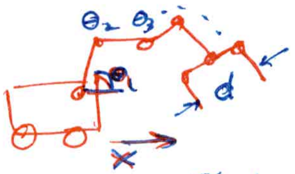
LaVallе Ch3

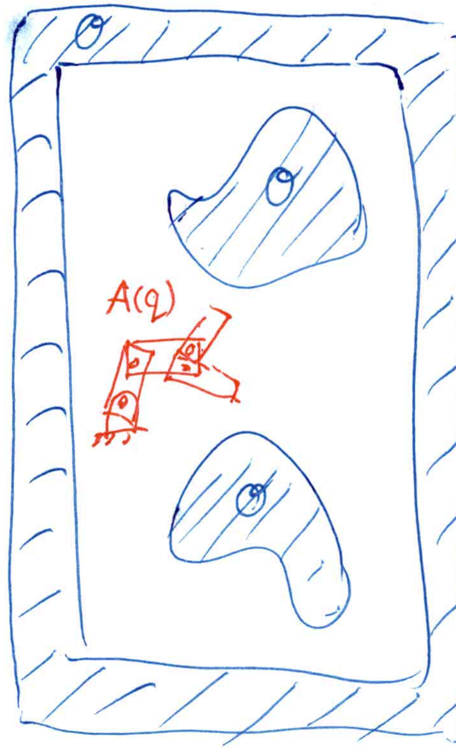
Sect 3.1 : Geometric Modeling

The "world," $\mathcal{W} = \mathbb{R}^2$ or $\mathbb{R}^3$

Obstacles, $\mathcal{O}$ = stationary objects

Robot, $\mathcal{A}(q)$ = moving object(s)
q is configuration





$\mathcal{O}$ & $\mathcal{A}$ will be defined as logical combinations of geometric primitives H_i

e.g. A point $a \in \mathcal{A}$ if

$$a \in H_1 \wedge (H_2 \vee H_3)$$

Recall set operations:

$\cup \triangleq$ union

$\cap \triangleq$ intersection

Logical analogues:

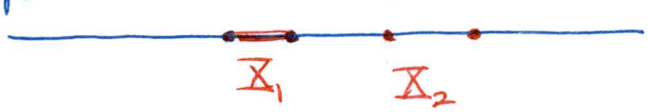
$\vee \triangleq$ or

$\wedge \triangleq$ and

Def: Convex Set

A subset X of $\mathbb{R}^n$ is convex iff for every pair of points $x_1, x_2 \in X$ the line segment joining x_1 & x_2 is in X

Example in $\mathbb{R}^1$



X_1 is convex

X_2 is convex

$X_1 \cup X_2$ is NOT convex

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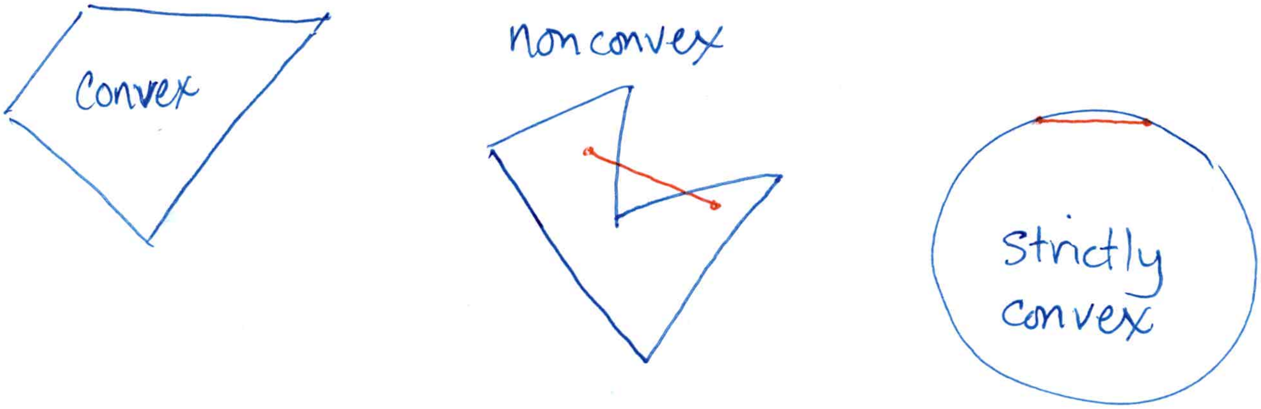
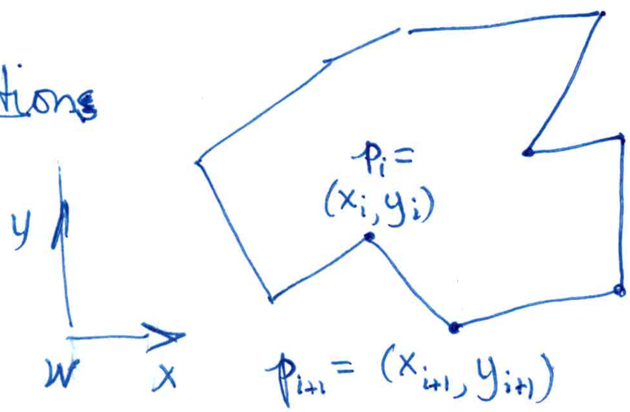


