Cooking with Complementarity: A Recipe Guide for Complementarity Based Rigid-Multi-Body Dynamics Simulation

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1 Introduction

The focus of this report is to clarify the complementarity based formulation of multi-rigid-body simulation through use of illustrative examples. We present an existing semi-implicit time stepping method with complementarity formulation for the contact [2]; and use this method on several 2D examples.

The report is organized as follows. In the next section, preliminary material and notation is presented. In section 3, the semi-implicit time stepper used for all the examples is presented. Section 4 begins the examples with a simple planar 2 bar pendulum and continues by adding more and more complexity to the subsequent examples.

2 Preliminaries and Notation

As the focus of this report is on implementation, all vectors will be written with respect to some frame. The frames are denoted as pre-superscrips appearing before the vector. For example

$$ {}^{A}\vec{u} \tag{1} $$

Denotes a vector $\vec{u}$ in the A frame. If no superscript is present, it is assumed the vector is represented in the fixed inertial frame F. The identity matrix is denoted $\mathbf{U}$, as I is reserved for the inertia tensor.

For a body B, the center of mass of the body is denoted B^*.

The rotation matrix used to transform a vector between frames, say A and B is denoted $\left[{}^{A}R^{B}\right]$. For example, the 2D rotation matrix is:

$$ \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{2} $$

where θ is the desired rotation amount.

The configuration of the body is denoted $\vec{q}$. For a 3D body, $\vec{q} = [x\ y\ z\ \vec{\varepsilon}]$, where $\vec{\varepsilon}$ is the orientation of the body. For 2D, $\vec{q} = [x\ y\ \theta]$. The velocity of a body $\dot{\vec{q}} = \frac{d\vec{q}}{dt}$ consists of its linear $\vec{v}$ and angular $\vec{\omega}$ components. For 2D systems, ω is a scalar.

For compactness, we will use velocity *twists* in the examples. The velocity of a body given in twist coordinates $\vec{\nu}$ is:

$$\vec{\nu} = \begin{Bmatrix} \vec{v} \\ \vec{\omega} \end{Bmatrix} \tag{3}$$

To transform velocity twists between coordinate systems, say A and B, one must use the *adjoint transform* denoted ${}^{A}\mathrm{Ad}^{B}$. The adjoint matrix for mapping twists between the two systems can be written as:

$${}^{A}\mathrm{Ad}^{B} = \begin{bmatrix} \left[{}^{A}R^{B}\right] & \left(\left[{}^{A}R^{B}\right]\vec{r}^{AB}\right)^{\wedge} \\ 0_{1\times 2} & 1 \end{bmatrix} \tag{4}$$

where $\vec{r}^{AB}$ is the position vector from the origin of frame A to the origin of frame B, and the symbol $\wedge$ denotes the linear cross product operator. For 3D vectors, it is the 3×3 skew symmetric matrix:

$$(\vec{a})^{\wedge} = \begin{bmatrix} 0 & -a_z & -a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix} \tag{5}$$

For 2D vectors, it becomes the perpendicular product:

$$(\vec{a})^{\wedge} = \begin{bmatrix} -a_y \\ a_x \end{bmatrix} \tag{6}$$

The velocity of any point p attached to a rigid body A can be written as

$${}^{A}\vec{v}^{p} = {}^{A}\vec{v}^{A^*} + {}^{A}\omega \times {}^{A}\vec{r}^{A^*p} \tag{7}$$

For 2D systems, the analog of the cross product, denoted by the symbol $\otimes$, is

$$\vec{u} \otimes \vec{v} = u_x v_y - u_y v_x;$$

The velocity of a point p attached to a rigid body can be written more compactly using twist notation:

$$\vec{v}^{p} = \begin{bmatrix} \mathbf{U} & \left(\left[{}^{F}R^{A}\right]\vec{r}^{A^*p}\right)^{\wedge} \end{bmatrix} \vec{\nu}^{A} \tag{8}$$

2.1 Complementarity

The standard *nonlinear complementarity problem* (NCP) [1] can be written as:

Definition 1. Given a vector function $w : \mathbb{R}^m \to \mathbb{R}^m$, find a vector $z \in \mathbb{R}^m$ such that:

$$0 \leq w(z) \perp z \geq 0$$

where $\perp$ indicates $w(z)^{\mathrm{T}} z = 0$.

In the special case that the function w is linear, we can write the definition of a *standard linear complementarity problem* (LCP) as:

Definition 2. Given a matrix $B \in \mathbb{R}^{m \times m}$ and a vector $b \in \mathbb{R}^m$ find a vector $z \in \mathbb{R}^m$ such that:

$$
\begin{aligned}
w &= Bz + b \\
0 \leq w &\perp z \geq 0
\end{aligned}
$$

2.2 Dynamics of multibody systems

The starting point for the dynamic model is the Newton-Euler equations.

$$
\vec{\mathfrak{f}}(\vec{q}, \dot{\vec{q}}, t) = m(\vec{q}, t)\dot{\vec{v}} \tag{9}
$$

$$
\vec{\tau}(\vec{q}, \dot{\vec{q}}, t) = I(\vec{q}, t)\dot{\vec{\omega}} \tag{10}
$$

where m is the mass of the body, $\dot{\vec{v}}$ is the linear acceleration of the body's center of mass, $\vec{\mathfrak{f}}$ is the sum of all forces acting on the system, $\vec{\tau}$ is the sum of all moments acting on the system, $I \in \mathbb{R}^{3\times 3}$ is the inertia matrix, and $\dot{\vec{\omega}}$ is the angular acceleration of the body. Note, for 2D systems Euler's equation is scalar.

Assume there are n bodies in the scene, we can compactly write equations (9) and (10) in matrix form as:

$$
M(\vec{q}, t)\dot{\vec{\nu}} = \lambda(\vec{q}, \dot{\vec{q}}, t) \tag{11}
$$

where $M \in \mathbb{R}^{6n\times 6n}$ is the generalized mass matrix of the bodies

$$
M = \begin{bmatrix} m_1\mathbf{U}_{3\times 3} & 0 & & 0 & 0 \\ 0 & I_1 & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & m_n\mathbf{U}_{3\times 3} & 0 \\ 0 & 0 & & 0 & I_n \end{bmatrix}
$$

where m_i is the mass of body i, and $I_i \in \mathbb{R}^{3\times 3}$ is the inertia matrix of body i. The vector $\vec{\nu} \in \mathbb{R}^{n_\nu}$ represents the velocity twists of the bodies in the system

$$
\vec{\nu} = \begin{bmatrix} \vec{v}_1^{\mathrm{T}} & \vec{\omega}_1^{\mathrm{T}} & \ldots & \vec{v}_n^{\mathrm{T}} & \vec{\omega}_n^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}
$$

and $\vec{\lambda} \in \mathbb{R}^{n_\nu}$ is the sum of all wrenches acting on the system of bodies.

We must also parameterize the configuration:

$$
\dot{\vec{q}} = G(\vec{q})\vec{\nu} \tag{12}
$$

where $\vec{q} \in \mathbb{R}^{n_q}$ represents the configuration of the body, $n_q = n(3 + \zeta)$, ζ is the number of parameters used to represent the orientation, and $G \in \mathbb{R}^{n_q \times n_\nu}$ is the representation Jacobian relating the system velocity $\vec{\nu}$ to the time-derivative of the system configuration $\dot{\vec{q}}$. For 2D systems, $n_q = 3n$ and $G(\vec{q}) = \mathbf{U}$.

2.3 Bilateral Constraints

Joints constrain the relative positions of two bodies, which in turn impose velocity and acceleration constraints. Mathematically, for each joint j, there is an equality constraint function, $\Phi_j(\vec{q}, t) = 0$. We can combine all the constraint functions into a single constraint matrix, represented by

$$0 = \Phi(\vec{q}, t) = \begin{bmatrix} \Phi_1(\vec{q}, t) \\ \Phi_2(\vec{q}, t) \\ \vdots \\ \Phi_n(\vec{q}, t) \end{bmatrix} \tag{13}$$

As written, these constraints are expressed at the position level. As we will see later, timestepping formulations will specify the bilateral constraints at the velocity level.

