Add Coulomb Friction

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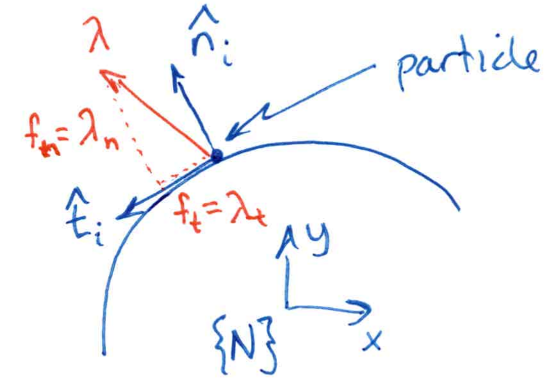
Planar Case:



body (particle) vel $\quad \nu = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}$

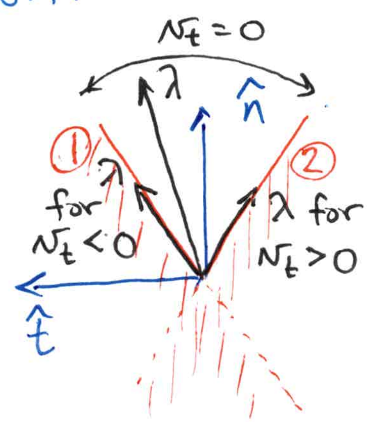
contact variables

rel. vel. $\begin{bmatrix} v_n \\ v_t \end{bmatrix} = v \qquad$ cont. force $\lambda = \begin{bmatrix} \lambda_n \\ \lambda_t \end{bmatrix} = \begin{bmatrix} f_n \\ f_t \end{bmatrix}$

Coulomb friction: $\quad -\mu\lambda_n \le \lambda_t \le \mu\lambda_n \longrightarrow \lambda_n \ge 0$

$\mu \ge 0$ is friction coeff.

Complementarity formulation for friction:



① $\mu\lambda_n - \lambda_t \ge 0$

② $\mu\lambda_n + \lambda_t \ge 0$

if ① & ② hold, $v_t \overset{\text{must}}{=} 0$

if $v_t > 0$, $\lambda_t = -\mu\lambda_n$ (② holds w/ equality)

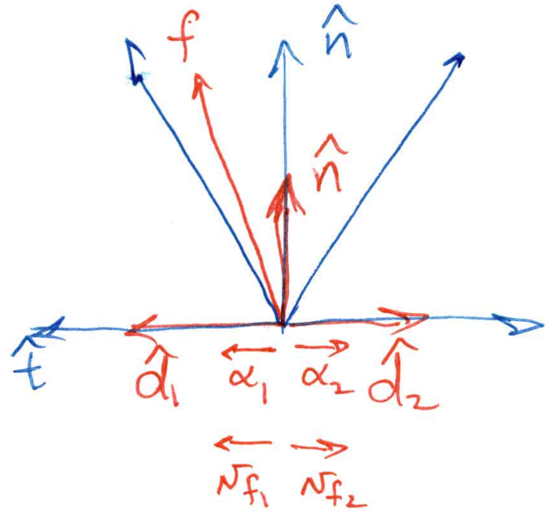
if $v_t < 0$, $\lambda_t = \mu\lambda_n$ (① holds w/ equality)

Introduce positive & negative parts of contact variables

component of friction force

$\lambda_t = \alpha_1 - \alpha_2$

$0 \le \alpha_1 \perp \alpha_2 \ge 0$



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component of relative sliding velocity

$v_t = v_{f_1} - v_{f_2}$

$0 \le v_{f_1} \perp v_{f_2} \ge 0$

Coupling conditions among friction force & rel. vel. components

$v_{f_1} > 0 \Rightarrow \alpha_2 = \mu f_n, \ \alpha_1 = 0, \ v_{f_2} = 0$ (slide left)

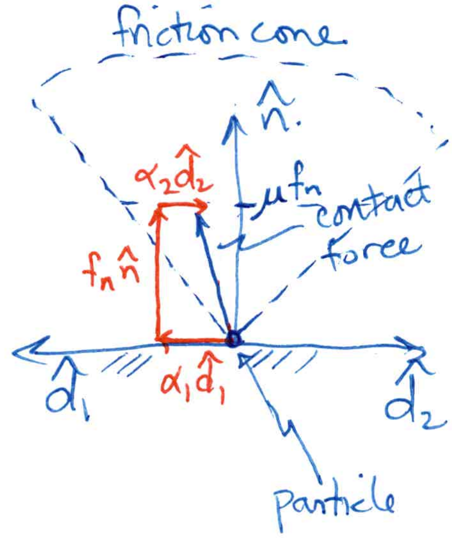
$v_{f_2} > 0 \Rightarrow \alpha_1 = \mu f_n, \ \alpha_2 = 0, \ v_{f_1} = 0$ (slide right)

$v_{f_1} = v_{f_2} = 0 \Rightarrow 0 \le \alpha_1, \alpha_2 \le \mu f_n$ (sticking)

Introduce sliding indicator variable, s, relative speed.

