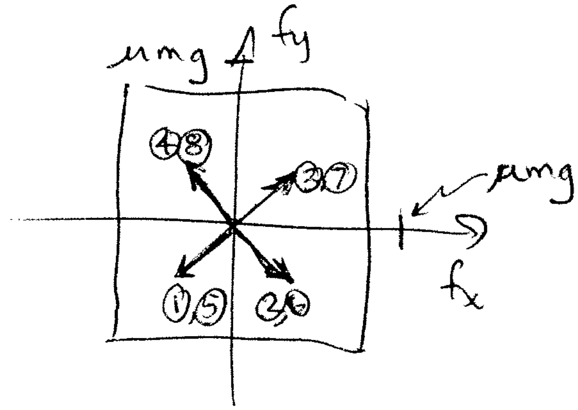
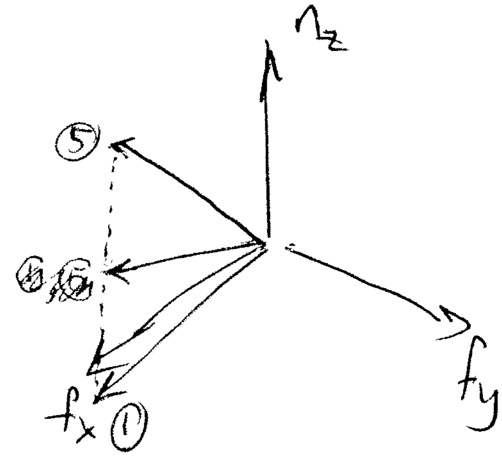
Relate to $W_f \lambda_{fn}$, $\lambda_f \geq 0$

4 frict directions per contact



L.S. becomes a poly hedron with 8 vertices.





Note that all moments are the same in magnitude.

$\therefore$ new LS is a cube inside LS from Coulomb model with 8 vertices on surface.

$W_f \lambda_f$, $\lambda_f \geq 0$ is a representation of Minkowski sum of individual frict cones