By differentiating the position level constraints (13), we arrive at the velocity level constraints:

$$0 = \frac{\partial \Phi(\vec{q}, t)}{\partial \vec{q}} \dot{\vec{q}} + \frac{\partial \Phi(\vec{q}, t)}{\partial t} \dot{t} \tag{14}$$

Looking back at equation (12), we can substitute in $G(\vec{q})\vec{\nu}$ for $\dot{\vec{q}}$ arriving at:

$$0 = W_b^{\mathrm{T}}(\vec{q}, t)\vec{\nu} + \frac{\partial \Phi(\vec{q}, t)}{\partial t} \tag{15}$$

where $W_b(\vec{q}, t)^{\mathrm{T}} = \frac{\partial \Phi(\vec{q},t)}{\partial \vec{q}} G(\vec{q})$ is the constraint Jacobian[1] matrix of $\Phi(\vec{q}, t)$ with respect to $\vec{q}$ times the representation Jacobian, and allows us to represent the constraint at the velocity level for various parameterizations of $SO(3)$. The constraint Jacobian $\frac{\partial \Phi(\vec{q},t)}{\partial \vec{q}}$ takes on different forms dependant on the desired joint.

2.4 Contact Constraints

Unlike the joint constraints mentioned above, contact constraints are unilateral and are represented as inequalities. Physically, contact forces are compressive, meaning the contact force cannot pull the two bodies together.

To describe the modelling of contact constraints, we begin by introducing some notation. When two bodies j and k ($j \neq k$) are in contact, we label the contact point as i, and consider it uniquely associated with the pair (j, k). Let n_c denote the total number of contact points at the current time t. Each contact point i of bodies j and k defines the origin of a contact frame Λ_i. Let $\hat{n}_i$ denote the unit contact normal. The other 2 axes[2] of contact frame Λ_i, denoted $\hat{t}_i$ and $\hat{o}_i$, span the contact tangent plane.

Furthermore, when discussing contact constraints, it becomes convenient to break the constraint into two components: the *normal* and *tangential*. We then define a normal wrench matrix, $W_{\mathrm{in}} \in \mathbb{R}^6$ and a two tangential wrench matrices $W_{\mathrm{io}}, W_{\mathrm{it}} \in \mathbb{R}^6$ for each contact force

[1] In robotics, the Jacobian is sometimes referred to as a "Wrench" matrix, since the rows of the matrix are unit wrenches.

[2] In 2D systems there is only one other axis. We would remove either the o or the t axis.

magnitude (wrench intensity) λ_{in}, λ_{it}, and λ_{io} respectively. Lastly, we can combine all the contact wrenches into a single wrench matrix, and all the contact force magnitudes into a single vector, arriving at expressions for the contact forces:

$W_{\mathrm{n}}\vec{\lambda}_{\mathrm{n}} \longrightarrow$ force along the $\hat{n}$ direction

$W_{\mathrm{t}}\vec{\lambda}_{\mathrm{t}} \longrightarrow$ force along the $\hat{t}$ direction

$W_{\mathrm{o}}\vec{\lambda}_{\mathrm{o}} \longrightarrow$ force along the $\hat{o}$ direction

2.4.1 Normal Contact (Nonpenetration) Constraints

The normal contact constraint prevents interpenetration of the bodies, but must also allow for separation. For each contact, we can define a signed distance function $\psi_{i\mathrm{n}}(\vec{q}, t)$ along the contact normal direction, $\hat{n}_i$, which equals 0 when bodies j and k are in contact, and is greater than 0 when the two bodies separate. Also, since no overlapping can occur[3] the function must be non-negative. Analogous to before, we stack all the active contact gap functions into a single vector, $\vec{\Psi}_{\mathrm{n}}$ obtaining the nonpenetration constraint:

$$\vec{\Psi}_{\mathrm{n}}(\vec{q}, t) \geq 0 \tag{16}$$

Unlike the joint constraints, the normal contact forces at a contact cannot pull the bodies together, $\lambda_{in} \geq 0$. Again, combining all the normal contact forces into a single vector, we obtain a constraint on the contact forces[4]:

$$\vec{\lambda}_{\mathrm{n}} \geq 0 \tag{17}$$

Lastly, at each contact there is a naturally occurring disjunctive relationship between the normal gap, ψ_{in}, and normal contact force, λ_{in}. Namely, if the contact is producing a normal contact force ($\lambda_{in} \geq 0$) then the normal distance between the two bodies must be zero ($\psi_{in} = 0$). Conversely, if there is a gap between the two bodies ($\psi_{in} \geq 0$), then the normal contact force must be zero ($\lambda_{in} = 0$). This final constraint can be written as:

$$\vec{\Psi}_{\mathrm{n}}(\vec{q}, t)^{\mathrm{T}}\vec{\lambda}_{\mathrm{n}} = 0 \tag{18}$$

Equations (16), (17), and (18) taken together represent the normal contact constraint.

[3] The distance function must be able to return a negative gap if two bodies are overlapping for numerical stability issues. Otherwise, if a small penetration occurs, it will snowball and break the simulation.

[4] The contact forces prevent the gap function from becoming negative and thus act as the lagrange multipliers

2.4.2 Tangential Contact Constraints (Friction)

Compactly (for more details see [3][4]), Coulomb's law for all contacts is:

$$0 = (U\vec{\lambda}_\mathrm{n}) \circ (W_\mathrm{t}^\mathrm{T}\vec{\nu} + \frac{\partial \vec{\Psi}_\mathrm{t}}{\partial t}) + \vec{\lambda}_\mathrm{t} \circ \vec{\sigma} \tag{19}$$

$$0 = (U\vec{\lambda}_\mathrm{n}) \circ (W_\mathrm{o}^\mathrm{T}\vec{\nu} + \frac{\partial \vec{\Psi}_\mathrm{o}}{\partial t}) + \vec{\lambda}_\mathrm{o} \circ \vec{\sigma} \tag{20}$$

$$0 \leq \vec{\sigma} \perp (U\vec{\lambda}_\mathrm{n}) \circ (U\vec{\lambda}_\mathrm{n}) - \vec{\lambda}_\mathrm{t} \circ \vec{\lambda}_\mathrm{t} - \vec{\lambda}_\mathrm{o} \circ \vec{\lambda}_\mathrm{o} \geq 0 \tag{21}$$

where U is the diagonal matrix with i^{th} diagonal element equal to μ_i and $\circ$ connotes the Hadamard product.

Some of the above equations are nonlinear in the unknowns (forces, configuration, and velocity), so their direct use in a time-stepping scheme would require the solution of mixed nonlinear complementarity problems (NCPs). In order to obtain a scheme based on mixed LCPs, a piecewise linear approximation of the quadratic friction cone with nonnegative force variables is needed.

Combining the tangential force vectors and relative slip velocity vectors at a contact into single vectors $\vec{\lambda_{i\mathrm{f}}} = [\lambda_{i\mathrm{t}} \;\; \lambda_{i\mathrm{o}}]^\mathrm{T}$, $\frac{\partial \vec{\psi_{i\mathrm{f}}}}{\partial t} = [\frac{\partial \psi_{i\mathrm{t}}}{\partial t} \;\; \frac{\partial \psi_{i\mathrm{o}}}{\partial t}]^\mathrm{T}$, maximum dissipation for all contacts can be written compactly (again see [3][4] for more details) as:

$$0 \leq \vec{\lambda}_\mathrm{f} \quad \perp \quad W_\mathrm{f}^\mathrm{T}\vec{\nu} + E\vec{\sigma} + \frac{\partial \vec{\Psi}_\mathrm{f}}{\partial t} \geq 0 \tag{22}$$

$$0 \leq \vec{\sigma} \quad \perp \quad U\vec{\lambda}_\mathrm{n} - E^\mathrm{T}\vec{\lambda}_\mathrm{f} \geq 0 \tag{23}$$

where now σ_i *approximates* the sliding speed at contact i, $e \in \mathbb{R}^{n_d}$ is a vector of ones, n_d is the number of sliding directions, and E is the block diagonal matrix with i^{th} block on the main diagonal given by e.