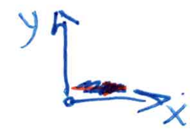
$$s \ge 0 \begin{cases} s > 0 \Rightarrow \text{sliding} & (\alpha_1 \text{ or } \alpha_2 = \mu f_n) \\ s = 0 \Rightarrow \text{sticking} & (\alpha_1 + \alpha_2 = \mu f_n) \end{cases}$$

Complementarity representation of friction cone constraints



$$0 \le \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} \perp \begin{bmatrix} \overbrace{\hat{d}_1^T v}^{v_{f_1}} + s \\ \underbrace{\hat{d}_2^T v}_{v_{f_2}} + s \end{bmatrix} \ge 0$$

$$0 \le s \quad \perp \quad \mu f_n - \alpha_1 - \alpha_2 \ge 0$$



Consider an example. Let the particle slide to the right with speed = 1.

$$\hat{d}_1^T = \begin{bmatrix} -1 & 0 \end{bmatrix} \qquad \hat{d}_2^T = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad v^T = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$0 \le \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ s \end{bmatrix} \perp \begin{bmatrix} -1 + s \\ 1 + s \\ \mu f_n - \alpha_1 - \alpha_2 \end{bmatrix} \ge 0$$

Rows 1 & 2: $\left.\begin{matrix} -1 + s \ge 0 \\ 1 + s \ge 0 \end{matrix}\right\} \Rightarrow \left.\begin{matrix} s \ge 1 \\ s \ge -1 \end{matrix}\right\} \Rightarrow s \ge 1 \xRightarrow{\text{row 2}} \boxed{\alpha_2 = 0}$

Row 3: $\begin{matrix} s > 0 \\ \alpha_2 = 0 \end{matrix} \Rightarrow \mu f_n - \alpha_1 - \cancel{\alpha_2} = 0 \Rightarrow \boxed{\alpha_1 = \mu f_n}$

Row 1: $\alpha_1 > 0 \Rightarrow -1 + s = 0 \Rightarrow \boxed{s = 1}$

Letting $\nu^T = [1 \; 0]$ yields

$\alpha_1 = 0 \;, \; \alpha_2 = \mu f_n \;, \; s = 1$ (sliding left)

Letting $\nu^T = [0 \; 0]$ yields

$s = 0, \; \alpha_1, \alpha_2 \geq 0 \quad \mu f_n - \alpha_1 - \alpha_2 \geq 0$

$\therefore$ contact force is inside friction cone

(Note: $s > 0$ is possible if $f_n = 0$, degenerate)

3D Case of Particle w/ Friction

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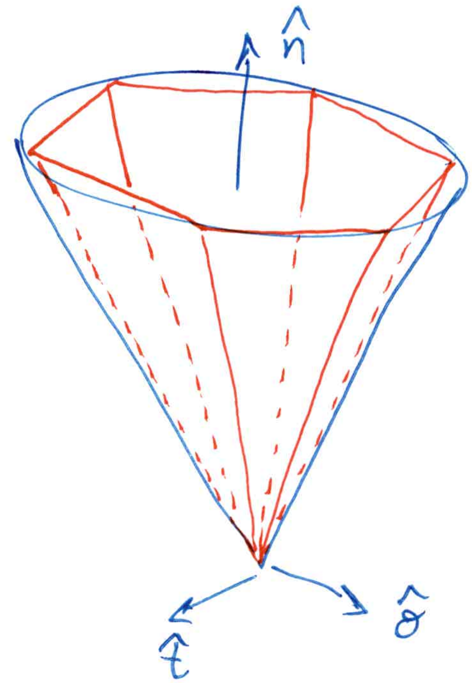
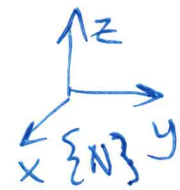
⑤

Linearize friction cone as convex polyhedral cone with friction directions $\hat{d}_j$





friction force is:

