Contact interactions & simulation

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Robots make contact with the world to perform tasks. In order for robots to locomote & manipulate effectively, they should understand (i.e., be able to predict) the effects their contacts have on themselves & the world.

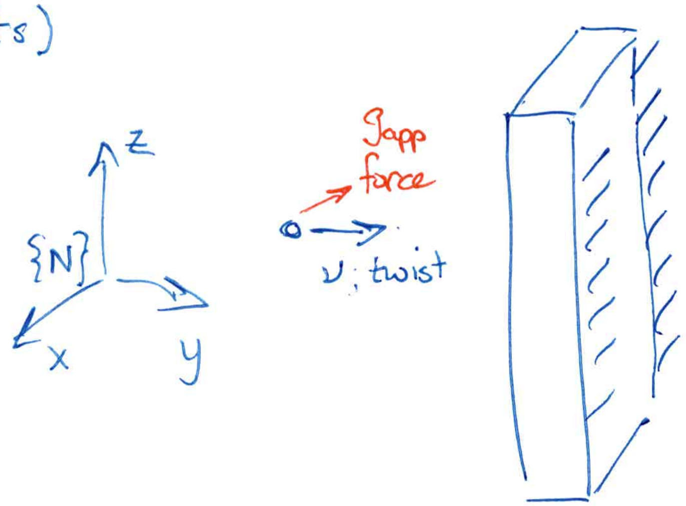
$\boxed{\text{Modeling body motions with intermittent contact}}$

(as occurs during walking, grasp acquisition, and assembly).

Start with simplest case; a particle and a wall (no rotation, no moments)

Configuration

$$q = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \nu = \begin{bmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{bmatrix} = \dot{q}$$



Newton's 2nd Law

$$\sum_i \text{forces} = m\dot{\nu} = \text{mass} \times \text{acceleration}$$

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Dynamic motion is described by two equations

dynamic model $\begin{cases} \dot{\nu} = \frac{1}{m} g_{app} = \frac{1}{m} g \\ \dot{q} = V\nu \end{cases}$

$\leftarrow$ Newton's 2nd Law Dynamics

$\leftarrow$ Velocity kinematics

in the particle case, $V = I_{(3\times 3)}$

Convert to discrete-time model.

Let h be a small time interval, $h \in \mathbb{R}^1$,

Current time is $t_\ell = h\ell$

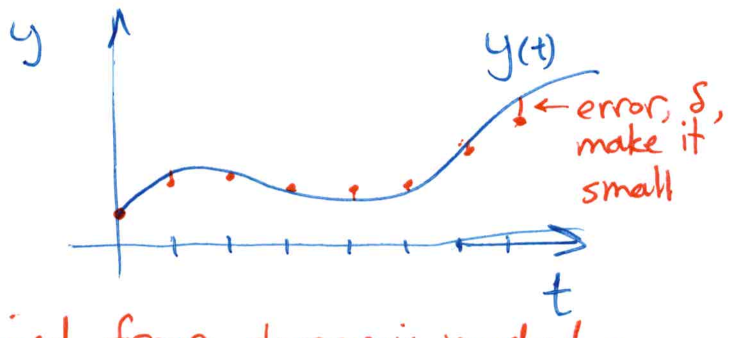
Current config is $q^\ell = q(t_\ell)$

Current twist is $\nu^\ell = \nu(t_\ell)$

Goal of a discrete-time model is to produce an approximation (point-wise) of the solution of the dynamic model

$\dot{y} = f(t, y(t))$

$y(t+h) = y(t) + \underline{\Delta y}$



quantity obtained from dynamic model.

Desired property of discrete-time model

<u>Convergence</u>: A discrete-time model, a.k.a. time-stepping method, is convergent if the maximum error δ^{ℓ} goes to zero as $h \to 0$.

<u>Order</u>: A discrete-time model has order p if the error δ of a single ~~time~~step is $O(h^{1+p})$ as $h \to 0$.