2.5 Instantaneous Form of Constrained Dynamics

Combining the above equations together, we arrive at the final set of equations for constrained multi-body dynamics.

$$\begin{aligned}
\dot{\vec{q}} &= G(\vec{q})\vec{\nu} \\
M(\vec{q},t)\dot{\vec{\nu}} &= W_b(\vec{q},t)\lambda_b + W_\mathrm{n}(\vec{q},t)\vec{\lambda}_\mathrm{n} + W_\mathrm{f}(\vec{q},t)\vec{\lambda}_\mathrm{f} + \lambda_{app}(\vec{q},\dot{\vec{q}},t) \\
\Phi(\vec{q},t) &= 0 \\
\vec{\Psi}_\mathrm{n}(\vec{q},t) &\geq 0 \\
\vec{\lambda}_\mathrm{n} &\geq 0 \\
\vec{\Psi}_\mathrm{n}(\vec{q},t)^\mathrm{T}\vec{\lambda}_\mathrm{n} &= 0 \\
0 \leq \vec{\lambda}_\mathrm{f} \perp W_\mathrm{f}^\mathrm{T}\vec{\nu} + E\vec{\sigma} + \frac{\partial \vec{\Psi}_\mathrm{f}}{\partial t} &\geq 0 \\
0 \leq \vec{\sigma} \perp U\vec{\lambda}_\mathrm{n} - E^\mathrm{T}\vec{\lambda}_\mathrm{f} &\geq 0
\end{aligned} \tag{24}$$

where λ_{app} represents all applied or non constraint forces.

3 Semi-Implicit Time-Stepping

Using an Euler step, we approximate derivatives as:

$$\dot{\vec{q}} = \frac{\vec{q}^{l+1} - \vec{q}^{l}}{h}$$

where $q^l = \vec{q}(t_l) = \vec{q}(lh)$ if time step h is constant.

The Newton-Euler and velocity kinematic equations (11), (12), and (15) can be written in discrete time form as follows:

$$M\vec{\nu}^{l+1} = M\vec{\nu}^{l} + h(\lambda_{app}(t_{l+1}) + W_{\mathrm{b}}\vec{\lambda}_{\mathrm{b}}^{l+1} + W_{\mathrm{n}}\vec{\lambda}_{\mathrm{n}}^{l+1} + W_{\mathrm{f}}\vec{\lambda}_{\mathrm{f}}^{l+1}) \tag{25}$$

$$0 = W_{\mathrm{b}}^{\mathrm{T}}\vec{\nu}^{l+1} \tag{26}$$

$$\vec{q}^{l+1} = \vec{q}^{l} + h\vec{\nu}^{l+1} \tag{27}$$

In equation (27), we cannot use $\vec{\nu}^l$ since that would determine q^{l+1} without dynamics

The discrete form of the nonpenetration and friction conditions (16), (17), (18), (22), and (23) can be written as:

$$0 \le \vec{\lambda}_{\mathrm{n}}^{l+1} \quad \perp \quad \vec{\Psi}_{\mathrm{n}}^{l} + \frac{\partial \vec{\Psi}_{\mathrm{n}}^{l}}{\partial \vec{q}}\Delta\vec{q} + \frac{\partial \vec{\Psi}_{\mathrm{n}}^{l}}{\partial t}\Delta t \ge 0 \tag{28}$$

$$0 \le \vec{\lambda}_{\mathrm{f}}^{l+1} \quad \perp \quad E\vec{\sigma}^{l+1} + \frac{\partial \vec{\Psi}_{\mathrm{f}}^{l}}{\partial \vec{q}}\Delta\vec{q} + \frac{\partial \vec{\Psi}_{\mathrm{f}}^{l}}{\partial t}\Delta t \ge 0 \tag{29}$$

$$0 \le \vec{\sigma}^{l+1} \quad \perp \quad U\vec{\lambda}_{\mathrm{n}}^{l+1} - E^T\vec{\lambda}_{\mathrm{f}}^{l+1} \ge 0 \tag{30}$$

where $\Delta\vec{q} = \vec{q}^{l+1} - \vec{q}^{l}$, $\Delta t = h$, $\frac{\partial \vec{\Psi}_{\mathrm{f}}^{l}}{\partial \vec{q}} = W_{\mathrm{f}}^T$, and $\frac{\partial \vec{\Psi}_{\mathrm{n}}^{l}}{\partial \vec{q}} = W_{\mathrm{n}}^T$. Note that $\frac{\partial \vec{\Psi}_{\mathrm{f}}^{l}}{\partial t}h$ represents the lateral position change of the frictional surface in one time step, i.e., imagine walking on a floor that remains planar, but moves, like people movers in large airports.

Dynamic time-stepping equations are written in terms of the generalized velocity vector $\vec{\nu}$. Let the contact impulse be denoted by $p_{(.)} = h\lambda_{(.)}$. Using equation (27), equations (28, 29, 30) can be rewritten as:

$$0 \le \vec{p}_{n}^{l+1} \quad \perp \quad \frac{\vec{\Psi}_{\mathrm{n}}^{l}}{h} + W_{\mathrm{n}}^T\vec{\nu}^{l+1} + \frac{\partial \vec{\Psi}_{\mathrm{n}}^{l}}{\partial t} \ge 0 \tag{31}$$

$$0 \le \vec{p}_{f}^{l+1} \quad \perp \quad \frac{E\vec{\sigma}^{l+1}}{h} + W_{\mathrm{f}}^T\vec{\nu}^{l+1} + \frac{\partial \vec{\Psi}_{\mathrm{f}}^{l}}{\partial t} \ge 0 \tag{32}$$

$$0 \le \vec{\sigma}^{l+1} \quad \perp \quad U\lambda_{n}^{l+1} - E^T\lambda_{f}^{l+1} \ge 0 \tag{33}$$

Equations (25, 26, 31, 32, 33) constitute an MCP. Equation (27) is used to update the configuration after solution of the MCP:

$$\begin{bmatrix} 0 \\ 0 \\ \vec{\rho}_n^{\,l+1} \\ \vec{\rho}_f^{\,l+1} \\ \vec{s}^{\,l+1} \end{bmatrix} = \begin{bmatrix} -M & W_{\mathrm{b}} & W_{\mathrm{n}} & W_{\mathrm{f}} & 0 \\ W_{\mathrm{b}}^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ W_{\mathrm{n}}^{\mathrm{T}} & 0 & 0 & 0 & 0 \\ W_{\mathrm{f}}^{\mathrm{T}} & 0 & 0 & 0 & E \\ 0 & 0 & U & -E^T & 0 \end{bmatrix} \begin{bmatrix} \vec{\nu}^{\,l+1} \\ \vec{p_b}^{\,l+1} \\ \vec{p_n}^{\,l+1} \\ \vec{p_f}^{\,l+1} \\ \vec{\sigma}^{\,l+1} \end{bmatrix} + \begin{bmatrix} M\vec{\nu}^{\,l} + \vec{p}_{ext} \\ \frac{\vec{\Phi}^l}{h} \\ \frac{\vec{\Psi}_{\mathrm{n}}^l}{h} + \frac{\partial \vec{\Psi}_{\mathrm{n}}^l}{\partial t} \\ \frac{\partial \vec{\Psi}_{\mathrm{f}}^l}{\partial t} \\ 0 \end{bmatrix} \tag{34}$$

$$0 \le \begin{bmatrix} \vec{\rho}_n^{\,l+1} \\ \vec{\rho}_f^{\,l+1} \\ \vec{s}^{\,l+1} \end{bmatrix} \perp \begin{bmatrix} \vec{p}_n^{\,l+1} \\ \vec{p}_f^{\,l+1} \\ \vec{\sigma}^{\,l+1} \end{bmatrix} \ge 0. \tag{35}$$

4 Examples

4.1 Planar 2 Bar Pendulum

This system consists of a planar simple double pendulum with all joint axes parallel to the Z-axis. It is constructed of two slender rods A and B with masses m_A and m_B respectively. Link A has length L_A and is connected to ground at point O. Link B has length L_B and is connected to link A at point P by point P'. Frames are attached to each body and are fixed in that body. The position and orientation of the frames are used as the generalized coordinates. Figure 1 illustrates a free body diagram of the system.

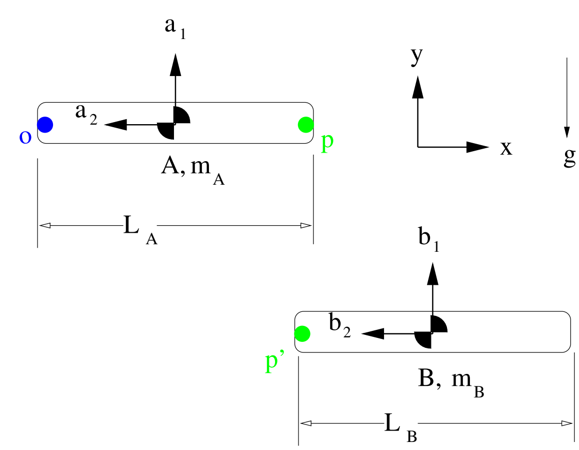

Figure 1: Free Body Diagram of Simple planar double pendulum.