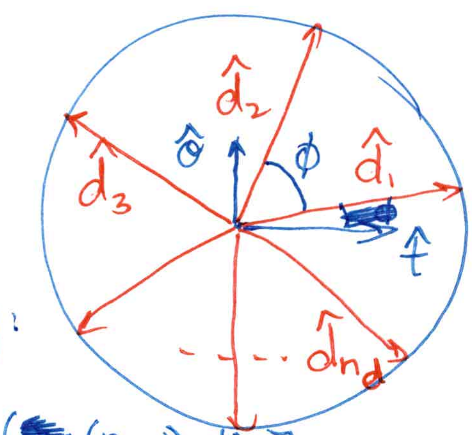
$$f_f = \sum_{j=1}^{n_d} \hat{d}_j \alpha_j = D\alpha$$

$$D = \left[\hat{d}_1 \cdots \hat{d}_{n_d}\right]$$

$$\alpha^T = \left[\alpha_1 \cdots \alpha_{n_d}\right]$$

Let $F = \left[\hat{t} \;\; \hat{o}\right]$ be the contact friction directions

~~the tangent~~ Assuming equal spacing:



$$\text{Let } A = \begin{bmatrix} 1 & \cos(\phi) & \cos(2\phi) & \cdots & \cos((n_d-1)\phi) \\ 0 & \sin(\phi) & \sin(2\phi) & \cdots & \sin((n_d-1)\phi) \end{bmatrix}$$

$$\underset{(3\times n_d)}{D} = \underset{(3\times 2)}{F} \; \underset{(2\times n_d)}{A}$$

Keep contact force in friction cone

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$$\sum_{j=1}^{n_d} \alpha_j \le \mu f_n \quad \text{rewrite as}$$

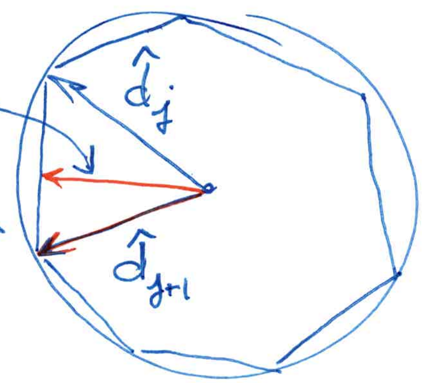
$$\mu f_n - e^T \alpha \ge 0$$

where $e^T = [1\ 1 \cdots 1]_{(1\times n_d)}$ $\qquad \alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{n_d} \end{bmatrix}$

If ~~one da~~ $\alpha_{j+1} = \mu f_n$

If $\alpha_j + \alpha_{j+1} = \mu f_n$

all other α's are zero.



Full model

$$\boxed{v^{\ell+1} = v^\ell + hM^{-1}(g^\ell + \hat{n}^\ell f_n^{\ell+1} + D\alpha^{\ell+1})}$$

$q^{\ell+1} = q^\ell + hVv^{\ell+1}$ ← recall $V = I_{3\times3}$ for spatial particle

$0 \le f_n^{\ell+1} \perp \psi_n^\ell + \hat{n}^{\ell T} v^{\ell+1} h + \frac{\partial \psi_n^\ell}{\partial t} h \ge 0$ ← impact with non penetration

$0 \le \alpha^{\ell+1} \perp D^T v^{\ell+1} + e\, s^{\ell+1} \ge 0$ ← friction force resists sliding

$0 \le s^{\ell+1} \perp \mu f_n^{\ell+1} - e^T \alpha^{\ell+1} \ge 0$ ← force in friction cone

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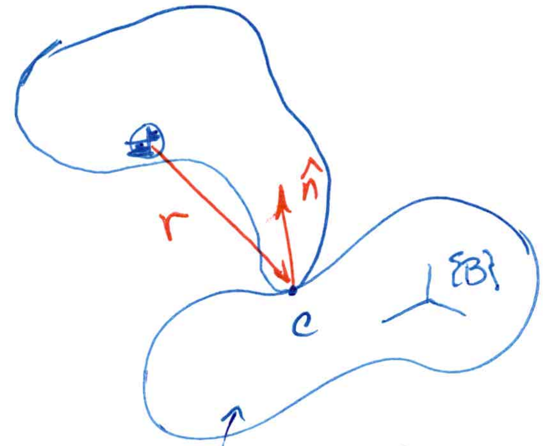
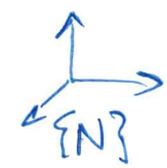
Replace particle with rigid body of finite extent



motion controlled

${}^{N}T_B(t)$ given



Replace $\hat{n}$ with G_n:

$$G_n = \begin{bmatrix} \hat{n} \\ r \times \hat{n} \end{bmatrix}_{(6\times1)}$$

Replace D with G_f

$$G_f = \begin{bmatrix} \hat{d}_1 & \cdots & d_{n_d} \\ r\times d_1 & \cdots & r \times d_{n_d} \end{bmatrix}_{(6\times n_d)}$$