<u>Polygonal Models</u>

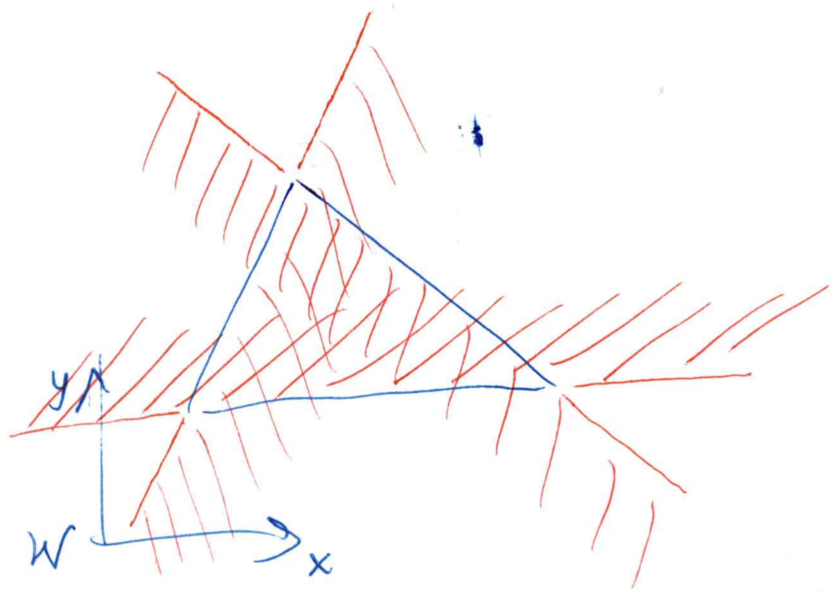
<u>Boundary representations</u>



- ccw set of ordered pairs of vertices (and edges they define)

<u>Solid representation</u>

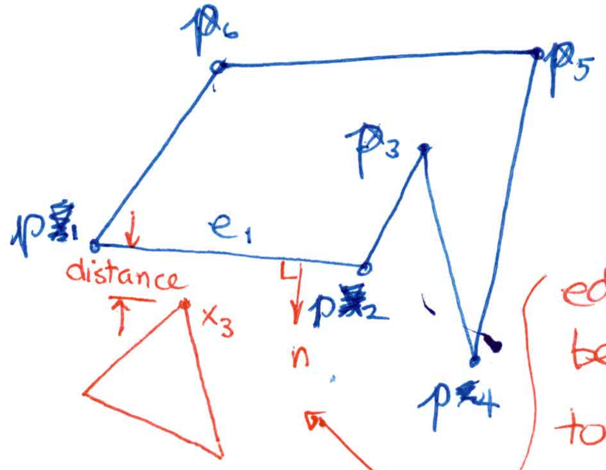
- Intersection of half-planes



Example

Two vertices, x_1 & x_2.

$$x_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \qquad x_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$



edges must be oriented to know outside & inside of polygon, e.g., Pgon = $\{x_1, x_2, \cdots, x_6\}$ ordered set.

bndry rep

e_1 = edge,

$e_1 = \lambda p_1 + (1-\lambda) p_2 \,, \quad 0 \le \lambda \le 1$

l_1 = line through p_1 & p_2.