The configuration q of the system is $\vec{q} = [\vec{q}^A \; \vec{q}^B]$ and similarly the velocity twist of the system is $\vec{\nu} = [\vec{\nu}^A \; \vec{\nu}^B]$.

For this system, the generalized mass matrix is

$$M = \begin{bmatrix} m_A & 0 & 0 & 0 & 0 & 0 \\ 0 & m_A & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{12}m_A L_A^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & m_B & 0 & 0 \\ 0 & 0 & 0 & 0 & m_B & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{12}m_B L_B^2 \end{bmatrix} \tag{36}$$

where $1/12 m_A L_A^2$ is the moment of inertia for a slender rod.

There are 2 bilateral constraints on this system, the postion of point O is fixed in the world frame and the position of points P and P' must lie in the same location. Let point O be $(0, 0)$ in the fixed world frame. Next, attach a "joint" frame α with its origin at point O. For computational simplicity align the joint frame axes with the world frame axes. We can mathematically write the first bilateral constraint ϕ_1 as

$$\phi_1 \triangleq O = \left[{}^{\alpha}R^{F}\right] A^* + \left[{}^{\alpha}R^{F}\right]\left[{}^{F}R^{A}\right] {}^{A}\vec{r}^{A^*O} = 0 \tag{37}$$

where A^* is the location of body A's center of mass and $\vec{r}^{A^*O}$ is a vector from the center of mass to the point O. However, since we aligned the joint frame with the fixed world frame, $\left[{}^{\alpha}R^{F}\right]$ is the identity matrix and the constraint simplifies to:

$$\phi_1 \triangleq O = A^* + \left[{}^{F}R^{A}\right] {}^{A}\vec{r}^{A^*O} = 0 \tag{38}$$

The second bilateral constraint is that points P and P' lie at the same location. Again, attach a "joint" frame β at points P and P' aligned with the world frame. The constraint that the points cannot separate can be written as:

$$\phi_2 \triangleq P - P' = A^* + \left[{}^{F}R^{A}\right] {}^{A}\vec{r}^{A^*P} - \left[B^* + \left[{}^{F}R^{B}\right] {}^{B}\vec{r}^{B^*P'}\right] = 0 \tag{39}$$

However, the unknowns in the time-stepping formulation are not position they are velocity. Therefore, the constraints must be enforced at the velocity level. For 2D revolute joints, the linear velocity is constrained to be zero by two scalar equations, while the angular velocity is unconstrained. We know from equation (8) the linear velocity of a point attached to a rigid body. Setting this equation equal to zero will satisfy the 2D revolute joint constraint, however, we will set up the joint constraint more generally in a framework that allows for other joint types, and for this revolute joint we will recover equation (8).

The constraint Jacobian is most easily specified in the joint's frame, however the velocity twist of the body is of the body's center of gravity in the fixed world frame. Therefore, to constrain the joint, we must first align the body frame with the world frame[5], followed by a

[5]It is important to realize that ${}^{A}\vec{\nu}^{A}$ is not 0. Another way of thinking of this velocity is as ${}^{A'}\vec{\nu}^{A}$, where A' is located at the origin of the fixed frame, but instantaneously aligned with the moving body frame A. This is why the transformation of ${}^{F}\vec{\nu}^{A}$ to ${}^{A}\vec{\nu}^{A}$ has 0 for the position vector in the top right block of the transform matrix defined in equation (4).

coordinate transform of the resulting velocity twist to the joint frame. With the velocity twist in the correct frame, we can apply the constraint Jacobian. Mathematically, the constraint can be written as:

$$\begin{aligned}0 &= \left[{}^{\alpha}J^{A}\right]\left[{}^{\alpha}\mathrm{Ad}^{A}\right]{}^{A}\vec{\nu}^{A} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\left[{}^{\alpha}R^{A}\right] & \left(\left[{}^{\alpha}R^{A}\right]{}^{A}\vec{r}^{A^{*}O}\right)^{\wedge}\\ 0_{1\times 2} & 1\end{bmatrix}\begin{bmatrix}\left[{}^{A}R^{F}\right] & 0\\ 0_{1\times 2} & 1\end{bmatrix}\vec{\nu}^{A}\\ &= \begin{bmatrix}\mathbf{U} & \left(\left[{}^{\alpha}R^{A}\right]{}^{A}\vec{r}^{A^{*}O}\right)^{\wedge}\end{bmatrix}\vec{\nu}^{A}\end{aligned} \tag{40}$$

where the matrix $\left[{}^{\alpha}J^{A}\right]$ is the constraint Jacobian for a 2D revolute joint specified at the joint frame. As stated earlier, the constraint equation reduced to the equation for the linear velocity of a point attached to the rigid body.

The second bilateral constraint (the revolute joint between links A and B) can also be written in matrix form as:

$$\left[{}^{\beta}J^{A}\right]\left[{}^{\beta}\mathrm{Ad}^{A}\right]{}^{A}\vec{\nu}^{A} + \left[{}^{\beta}J^{B}\right]\left[{}^{\beta}\mathrm{Ad}^{B}\right]{}^{B}\vec{\nu}^{B} = 0 \tag{41}$$

Expanding equation (41):

$$\begin{aligned}0 &= \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}\left[{}^{\beta}R^{A}\right] & \left(\left[{}^{\beta}R^{A}\right]{}^{A}\vec{r}^{A^{*}P}\right)^{\wedge}\\ 0_{1\times 2} & 1\end{bmatrix}\begin{bmatrix}\left[{}^{A}R^{F}\right] & 0\\ 0_{1\times 2} & 1\end{bmatrix}\vec{\nu}^{A}\\ &+ \begin{bmatrix}-1 & 0 & 0\\ 0 & -1 & 0\end{bmatrix}\begin{bmatrix}\left[{}^{\beta}R^{B}\right] & \left(\left[{}^{\beta}R^{B}\right]\vec{r}^{B^{*}P'}\right)^{\wedge}\\ 0_{1\times 2} & 1\end{bmatrix}\begin{bmatrix}\left[{}^{B}R^{F}\right] & 0\\ 0_{1\times 2} & 1\end{bmatrix}\vec{\nu}^{B}\\ &= \begin{bmatrix}\mathbf{U} & \left(\left[{}^{\beta}R^{A}\right]{}^{A}\vec{r}^{A^{*}P}\right)^{\wedge}\end{bmatrix}\vec{\nu}^{A} + \begin{bmatrix}-\mathbf{U} & -\left(\left[{}^{\beta}R^{B}\right]{}^{B}\vec{r}^{B^{*}P'}\right)^{\wedge}\end{bmatrix}\vec{\nu}^{B}\end{aligned} \tag{42}$$

Using our notation,

$$W_{b1}^{\mathrm{T}} = \begin{bmatrix}\mathbf{U} & \left(\left[{}^{\alpha}R^{A}\right]{}^{A}\vec{r}^{A^{*}O}\right)^{\wedge}\end{bmatrix} \qquad W_{b2}^{\mathrm{T}} = \begin{bmatrix}\mathbf{U} & \left(\left[{}^{\beta}R^{A}\right]{}^{A}\vec{r}^{A^{*}P}\right)^{\wedge}\end{bmatrix} \tag{43}$$

$$W_{b3}^{\mathrm{T}} = \begin{bmatrix}-\mathbf{U} & -\left(\left[{}^{\beta}R^{B}\right]{}^{B}\vec{r}^{B^{*}P'}\right)^{\wedge}\end{bmatrix} \tag{44}$$

Combining the 3 constraints into a single matrix $W_{\mathrm{b}}^{\mathrm{T}}$ produces the constraint wrench for the system:

$$W_{\mathrm{b}}^{\mathrm{T}} = \begin{bmatrix}W_{b1}^{\mathrm{T}} & 0\\ W_{b2}^{\mathrm{T}} & W_{b3}^{\mathrm{T}}\end{bmatrix} \tag{45}$$