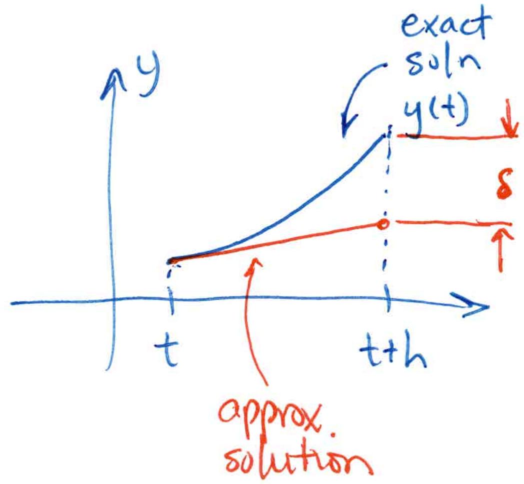
Explicit Euler

$$\dot{y} \simeq \frac{y(t+h) - y(t)}{h}$$

$$\Rightarrow \quad y(t+h) = y(t) + h f(t, y(t))$$

$$y^{\ell+1} = y^{\ell} + h f^{\ell}$$

slope at t



Notes: • The new value of y is a function of known quantities.

when $f(t)$ is smooth →
- Explicit Euler is a first-order method, $\delta = O(h^2)$.
- Explicit Euler is convergent.

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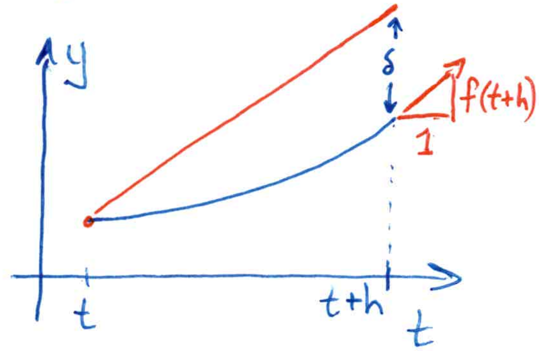
Backward Euler (a.k.a. $\boxed{\text{Implicit Euler}}$)

$$\dot{y} \simeq \frac{y(t) - y(t-h)}{h}$$

$$\Rightarrow \quad y(t) = y(t-h) + hf(t, y(t))$$

or $\quad y^{l+1} = y^{l} + hf^{l+1}$

unknown quantity



Notes:

- New value, y^{l+1}, is function of unknown quantities. f^{l+1} is function of y^{l+1}, so determining y^{l+1} can involve solving a system of nonlinear equations.
- When $f(t)$ is smooth, Implicit Euler is <u>first-order</u> and <u>convergent</u>.

Euler's Method Applied to Particle

$$\dot{q} \simeq \frac{q^{l+1} - q^{l}}{h} \qquad \dot{\nu} \simeq \frac{\nu^{l+1} - \nu^{l}}{h}$$

$$\nu^{l+1} = \nu^{l} + \frac{h}{m} g$$

$l, l+1$?

$$q^{l+1} = q^{l} + hV\nu$$

Explicit Euler for both equations

$$\nu^{l+1} = \nu^{l} + \frac{h}{m} g^{l}$$

$$q^{l+1} = q^{l} + hV\nu^{l}$$

Consider 1D example

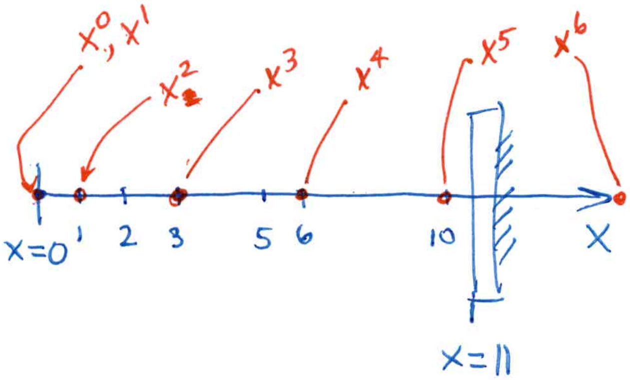
$q = x \ , \ \nu = v_x$

for simplicity, let $m = h = g = 1$

let $x^0 = 0 \ , \ v_x^0 = 0$

$$v_x^{l+1} = v_x^{l} + 1$$

$$x^{l+1} = x^{l} + v_x^{l}$$



l	x^l	$v_x{}^l$
0	0	0
1	0	1
2	1	2
3	3	3
4	6	4
5	10	5
6	15	6

collision w/ wall

<u>Note</u>: Explicit Euler uses info at beginning of time step, so it doesn't anticipate wall