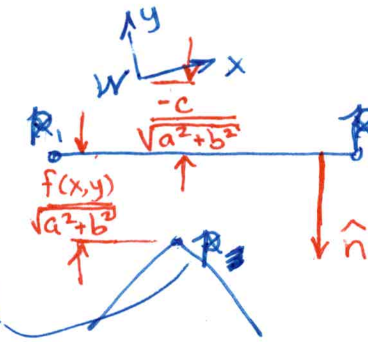
$l_1 = f(x,y) \Rightarrow ax + by + c = 0$

Sometime important to know distance of one polygon to another.

$$\text{distance} = f(x_3, y_3) / \sqrt{a^2+b^2}$$

$$\text{normal to edge} = \nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

geometric interpretation



$$\hat{n} = \frac{\begin{bmatrix} a \\ b \end{bmatrix}}{\sqrt{a^2+b^2}}$$

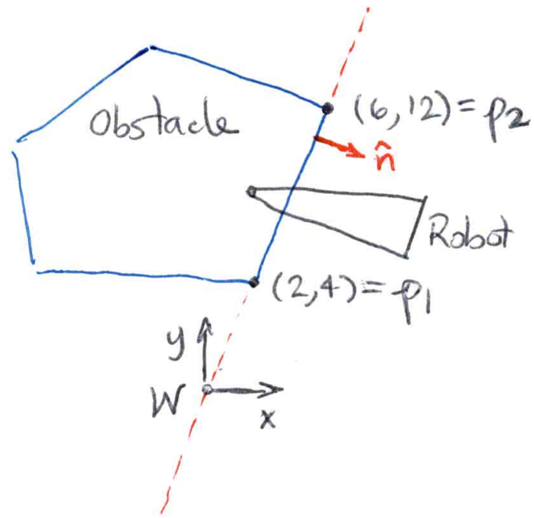
Example

Relevant edge

$(2,4) \rightarrow (6,12)$

$$f(x,y) = \frac{2}{\sqrt{5}}x - \frac{1}{\sqrt{5}}y + 0$$

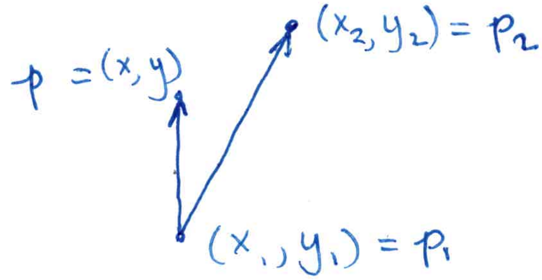
$$\hat{n} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \frac{1}{\sqrt{5}}$$



Formulas for a, b, c.

Given (x_1, y_1) & (x_2, y_2), determine a, b, c.

From plane geometry



point p is on the line defined by p_1 & p_2 iff

$$(p - p_1) \times (p_2 - p_1) = 0$$

Assume $p_1 \neq p_2$

$p_1 \neq p$

Substitute vectors for ~~p, p_1 & p_2~~ $\overrightarrow{p_1 p}$ and $\overrightarrow{p_1 p_2}$

$(x - x_1, y - y_1)$ and $(x_2 - x_1, y_2 - y_1)$.

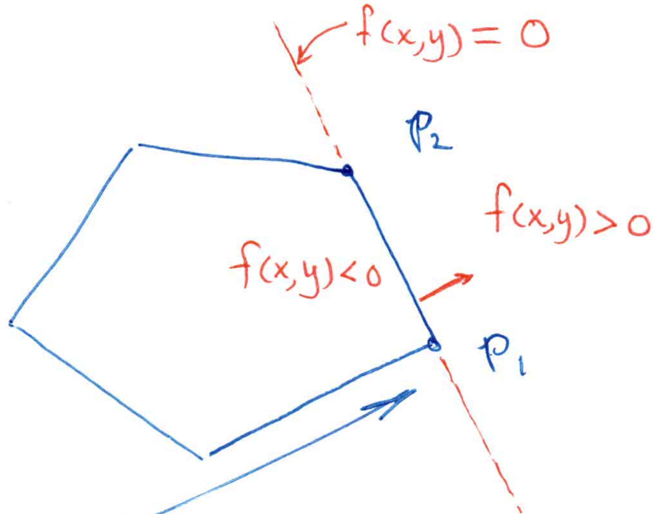
Expand and collect terms to get:

$$x\underbrace{(y_2 - y_1)}_{a} + y\underbrace{(x_1 - x_2)}_{b} + \underbrace{x_2 y_1 - y_2 x_1}_{c} = 0$$

The distance computation has led us to a ~~boundary~~ solid representation

Order points ccw.