We can now formulate the mixed complementarity problem:

$$\begin{bmatrix}0_{6\times 1}\\ 0_{4\times 1}\end{bmatrix} = \begin{bmatrix}-M & W_{\mathrm{b}}\\ W_{\mathrm{b}}^{\mathrm{T}} & 0\end{bmatrix}\begin{bmatrix}\vec{\nu}^{l+1}\\ \vec{p_b}^{l+1}\end{bmatrix} + \begin{bmatrix}M\vec{\nu}^{l} + h\vec{\lambda}_{app}\\ \frac{\vec{\phi}^{l}}{h}\end{bmatrix} \tag{46}$$

where $\frac{\vec{\phi}^{l}}{h}$ is the constraint stabilization term for the bilateral constraints and $\vec{\phi}^{l}$ is obtained from equations (38) and (39):

$$\begin{bmatrix}A^{*} + \left[{}^{F}R^{A}\right]{}^{A}\vec{r}^{A^{*}O}\\ A^{*} + \left[{}^{F}R^{A}\right]{}^{A}\vec{r}^{A^{*}P} - \left[B^{*} + \left[{}^{F}R^{B}\right]{}^{B}\vec{r}^{B^{*}P'}\right]\end{bmatrix} \tag{47}$$

4.2 Planar 2 Bar Pendulum Attached to Block

This system consists of a planar simple double pendulum attached to a block with all joint axes parallel to the Z-axis. The pendulum is constructed of two slender rods A and B with masses m_A and m_B respectively. Link A has length L_A and is connected to the block at point O by point O'. Link B has length L_B and is connected to link A at point P by point P'. Initially, the block is at rest on a horizontal surface. Frames are attached to each body and are fixed in that body. The position and orientation of the frames are used as the generalized coordinates. Figure 2 illustrates the system.

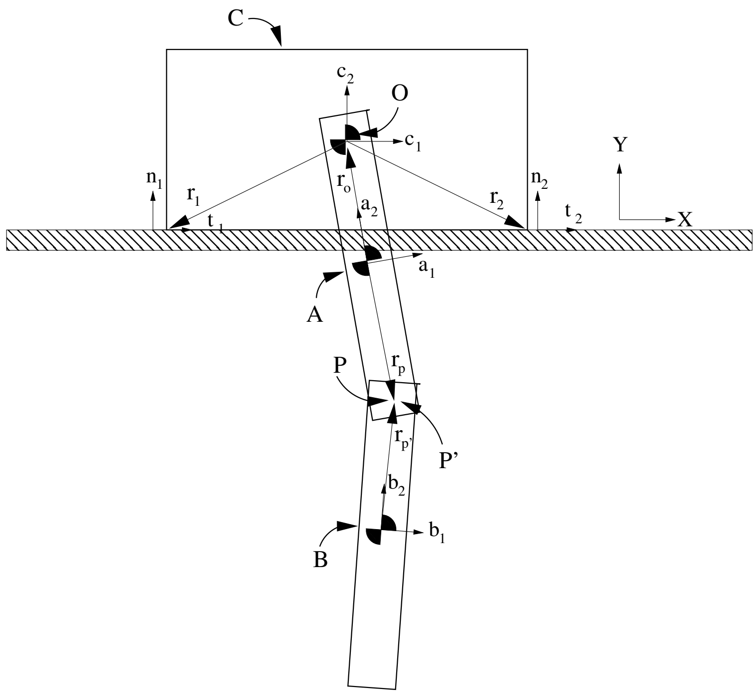

Figure 2: A planar simple double pendulum with all joint axes parallel to the Z-axis attached to a block on a surface.

The configuration q of the system is $\vec{q} = [\vec{q}^C\ \vec{q}^A\ \vec{q}^B]$ and similarly the velocity twist of the system is $\vec{\nu} = [\vec{\nu}^C\ \vec{\nu}^A\ \vec{\nu}^B]$.

The bilateral constraint between links A and B of the pendulum is identical to the previous example. The constraint between the pendulum and the block at point O must now be handled, it can be written in matrix form as:

$$\left[\mathbf{U} \quad \left(\left[{}^{\alpha}R^C\right] {}^C\vec{r}^{C^*O}\right)^{\wedge}\right] \vec{\nu}^C + \left[-\mathbf{U} \quad -\left(\left[{}^{\alpha}R^A\right] {}^A\vec{r}^{A^*O'}\right)^{\wedge}\right] \vec{\nu}^A \tag{48}$$

Partitioning into the bilateral constraint wrenches:

$$W_{b1}^{\mathrm{T}} = \begin{bmatrix}\mathbf{U} & \left(\left[{}^{\alpha}R^{C}\right]{}^{C}\vec{r}^{C^*O}\right)^{\wedge}\end{bmatrix} \qquad W_{b2}^{\mathrm{T}} = \begin{bmatrix}-\mathbf{U} & -\left(\left[{}^{\alpha}R^{A}\right]{}^{A}\vec{r}^{A^*O'}\right)^{\wedge}\end{bmatrix} \tag{49}$$

$$W_{b3}^{\mathrm{T}} = \begin{bmatrix}\mathbf{U} & \left(\left[{}^{\beta}R^{A}\right]{}^{A}\vec{r}^{A^*P}\right)^{\wedge}\end{bmatrix} \qquad W_{b4}^{\mathrm{T}} = \begin{bmatrix}-\mathbf{U} & -\left(\left[{}^{\beta}R^{B}\right]{}^{B}\vec{r}^{B^*P'}\right)^{\wedge}\end{bmatrix} \tag{50}$$

The nonpenetration constraints between the block and floor are written as:

$$\psi_{1\mathrm{n}} = \left[{}^{n}R^{c}\right]{}^{c}\vec{r}_1 \geq 0 \tag{51}$$

$$\psi_{2\mathrm{n}} = \left[{}^{n}R^{c}\right]{}^{c}\vec{r}_2 \geq 0 \tag{52}$$

where $\vec{r}_1$ and $\vec{r}_2$ are shown in figure 2. Vectors $\vec{r}_1$ and $\vec{r}_2$ are constant in the body fixed frame C, and must be transformed into the inertial frame F to determine the gap above the floor. Each gap function has a corresponding multiplier λ_{in} which is the contact force between contact i and the floor.

4.2.1 System Dynamics

Assuming the two contacts are included in the active set every time step, the MCP is size 15 and the various quantities appearing in it are:

$$M = \begin{bmatrix} M_c & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & M_a & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & M_b \end{bmatrix} \tag{53}$$

where $M_c = \begin{bmatrix} m_c & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & I_c \end{bmatrix}$, $M_a = \begin{bmatrix} m_a & 0 & 0 \\ 0 & m_a & 0 \\ 0 & 0 & I_a \end{bmatrix}$, and $M_b = \begin{bmatrix} m_b & 0 & 0 \\ 0 & m_b & 0 \\ 0 & 0 & I_b \end{bmatrix}$. The scalar $I_c = \frac{1}{12}m_c(l_c^2 + w_c^2)$ is the moment of inertia for a 2D block and the scalars $I_a = \frac{1}{12}m_a l_a^2$ and $I_b = \frac{1}{12}m_b l_b^2$ are the respective moments of inertia for rod's A and B.

The system constraint wrenches are:

$$W_{\mathrm{b}}^{\mathrm{T}} = \begin{bmatrix} W_{b1}^{\mathrm{T}} & \vdots & W_{b2}^{\mathrm{T}} & \vdots & 0_{2\times3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_{2\times3} & \vdots & W_{b3}^{\mathrm{T}} & \vdots & W_{b4}^{\mathrm{T}} \end{bmatrix} \qquad W_{\mathrm{n}} = \begin{bmatrix} \hat{n}_1 & \vdots & \hat{n}_2 \\ \left[{}^{F}R^{C}\right]{}^{C}\vec{r}_1 \otimes \hat{n}_1 & \vdots & \left[{}^{F}R^{C}\right]{}^{c}\vec{r}_2 \otimes \hat{n}_2 \\ \cdots & \cdots & \cdots \\ 0_{3\times1} & \vdots & 0_{3\times1} \\ \cdots & \cdots & \cdots \\ 0_{3\times1} & \vdots & 0_{3\times1} \end{bmatrix} \tag{54}$$

For the terms appearing in the MCP's b vector:

$$\lambda_{app} = \begin{bmatrix} 0 & -m_c g & 0 & 0 & -m_a g & 0 & 0 & -m_b g & 0 \end{bmatrix}^{\mathrm{T}} \tag{55}$$

where g is the gravitational acceleration constant.