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Semi-Implicit Euler

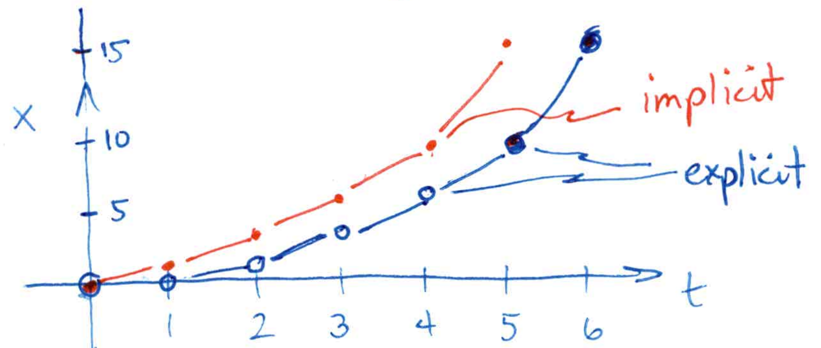
$$v^{l+1} = v^l + \frac{h}{m} g^{l}$$ Explicit Euler

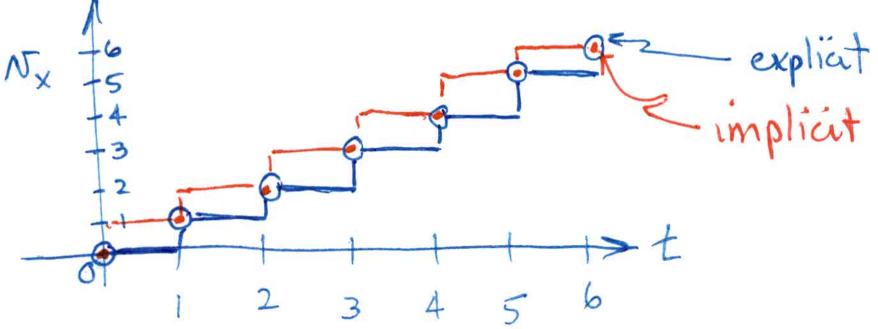
$$q^{l+1} = q^l + hVv^{l+1}$$ Implicit Euler. Same info at end of timestep is used.

$$v_x^{l+1} = v_x^l + 1$$

$$x^{l+1} = x^l + v_x^{l+1}$$

l	x^l	v_x^l
0	0	0
1	1	1
2	3	2
3	6	3
4	10	4
5	15	5
⋮	⋮	⋮





Implicit method assumes velocity throughout time step is same as velocity at end of time step!

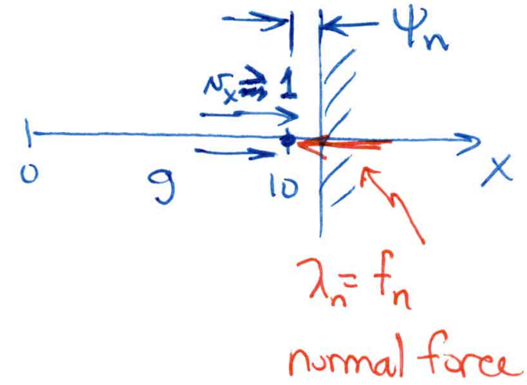
Explicit method assumes velocity throughout time step is same as velocity at start of time step!

How do we handle collisions?

Let ψ_n be the gap distance

$$\psi_n = x_{wall} - x$$

$$= 11 - x$$



Time stepper should not produce any $x^l > 11$, i.e. $\psi_n \geq 0$

We need a contact force f_n to stop the particle. before it passes the wall.

$f_n \geq 0$ ← force can push, not pull

Also force should be zero if collision is not imminent ∴ we need:

$$\boxed{\begin{aligned} f_n &\geq 0 \\ \psi_n &\geq 0 \\ f_n \psi_n &= 0 \end{aligned}}$$

These 3 conditions are known as complementarity conditions.

$$\equiv$$

Shorthand notation $\boxed{0 \leq f_n \perp \psi_n \geq 0}$

New Dynamic Model

$$\dot{\nu} = \frac{1}{m}(g - \lambda_n)$$

$$\dot{q} = V\nu$$

$$0 \leq \lambda_n \perp \psi_n(q) \geq 0$$

Fully explicit Euler fails!
(with constant h)

$$\nu_x^{\ell+1} = \nu_x^{\ell} + 1 - f_n^{\ell}$$

$$x^{\ell+1} = x^{\ell} + \nu_x^{\ell}$$

$$f_n^{\ell+1} \geq 0$$

$$\psi_n^{\ell+1} = x^{\ell} + \nu_x^{\ell} - 11 \geq 0$$

$$f_n^{\ell+1} \cdot \psi_n^{\ell+1} = 0$$

ℓ	x^ℓ	ν_x^ℓ	ψ_n^ℓ	f_n^ℓ
0	0	0	11	0
1	0	1	11	0
2	1	2	10	0
3	3	3	8	0
4	6	4	5	0
5	10	5	1	0
6	15	6	−4	

Need to use some info at end of timestep to avoid failure!