ν becomes $6\times1 = \begin{bmatrix} v \\ \omega \end{bmatrix}$

M becomes $6\times6 = \begin{bmatrix} mI_{3\times3} & 0 \\ 0 & J_{3\times3} \end{bmatrix}$

inertia matrix not Jacobian

q become $7\times1 = [x \; y \; z \; \underbrace{a \; b \; c \; d}]$

unit quaternion

$$V = \begin{bmatrix} I_{(3\times3)} & 0 \\ 0 & B_{(4\times3)} \end{bmatrix} \text{ where } B = \begin{bmatrix} -b & -c & -d \\ a & -d & c \\ d & a & -b \\ -c & b & a \end{bmatrix}$$

Multiple bodies w/ multiple contacts

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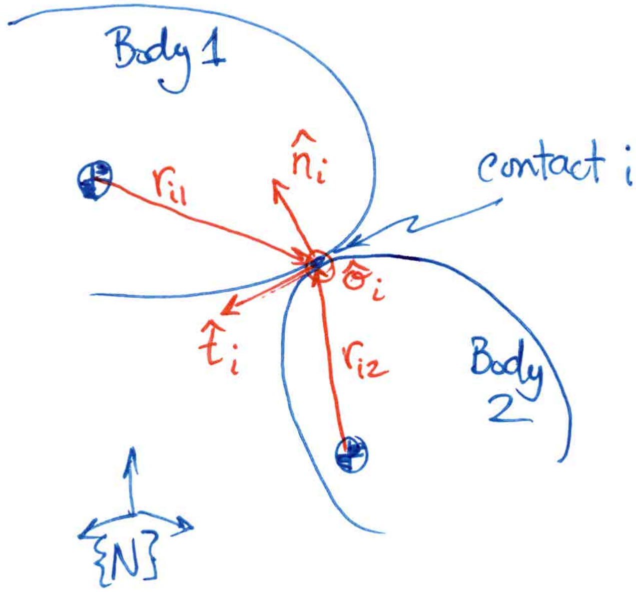
Choose one contact frame at each contact.



One body has inward contact normal.

Define for contact i, body 1:

$$F_{i1} = \begin{bmatrix} \hat{t}_i & \hat{\theta}_i \\ r_{i1} \times \hat{t}_i & r_{i1} \times \hat{\theta}_i \end{bmatrix}$$

$$G_{in1} = \begin{bmatrix} \hat{n}_i \\ r_{i1} \times \hat{n}_i \end{bmatrix}$$

for body 2

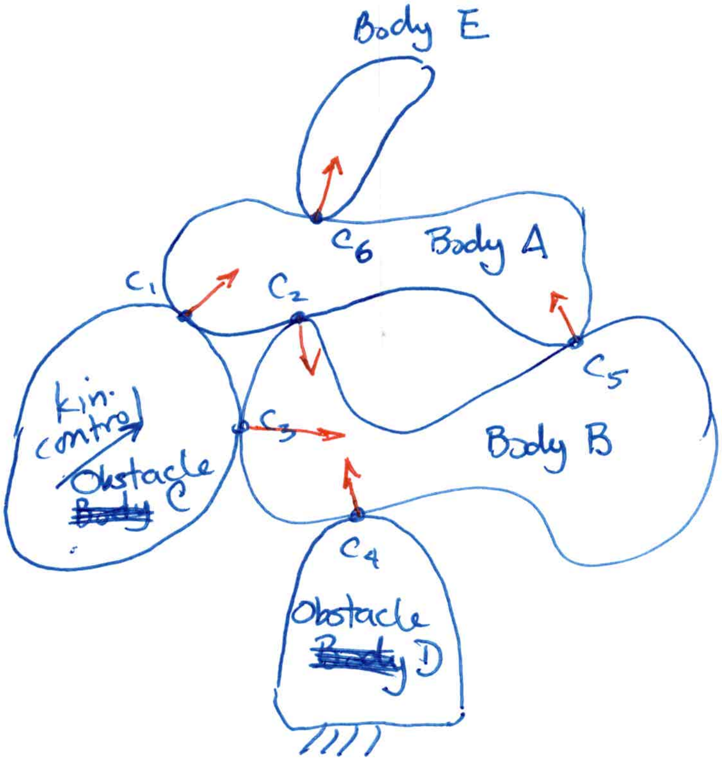
$$F_{i2} = \begin{bmatrix} -\hat{t}_i & -\hat{\theta}_i \\ -r_{i2} \times \hat{t}_i & -r_{i2} \times \hat{\theta}_i \end{bmatrix}$$

$$G_{in2} = \begin{bmatrix} -\hat{n}_i \\ -r_{i2} \times \hat{n}_i \end{bmatrix}$$

Constructed block matrix.