The bilateral constraint stabilization terms are again obtained from the position level constraint functions:

$$\vec{\phi}^l = \begin{bmatrix} C^* + \left[{}^F R^C\right] {}^C\vec{r}^{C^*O} - \left(A^* + \left[{}^F R^A\right] {}^A\vec{r}^{A^*O'}\right) \\ A^* + \left[{}^F R^A\right] {}^A\vec{r}^{A^*P} - \left(B^* + \left[{}^F R^B\right] {}^B\vec{r}^{B^*P'}\right) \end{bmatrix} \tag{56}$$

We now also have unilateral constraint terms, and these are obtained from equations (51) and (52):

$$\vec{\Psi}^l_{\mathrm{n}} = \begin{bmatrix} \left[{}^n R^c\right] {}^c\vec{r}_1 \\ \left[{}^n R^c\right] {}^c\vec{r}_2 \end{bmatrix} \tag{57}$$

Putting it all together, we can now formulate the mixed complementarity problem for this example:

$$\begin{bmatrix} 0_{9\times1} \\ 0_{4\times1} \\ \vec{\rho}_n^{l+1} \end{bmatrix} = \begin{bmatrix} -M & W_{\mathrm{b}} & W_{\mathrm{n}} \\ W_{\mathrm{b}}^{\mathrm{T}} & 0 & 0 \\ W_{\mathrm{n}}^{\mathrm{T}} & 0 & 0 \end{bmatrix} \begin{bmatrix} \vec{\nu}^{l+1} \\ \vec{p_b}^{l+1} \\ \vec{p_n}^{l+1} \end{bmatrix} + \begin{bmatrix} M\vec{\nu}^l + h\vec{\lambda}_{app} \\ \frac{\vec{\phi}^l}{h} \\ \frac{\vec{\Psi}_{\mathrm{n}}}{h} \end{bmatrix} \tag{58}$$

$$0 \leq \vec{\rho}_n^{l+1} \perp \vec{p}_n^{l+1} \geq 0 \tag{59}$$

4.3 Planar 2 Bar Pendulum Attached to Block with Friction

This problem is identical to the previous "Planar 2 Bar Pendulum Attached to Block" example, with the addition of a friction force between the block and surface. We can skip right to the system dynamics.

4.3.1 System Dynamics

Assuming the two contacts are included in the active set every time step, the MCP is size 21 and the following quantities appearing in it are:

$$M = \begin{bmatrix} M_c & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & M_a & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & M_b \end{bmatrix} \qquad W_{\mathrm{b}}^{\mathrm{T}} = \begin{bmatrix} W_{b1}^{\mathrm{T}} & \vdots & W_{b2}^{\mathrm{T}} & \vdots & 0_{2\times3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_{2\times3} & \vdots & W_{b3}^{\mathrm{T}} & \vdots & W_{b4}^{\mathrm{T}} \end{bmatrix} \tag{60}$$

$$W_{\mathrm{n}} = \begin{bmatrix} \hat{n}_1 & \vdots & \hat{n}_2 \\ \left[{}^F R^C\right] {}^C\vec{r}_1 \otimes \hat{n}_1 & \vdots & \left[{}^F R^C\right] {}^C\vec{r}_2 \otimes \hat{n}_2 \\ \cdots & \cdots & \cdots \\ 0_{3\times1} & \vdots & 0_{3\times1} \\ \cdots & \cdots & \cdots \\ 0_{3\times1} & \vdots & 0_{3\times1} \end{bmatrix} \qquad \lambda_a = \begin{bmatrix} 0 \\ -m_c g \\ 0 \\ 0 \\ -m_a g \\ 0 \\ 0 \\ -m_b g \\ 0 \end{bmatrix} \tag{61}$$

The new frictional quantities to appear are:

$$W_{\mathrm{f}} = \begin{bmatrix} \hat{t}_1 & -\hat{t}_1 & \vdots & \hat{t}_2 & -\hat{t}_2 \\ \left[{}^F R^C\right] {}^C\vec{r}_1 \otimes \hat{t}_1 & \left[{}^F R^C\right] {}^C\vec{r}_1 \otimes -\hat{t}_1 & \vdots & \left[{}^F R^C\right] {}^C\vec{r}_2 \otimes \hat{t}_2 & \left[{}^F R^C\right] {}^C\vec{r}_2 \otimes -\hat{t}_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_{3\times1} & 0_{3\times1} & \vdots & 0_{3\times1} & 0_{3\times1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_{3\times1} & 0_{3\times1} & \vdots & 0_{3\times1} & 0_{3\times1} \end{bmatrix} \tag{62}$$

$$E = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix} \tag{63}$$

where μ_i is the coefficient of friction at contact i.

Putting it all together, we can now formulate the mixed complementarity problem:

$$\begin{bmatrix} 0_{9\times1} \\ 0_{4\times1} \\ \vec{\rho}_n^{\,l+1} \\ \vec{\rho}_f^{\,l+1} \\ \vec{s}^{\,l+1} \end{bmatrix} = \begin{bmatrix} -M & W_{\mathrm{b}} & W_{\mathrm{n}} & W_{\mathrm{f}} & 0 \\ W_{\mathrm{b}}^{\mathrm{T}} & 0 & 0 & 0 \\ W_{\mathrm{n}}^{\mathrm{T}} & 0 & 0 & 0 \\ W_{\mathrm{f}}^{\mathrm{T}} & 0 & 0 & E \\ 0 & U & -E^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \vec{\nu}^{\,l+1} \\ \vec{p}_b^{\,l+1} \\ \vec{p}_n^{\,l+1} \\ \vec{p}_f^{\,l+1} \\ \vec{\sigma}^{\,l+1} \end{bmatrix} + \begin{bmatrix} M\vec{\nu}^{\,l} + h\vec{\lambda}_{app} \\ \frac{\vec{\phi}^l}{h} \\ \frac{\vec{\Psi}_{\mathrm{n}}}{h} \\ 0_{4\times1} \\ 0_{2\times1} \end{bmatrix} \tag{64}$$

$$0 \leq \begin{bmatrix} \vec{\rho}_n^{\,l+1} \\ \vec{\rho}_f^{\,l+1} \\ \vec{s}^{\,l+1} \end{bmatrix} \perp \begin{bmatrix} \vec{p}_n^{\,l+1} \\ \vec{p}_f^{\,l+1} \\ \vec{\sigma}^{\,l+1} \end{bmatrix} \geq 0 \tag{65}$$

4.4 Planar 2 Bar Pendulum Attached to Block with Friction and Position Controlled Body

We extend the previous example with the introduction of a position controlled body manipulating the block. For simplification, we assume the pusher is a particle and interacts with the block through a single frictional point contact. The configuration of this system is unchanged as the new body is not force controlled. Figure 3 illustrates the problem.

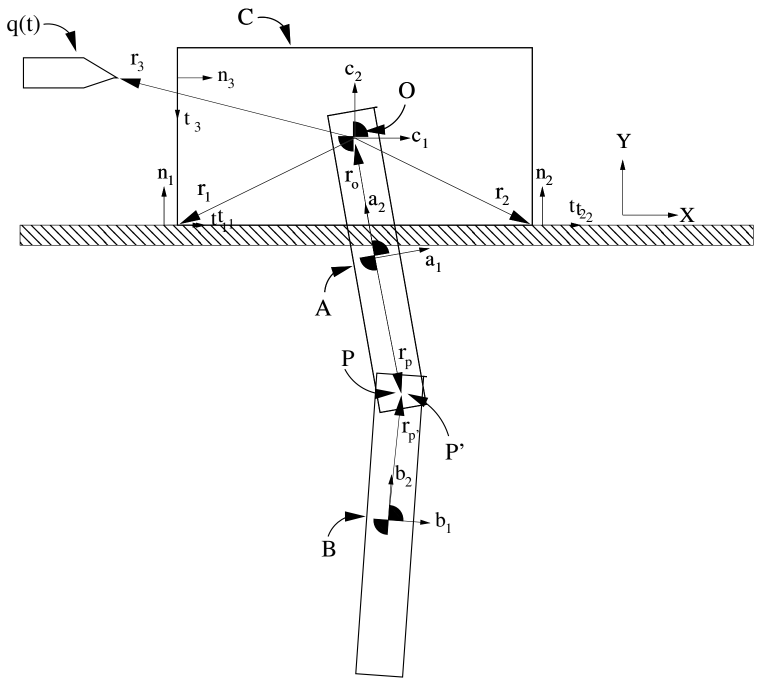

Figure 3: A planar simple double pendulum with all joint axes parallel to the Z-axis attached to a block on a surface. In addition, there is a position controlled body manipulating the block.