Semi-Implicit Euler allows anticipation of collision.

$$v_x^{l+1} = v_x^l + 1 - f_n$$

$$x^{l+1} = x^l + v_x^{l+1}$$

This term was the culprit.

Note: 1 is hg = impulse applied over $hl \le t \le h(l+1)$

Similarly $f_n h$ = " " by contact over "

Let's use hf_n^{l+1} to denote this impulse

$$v_x^{l+1} = v_x^l + 1 - f_n^{l+1}$$

$$x^{l+1} = x^l + (v_x^l + 1 - f_n^{l+1})$$

Now a chance for f_n to prevent penetration is built into the time-stepping equations!

$$f_n^{l+1} \geq 0$$

$$\psi_n^{l+1} \geq 0$$

$$f_n^{l+1} \cdot \psi_n^{l+1} = 0$$

First solve these 3 eqs. for f_n^{l+1}

Then solve for v_x^{l+1} & x^{l+1}

$$\psi_n^{l+1} = 11 - x^{l+1} \geq 0$$

$$= 11 - (x^l + v_x^l + 1 - f_n^{l+1}) \geq 0$$

The Linear Complementarity Problem (LCP) is:

$$f_n^{l+1} \geq 0$$

$$\psi_n^{l+1} = f_n^{l+1} + 10 - x^l - v_x^l \geq 0$$

$$f_n^{l+1} \cdot \psi_n^{l+1} = 0$$

l	x^l	v_x^l	ψ_n^l	f_n^l
0	0	0	11	0
1	1	1	10	0
2	3	2	8	0
3	6	3	5	0
4	10	4	1	0
5	11	1	0	4
6	11	0	0	2
7	11	0	0	1
8	11	0	0	1
⋮	⋮	⋮	⋮	⋮

Consider first time step.

$$0 \leq f_n^1 \perp f_n^1 + 10 - 0 - 0 \geq 0$$

$f_n^1 = 0$ is unique soln.

$$v_x^1 = v_x^0 + 1 - 0 = 1$$

$$x^1 = x^0 + v_x^1 = 1$$

Fifth time step.

$$0 \leq f_n^5 \perp f_n^5 + 10 - 10 - 4 \geq 0$$

$f_n^5 = 4$ is unique soln.

$$v_x^5 = v_x^4 + 1 - 4 = 1$$

$$x^5 = x^4 + v_x^5 =$$

Notice that two time steps were needed to model the impact.

- The first reduced the speed to avoid penetration
- The second matched the particle's speed to that of the wall.

After the impact, the contact force exactly balanced the external force.

Now consider the case of a moving wall whose motion is specified as a function of time, e.g. $x_{wall} = 11 - t$.

The Euler approximation is closely related to Taylor Series Expansion

$$x(t+h) \simeq x(t) + \frac{dx}{dt} h = \underline{x(t) + v_x(t)\, h}$$

$$\underline{x^{l+1} = x^{l} + v_x^{l+1} h}$$

This approx. is in our model.

Now bring the wall into the picture

$$\psi_n(t) = x_{wall}(t) - x(t)$$

$$\psi_n(t+h) \simeq \psi_n(t) + \frac{d\psi_n}{dt} h$$

$$\simeq x_{wall}(t) - x(t) + \boxed{\frac{dx_{wall}}{dt} h} - \frac{dx}{dt} h$$

let $t = hl$

new term due to motion of wall

$$\psi_n^{l+1} \simeq \psi_n^l + \frac{d\psi_n^l}{dt} h$$

$$\simeq x_{wall}^l - x^l + \boxed{\frac{dx_{wall}^l}{dt} h} - v_x^l h$$

Extend previous example.