6 rows for each dynamic body

1 column for each contact



$$G_n = \begin{array}{c l}
\left[\begin{array}{c|c|c|c|c|c}
G_{1n1} & G_{2n2} & 0 & 0 & G_{5n1} & G_{6n2} \\
\hline
0 & G_{2n1} & G_{3n1} & G_{4n1} & G_{5n2} & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & G_{6n1}
\end{array}\right] & \begin{array}{l} \text{Body A} \\ \text{Body B} \\ \text{Body E} \end{array} \\
\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} &
\end{array}$$

contacts: 1 2 3 4 5 6

$(18 \times n_c)$

G_f is constructed in exactly the same way.
$(18 \times n_c n_d)$

$M = \text{block diagonal}\,(M_A, M_B, M_E)_{(18 \times 18)}$ for this example.

$g_{app}^T = [\,g_A^T \;\; g_B^T \;\; g_E^T\,]_{(1 \times 18)}$

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μ becomes $\underset{(n_c \times n_c)}{U} = \text{diagonal}(\mu_1, \mu_2, \dots, \mu_{n_c})$

f_n^T becomes $f_n^T = [f_{1n} \; f_{2n} \; \cdots \; f_{n_c n}]_{(1 \times n_c)}$

e becomes $E = \text{blockdiag}(e, e, \dots, e)_{(n_c n_d \times n_c)}$

α^T becomes $\alpha^T = [\alpha_1^T \; \alpha_2^T \; \cdots \; \alpha_{n_c}^T]$

ψ_n^T becomes $\psi_n^T = [\psi_{1n}^T \; \cdots \; \psi_{n_c n}^T]$

$\dfrac{\partial \psi_n}{\partial t}^T$ becomes $\dfrac{\partial \psi_n}{\partial t}^T = \left[\dfrac{\partial \psi_{1n}}{\partial t} \; \dfrac{\partial \psi_{2n}}{\partial t} \; \cdots \; \dfrac{\partial \psi_{n_c n}}{\partial t}\right]$

V becomes $V = \text{blockdiag}(V_A, V_B, V_E)$

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Final Formulation:

Let $p_n = h f_n$, $p_f = h\alpha$, $p_{app} = h g_{app}$

p_{app} — impulse of external wrench

p_f — impulses of friction wrenches

p_n — impulses of normal wrenches.

$$\nu^{l+1} = \nu^l + hM^{-1}\left(p_{app} + G_n p_n^{l+1} + G_f p_f^{l+1}\right)$$

$$q^{l+1} = q^l + hV\nu^{l+1}$$

$$0 \le \underbrace{\begin{bmatrix} p_n^{l+1} \\ p_f^{l+1} \\ s^{l+1} \end{bmatrix}}_{\text{unknowns}} \perp \begin{bmatrix} G_n^T M^{-1} G_n & G_n^T M^{-1} G_f & 0 \\ G_f^T M^{-1} G_n & G_f^T M^{-1} G_f & E \\ U & -E^T & 0 \end{bmatrix} \underbrace{\begin{bmatrix} p_n^{l+1} \\ p_f^{l+1} \\ s^{l+1} \end{bmatrix}}_{\text{unknowns}} + \begin{bmatrix} G_n^T\left(\nu^l + M^{-1}(p_{app}^l + T^l)\right) + \frac{\Psi_n^l}{h} + \frac{\partial \Psi_n^l}{\partial t} \\ G_f^T\left(\nu^l + M^{-1}(p_{app}^l + T^l)\right) \\ 0 \end{bmatrix} \ge 0$$

everything else is known.

where $T^T = \left(T_A^T \; T_B^T \; \cdots \; T_E^T\right)$ and $T_i^l = \begin{bmatrix} 0_{(3\times1)} \\ -\omega_i^l \times J_i^l \omega_i^l \end{bmatrix}_{(6\times1)}$

$\underset{(3\times1)}{\omega_i}$ is angular velocity of body i

$\underset{(3\times1)}{\omega_i} = \underset{(6\times1)}{\nu_i}$ elements 4-6.

$\underset{(3\times3)}{J_i}$ is the inertia matrix of body i