4.4.1 System Dynamics

The following quantities from before still appear:

$$M = \begin{bmatrix} M_c & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & M_a & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & M_b \end{bmatrix} \qquad W_{\mathrm{b}}^{\mathrm{T}} = \begin{bmatrix} W_{b1}^{\mathrm{T}} & \vdots & W_{b2}^{\mathrm{T}} & \vdots & 0_{2\times3} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_{2\times3} & \vdots & W_{b3}^{\mathrm{T}} & \vdots & W_{b4}^{\mathrm{T}} \end{bmatrix} \qquad \lambda_a = \begin{bmatrix} 0 \\ -m_c g \\ 0 \\ 0 \\ -m_a g \\ 0 \\ 0 \\ -m_b g \\ 0 \end{bmatrix} \tag{66}$$

Assuming the 3 contacts are included in the active set every time step, the MCP is size

25, and the other quantities are:

$$W_{\mathrm{n}} = \begin{bmatrix} \hat{n}_1 & \vdots & \hat{n}_2 & \vdots & {}^c\hat{n}_3 \\ \left[{}^F R^C\right] {}^C\vec{r}_1 \otimes \hat{n}_1 & \vdots & \left[{}^F R^C\right] {}^C\vec{r}_2 \otimes \hat{n}_2 & \vdots & \left[{}^C R^F\right] {}^F\vec{r}_3 \otimes {}^c\hat{n}_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_{3\times1} & \vdots & 0_{3\times1} & \vdots & 0_{3\times1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_{3\times1} & \vdots & 0_{3\times1} & \vdots & 0_{3\times1} \end{bmatrix} \tag{67}$$

The friction constraint wrench (for space constraints the rotation matrices performing the frame transforms have been dropped):

$$W_{\mathrm{f}} = \begin{bmatrix} \hat{t}_1 & -\hat{t}_1 & \vdots & \hat{t}_2 & -\hat{t}_2 & \vdots & {}^C\hat{t}_3 & -{}^C\hat{t}_3 \\ \vec{r}_1 \otimes \hat{t}_1 & \vec{r}_1 \otimes -\hat{t}_1 & \vdots & \vec{r}_2 \otimes \hat{t}_2 & \vec{r}_2 \otimes -\hat{t}_2 & \vdots & {}^C\vec{r}_3 \otimes {}^C\hat{t}_3 & {}^C\vec{r}_3 \otimes -{}^C\hat{t}_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_{3\times1} & 0_{3\times1} & \vdots & 0_{3\times1} & 0_{3\times1} & \vdots & 0_{3\times1} & 0_{3\times1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_{2\times1} & 0_{2\times1} & \vdots & 0_{2\times1} & 0_{2\times1} & \vdots & 0_{2\times1} & 0_{2\times1} \end{bmatrix} \tag{68}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix} \tag{69}$$

Since the position of the pusher is a time-dependent function $f(t)$, we must remember to include the partial derivative of that function in the b vector.

Putting it all together, we can now formulate the mixed complementarity problem:

$$\begin{bmatrix} 0_{9\times1} \\ 0_{4\times1} \\ \vec{\rho}_n^{\,l+1} \\ \vec{\rho}_f^{\,l+1} \\ \vec{s}^{\,l+1} \end{bmatrix} = \begin{bmatrix} -M & W_{\mathrm{b}} & W_{\mathrm{n}} & W_{\mathrm{f}} & 0 \\ W_{\mathrm{b}}^{\mathrm{T}} & 0 & 0 & 0 \\ W_{\mathrm{n}}^{\mathrm{T}} & 0 & 0 & 0 \\ W_{\mathrm{f}}^{\mathrm{T}} & 0 & 0 & E \\ 0 & U & -E^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \vec{\nu}^{\,l+1} \\ \vec{p}_b^{\,l+1} \\ \vec{p}_n^{\,l+1} \\ \vec{p}_f^{\,l+1} \\ \vec{\sigma}^{\,l+1} \end{bmatrix} + \begin{bmatrix} M\vec{\nu}^{\,l} + h\vec{\lambda}_{app} \\ \frac{\vec{\phi}^l}{h} \\ \frac{\vec{\Psi}_{\mathrm{n}}^l}{h} + \frac{\partial \vec{\Psi}_{\mathrm{n}}^l}{\partial t} \\ 0_{6\times1} \\ 0_{3\times1} \end{bmatrix} \tag{70}$$

$$0 \le \begin{bmatrix} \vec{\rho}_n^{\,l+1} \\ \vec{\rho}_f^{\,l+1} \\ \vec{s}^{\,l+1} \end{bmatrix} \perp \begin{bmatrix} \vec{p}_n^{\,l+1} \\ \vec{p}_f^{\,l+1} \\ \vec{\sigma}^{\,l+1} \end{bmatrix} \ge 0 \tag{71}$$

4.5 Planar 2 Bar Pendulum Attached to Block with Friction, Position Controlled Body, and Moving Floor

This example is identical to the previous example with the simple addition of $\frac{\partial \vec{\Psi}_\mathrm{f}}{\partial t}$ in the b vector. This term represents the lateral position change of the frictional surface in one time step, i.e., people movers in airports.

$$b = \begin{bmatrix} M\vec{\nu}^l + h\vec{\lambda}_{app} \\ \dfrac{\vec{\phi}^l}{h} \\ \dfrac{\vec{\Psi}^l_\mathrm{n}}{h} + \frac{\partial \vec{\Psi}^l_\mathrm{n}}{\partial t} \\ \frac{\partial \vec{\Psi}^l_\mathrm{f}}{\partial t} \\ 0_{3\times1} \end{bmatrix} \tag{72}$$

4.6 Full Planar Model

We extend the previous example by replacing the bottom rod of the pendulum with a spring and particle mass. Figure 4 illustrates the problem. For this example, vectors $\vec{r}_1$ and $\vec{r}_2$ are constant in the blocks body frame C and vector $\vec{r}_3$ is most naturally represented in the fixed inertial frame. Similarly, vectors $\hat{n}_1,\ \hat{n}_2,\ \hat{t}_1,$ and $\hat{t}_2$ are constant in the fixed inertial frame, and vectors $\hat{n}_3$ and $\hat{t}_3$ are constant in the C frame.

Assuming the 3 contacts are included in the active set every time step, the MCP is size 22, and the quantities appearing in the matrix are presented next.

The mass matrices of the three force controlled bodies are:

$$M_c = \begin{bmatrix} m_c & 0 & 0 \\ 0 & m_c & 0 \\ 0 & 0 & I_c \end{bmatrix} \qquad M_a = \begin{bmatrix} m_a & 0 & 0 \\ 0 & m_a & 0 \\ 0 & 0 & I_a \end{bmatrix} \qquad M_b = \begin{bmatrix} m_b & 0 \\ 0 & m_b \end{bmatrix} \tag{73}$$

resulting in a system mass matrix of

$$M = \begin{bmatrix} M_c & 0_{3\times3} & 0_{3\times2} \\ 0_{3\times3} & M_a & 0_{3\times2} \\ 0_{2\times3} & 0_{2\times3} & M_b \end{bmatrix} \tag{74}$$

Next, we must deal with the single bilateral constraint of the system, the revolute joint at point O. From before, we know we must constrain the relative velocity at point O between bodies C and A to be zero.

$$W_{b1}^\mathrm{T} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{U}_{2\times2} & 0 \\ 0_{1\times2} & 1 \end{bmatrix} \quad W_{b2}^\mathrm{T} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \mathbf{U}_{2\times2} & \left[{}^F R^A\right] {}^A\vec{r}_o \\ 0_{1\times2} & 1 \end{bmatrix} \quad W_\mathrm{b} = \begin{bmatrix} W_{b1} \\ W_{b2} \\ 0_{2\times2} \end{bmatrix} \tag{75}$$