Let $x_{wall} = 11 - t \rightarrow \frac{dx_{wall}}{dt} = -1$

$$v_x^{l+1} = v_x^l + 1 - f_n^{l+1}$$

$$x^{l+1} = x^l + v_x^{l+1}$$

$$f_n^{l+1} \geq 0$$

$$\psi_n^{l+1} \geq 0$$

$$f_n^{l+1} \cdot \psi_n^{l+1} = 0$$

where $\psi_n^{l+1} \simeq$

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Substitute v_x^{l+1} & x^{l+1} into ψ_n^{l+1} to obtain LCP.

$$\psi_n^{l+1} = x_{wall}^{l+1} - x^{l+1}$$

$$\cong x_{wall}^{l} + \frac{dx_{wall}^{l}}{dt} h - \left(x^l + v_x^{l+1} h\right)$$

For our example $= -1$, $h = 1$

$$\psi_n^{l+1} \simeq f_n^{l+1} + x_{wall}^{l} + \cancelto{-1}{\frac{dx_{wall}}{dt}} - \left(x^l + v_x^l + 1\right)$$

LCP: $0 \le f_n^{l+1} \perp f_n^{l+1} + x_{wall}^{l} - x^l - v_x^l - 2 \ge 0$

Solve for f_n^{l+1} then compute

$$v_x^{l+1} = v_x^l + 1 - f_n^{l+1}$$

$$x^{l+1} = x^l + v_x^{l+1}$$

$$x_{wall}^{l+1} = x_{wall}^{l} - 1$$

then solve next LCP

	l	x^l	v_x^l	f_n^l	ψ_n^l	x_{wall}^l
no contact	0	0	0	0	11	11
	1	1	1	0	9	10
	2	3	2	0	6	9
	3	6	3	0	2	8
impact	4	7	1	3	0	7
	5	6	-1	3	0	6
steady contact	6	5	-1	1	0	5
	7	4	-1	1	0	4
	8	3	-1	1	0	3
	⋮	⋮	⋮	⋮	⋮	⋮

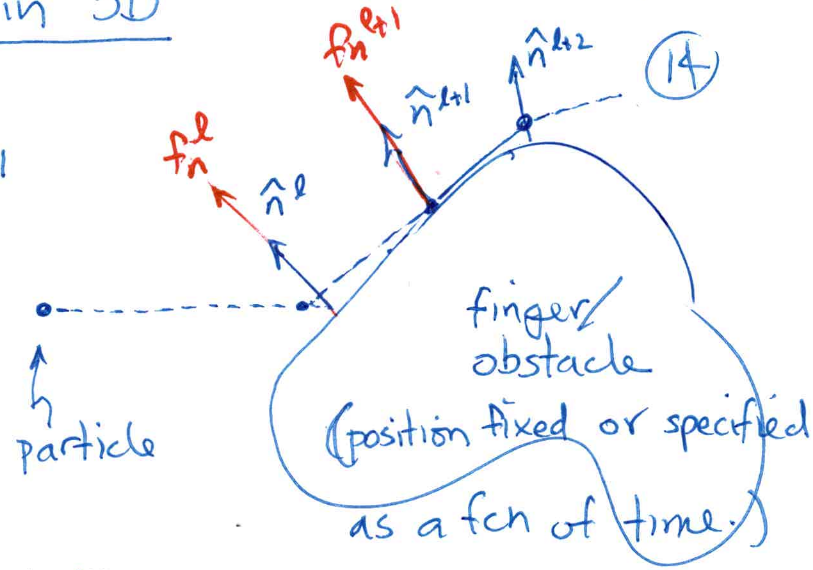
Frictionless particle in 3D

In 3D,

q, ν, g, n are now 3×1

$$M = m I_{3\times3}$$

mass of particle



$$\nu^{l+1} = \nu^l + hM^{-1}(g^l + \hat{n}^l f_n^{l+1})$$

$$q^{l+1} = q^l + hV\nu^{l+1} \quad \longleftarrow V = I_{(3\times3)}$$

$$0 \le f_n^{l+1} \perp \psi_n^{l+1} \ge 0$$

To make problem easier to solve linearize ψ_n^{l+1}

Taylor series

$$\psi_n^{l+1} \simeq \psi_n^l + \frac{\partial \psi_n^l}{\partial q}\Delta q + \frac{\partial \psi_n}{\partial t} h \ge 0$$

substitute $\Delta q = hV\nu^{l+1}$

$$\psi_n^{l+1} \simeq \psi_n^l + \underbrace{\frac{\partial \psi_n^l}{\partial q}V}_{\hat{n}^{l\,T}} h\nu^{l+1} + \frac{\partial \psi_n^l}{\partial t} h$$

It turns out that $\frac{\partial \psi_n}{\partial q} V = n^T$.

(In general, with rigid bodies, $\frac{\partial \psi_n}{\partial q} V = G_n^T$)

Now approximate $\dfrac{\partial \psi_n^l}{\partial t}$

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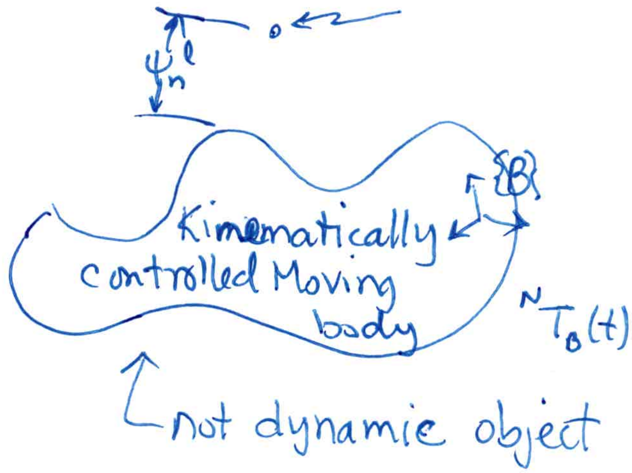
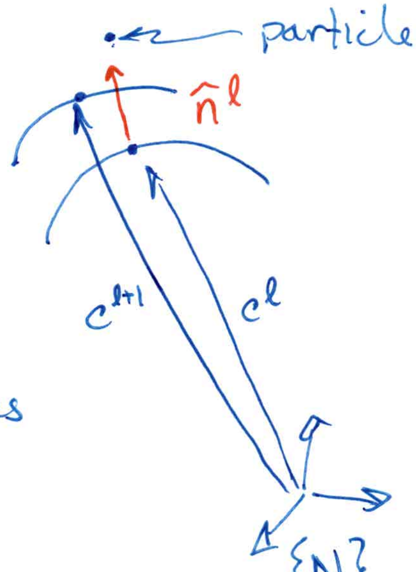
q is not an explicit fcn of time. It's motion is handled by $\dfrac{\partial \psi_n^l}{\partial q} \Delta q$.





$$\frac{\partial \psi_n^l}{\partial t} \simeq \hat{n}^{l^T} (c^{l+1} - c^l)$$

Now our approximation for ψ_n^{l+1} becomes

$$\psi_n^{l+1} \simeq \psi_n^l + \hat{n}^l \left(\nu^{l+1} h + \frac{c^{l+1} - c^l}{\cancel{h}} \cancel{h} \right)$$

$$\simeq \psi_n^l + \hat{n}^l \left(\nu^{l+1} h + c^{l+1} - c^l \right)$$

Note: when particle is replaced by a body of finite extent, $\hat{n}^T$ will be replaced by $G_n^T = \begin{bmatrix} \vdots & \vdots \\ \hat{n}^T & (r \times \hat{n})^T \\ \vdots & \vdots \end{bmatrix}$.

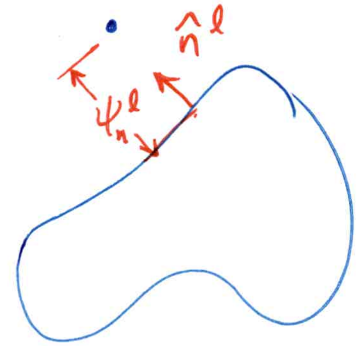
Substitute in expression for v^{l+1} to get a pure LCP.