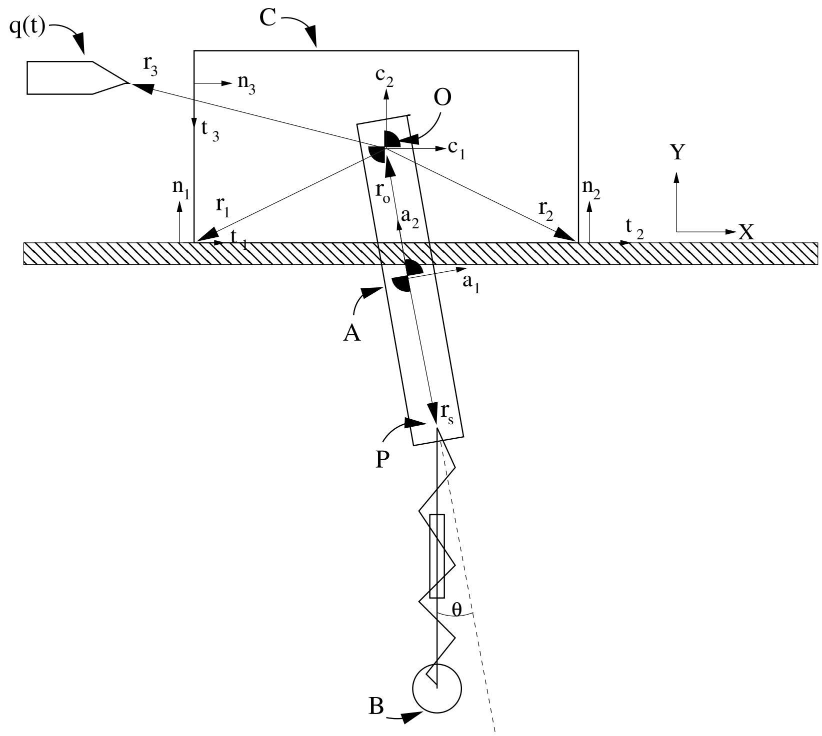

Figure 4: A force controlled block is sitting on a conveyor belt with a pendulum attached at its center of mass. At the other end of the pendulum, a spring and particle are attached. The block is interacted along its left edge with a position controlled pusher.

Now, the unilateral constraint wrench:

$$W_{\mathrm{n}} = \begin{bmatrix} \hat{n}_1 & \vdots & \hat{n}_2 & \vdots & {}^C\hat{n}_3 \\ \left[{}^F R^C\right] {}^C\vec{r}_1 \otimes \hat{n}_1 & \vdots & \left[{}^F R^C\right] {}^C\vec{r}_2 \otimes \hat{n}_2 & \vdots & \left[{}^C R^F\right] {}^F\vec{r}_3 \otimes {}^C\hat{n}_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_{3\times1} & \vdots & 0_{3\times1} & \vdots & 0_{3\times1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_{2\times1} & \vdots & 0_{2\times1} & \vdots & 0_{2\times1} \end{bmatrix} \tag{76}$$

The friction constraint wrench (for space constraints the rotation matrices performing the frame transforms have been dropped):

$$W_{\mathrm{f}} = \begin{bmatrix} \hat{t}_1 & -\hat{t}_1 & \vdots & \hat{t}_2 & -\hat{t}_2 & \vdots & {}^C\hat{t}_3 & -{}^C\hat{t}_3 \\ \vec{r}_1 \otimes \hat{t}_1 & \vec{r}_1 \otimes -\hat{t}_1 & \vdots & \vec{r}_2 \otimes \hat{t}_2 & \vec{r}_2 \otimes -\hat{t}_2 & \vdots & {}^C\vec{r}_3 \otimes {}^C\hat{t}_3 & {}^C\vec{r}_3 \otimes -{}^C\hat{t}_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_{3\times1} & 0_{3\times1} & \vdots & 0_{3\times1} & 0_{3\times1} & \vdots & 0_{3\times1} & 0_{3\times1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0_{2\times1} & 0_{2\times1} & \vdots & 0_{2\times1} & 0_{2\times1} & \vdots & 0_{2\times1} & 0_{2\times1} \end{bmatrix} \tag{77}$$

The other matrices required for Coulomb's friction:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{bmatrix} \tag{78}$$

In this example, gravity is not the only external force acting on the system, we also have the spring force and damping force acting on bodies A and B.

For the spring, $\sin(\theta)$ and $\cos(\theta)$ are functions of rod A's endpoint (point P) and the particle B's position.

$$L = \sqrt{(B_x - P_x)^2 + (B_y - P_y)^2} \tag{79}$$

$$\sin(\theta) = -(B_x - P_x)/L \tag{80}$$

$$\cos(\theta) = -(P_y - B_y)/L \tag{81}$$

where L is the length of the spring. This allows us to construct the rotation matrix from the springs frame s into the fixed world frame n:

$$[{}^nR^s] = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \tag{82}$$

The force produced by the spring acts along the spring's y-axis and is equal to $-K(L-L_0)$, where L_0 is the rest length of the spring and K is the spring constant. This allows us to write down the spring force in the springs frame as: ${}^sf_s = [0 \;\; -K(L - L_0)]^{\mathrm{T}}$.

To compute the force of the spring on the particle, a simple change of frame is all that is required: $f_s = [{}^nR^s]\,{}^sf_s$.

For the pendulum, slightly more work is needed. We first compute the spring force acting at point P, which we know must be opposite and equal the force acting on the particle B: $-f_s$. Now that we know the force acting at point P, we need to compute the wrench acting at A^*. Similar to contact forces, the wrench associated with the spring $\vec{W}_s$ for body A is:

$$\vec{W}_s = \left[-\vec{f}\,[{}^nR^a]\,{}^ar_s \otimes -\vec{f}_s\right] \tag{83}$$

For the damping force, we need to first compute the relative velocity between the particle B and the point P. The velocity of point P can be obtained from the velocity twist of the pendulum:

$$\vec{v}^p = \vec{v}^a + \omega^a\,(\vec{r}_p)^\wedge \tag{84}$$

The damping force acting on the particle B therefore becomes:

$$\vec{f}_d = -C(\vec{v}^b - \vec{v}^p) \tag{85}$$

and the force acting at point P is $-\vec{f}_d$ (opposite and equal). For the body A, we again have to transform the damping force acting at point P into the corresponding wrench acting at A^*.

$$\vec{W}_d = \begin{bmatrix} -\vec{f}_d \\ \vec{r}_s \otimes -\vec{f}_d \end{bmatrix} \tag{86}$$

We can group all the external forces/wrenches into a single vector λ_{app}:

$$\lambda_{app} = \begin{bmatrix} \vec{g}_c \\ \vec{g}_a + \vec{W}_s + \vec{W}_d \\ \vec{g}_b + \vec{f}_s + \vec{f}_d \end{bmatrix} \tag{87}$$

The other elements of b are the constraint stabilization terms and the partial derivative appearing from the position controlled body. Putting it all together, we can formulate the mixed linear complementarity problem:

$$\begin{bmatrix} 0_{8\times 1} \\ 0_{2\times 1} \\ \vec{\rho}_n^{\,l+1} \\ \vec{\rho}_f^{\,l+1} \\ \vec{s}^{\,l+1} \end{bmatrix} = \begin{bmatrix} -M & W_{\mathrm{b}} & W_{\mathrm{n}} & W_{\mathrm{f}} & 0 \\ W_{\mathrm{b}}^{\mathrm{T}} & 0 & 0 & 0 & \\ W_{\mathrm{n}}^{\mathrm{T}} & 0 & 0 & 0 & \\ W_{\mathrm{f}}^{\mathrm{T}} & 0 & 0 & E & \\ 0 & U & -E^{\mathrm{T}} & 0 & \end{bmatrix} \begin{bmatrix} \vec{\nu}^{\,l+1} \\ \vec{p}_b^{\,l+1} \\ \vec{p}_n^{\,l+1} \\ \vec{p}_f^{\,l+1} \\ \vec{\sigma}^{\,l+1} \end{bmatrix} + \begin{bmatrix} M\vec{\nu} + h(\lambda_{app}) \\ \frac{\vec{\phi}^l}{h} \\ \frac{\vec{\Psi}_{\mathrm{n}}}{h} + \frac{\partial \vec{\Psi}_{\mathrm{n}}}{\partial t} \\ \frac{\partial \vec{\Psi}_{\mathrm{f}}}{\partial t} \\ 0_{3x1} \end{bmatrix} \tag{88}$$

$$0 \le \begin{bmatrix} \vec{\rho}_n^{\,l+1} \\ \vec{\rho}_f^{\,l+1} \\ \vec{s}^{\,l+1} \end{bmatrix} \perp \begin{bmatrix} \vec{p}_n^{\,l+1} \\ \vec{p}_f^{\,l+1} \\ \vec{\sigma}^{\,l+1} \end{bmatrix} \ge 0 \tag{89}$$

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