To simulate, perform distance computation to get $\psi_n^l, \hat{n}^l$, compute c^{l+1} from known body motion, then formulate LCP, & solve.

$$0 \le f_n^{l+1} \perp \hat{n}^{lT} M^{-1} \hat{n} f_n^{l+1} + \hat{n}^{lT}\left(M^{-1} g^l h + v^l h + c^{l+1} - c^l\right) + \psi_n^l \ge 0$$

Update particle velocity

$$v^{l+1} = v^l + hM^{-1}\left(g^l + \hat{n}^{lT} f_n^{l+1}\right)$$



Update particle position

$$q^{l+1} = q^l + hVv^{l+1}$$

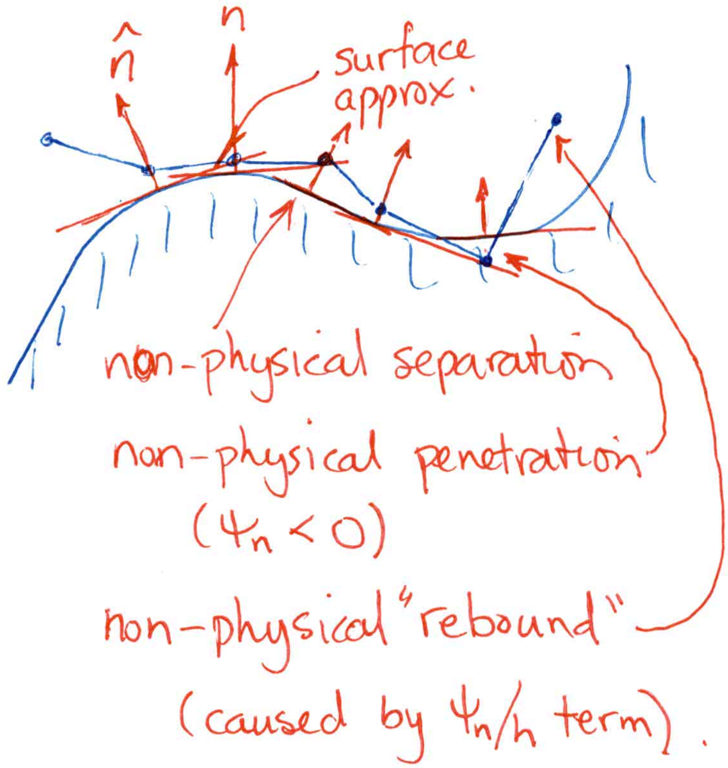
(recall $V = I_{3\times3}$ for particle)

Update pose of kinematically controlled body ~~from~~ as:

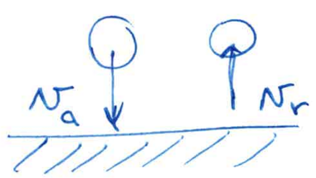
$${}^N T_B(t^{l+1})$$

Undesirable side effects of linearization

What we derived so far assumes (implicitly) that the coeff. of restitution is zero, but we can see "bounce" anyway.



<u>Coeff. of rest.</u>



$v_a = v_n$ approaching surface

$v_r = v_n$ rebounding from surface

Newton's hypothesis:

$$v_r = -\epsilon v_a$$

$\epsilon = 0 \Rightarrow$ all K.E. is lost (like soft clay ball).

$\epsilon = 1 \Rightarrow$ all K.E. is recovered. (like superball).

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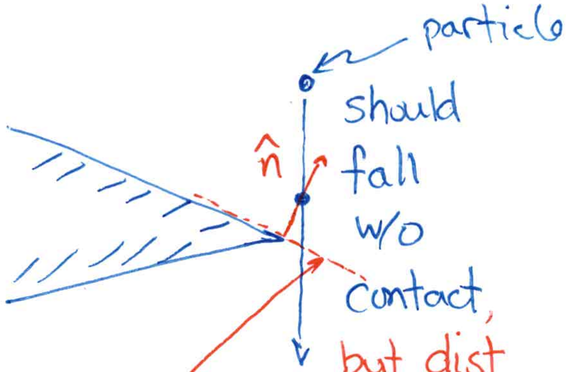
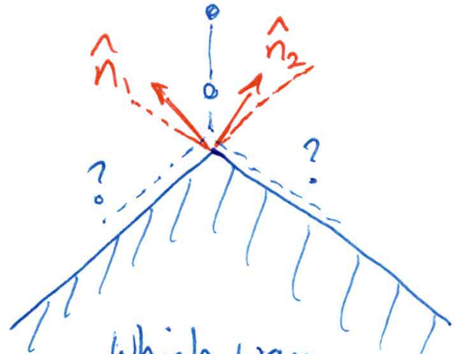
Side effects of polyhedral geometry



should fall w/o contact, but dist computation can find $\hat{n}$. This effectively extends the facet.



Which way will particle go? left or right?

It could even get trapped if both contacts are used!