Apply formulas from previous page to obtain $f(x,y)$ for each edge.



Let $f_i(x,y)$ be the function for edge i; $i = 1, \ldots, m$

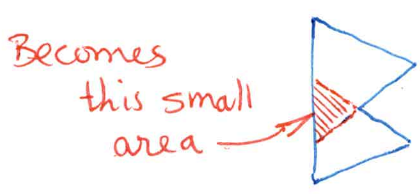
Let H_i be the half plane:

$$H_i = \{(x,y) \in W \mid f_i(x,y) \leq 0\}$$

An m-sided convex polygonal ~~obstacle~~ object is expressed as an intersection of half-plane primitives.

$$\mathcal{O} = H_1 \cap H_2 \cap \cdots \cap H_m$$

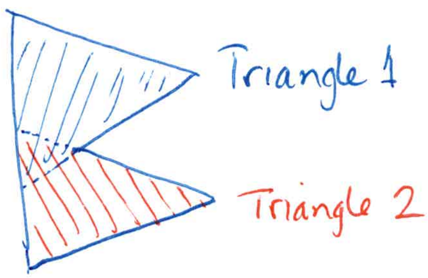
What goes wrong for non-convex object?



Representing Nonconvex Polygons

$$\mathcal{O} = \mathcal{O}_1 \cup \mathcal{O}_2 \cup \cdots \cup \mathcal{O}_n$$

Example: $\longrightarrow$



A nonconvex polygon has many possible ~~B~~ representations as a union of convex polygons.

It is always possible to represent polygons as a union of convex polygons, i.e., $\mathcal{O}_j = \bigcap_{i=1}^{m} H_i$ & $\mathcal{O} = \bigcup_{i=1}^{n} \mathcal{O}_i$

Collision detection predicate, ϕ.

Let $\phi : \mathcal{W} \to \{\text{TRUE}, \text{FALSE}\}$ return TRUE for each point in $\mathcal{W}$ that is in $\mathcal{O}$

Let $e_i(x,y)$ return ~~TRUE~~ if $f_i(x,y) \leq 0$, FALSE otherwise

$\alpha(x,y) = e_1(x,y) \wedge e_2(x,y) \wedge \cdots \wedge e_m(x,y)$ is a logical predicate that returns TRUE if (x,y) is <u>in or on</u> ~~the~~ a convex polygon.

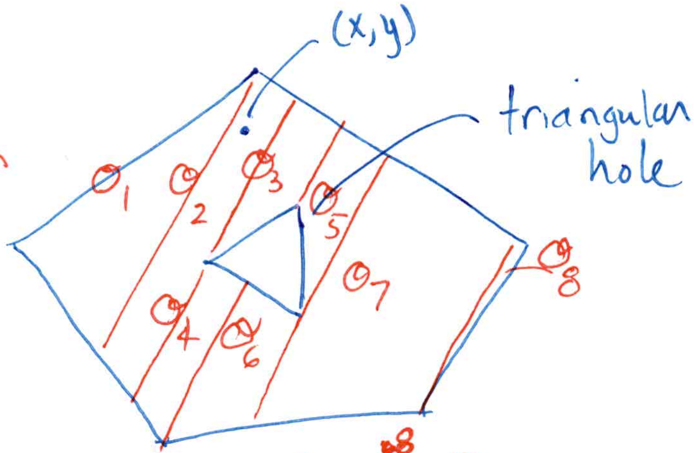
Given a convex decomposition of a polygon, $\exists$ a collision if $\phi(x,y) = \text{TRUE}$

$$\phi(x,y) = \alpha_1(x,y) \vee \alpha_2(x,y) \vee \cdots \vee \alpha_n(x,y).$$

Convex decomposition is not unique!

Pick a direction and make trapezoids

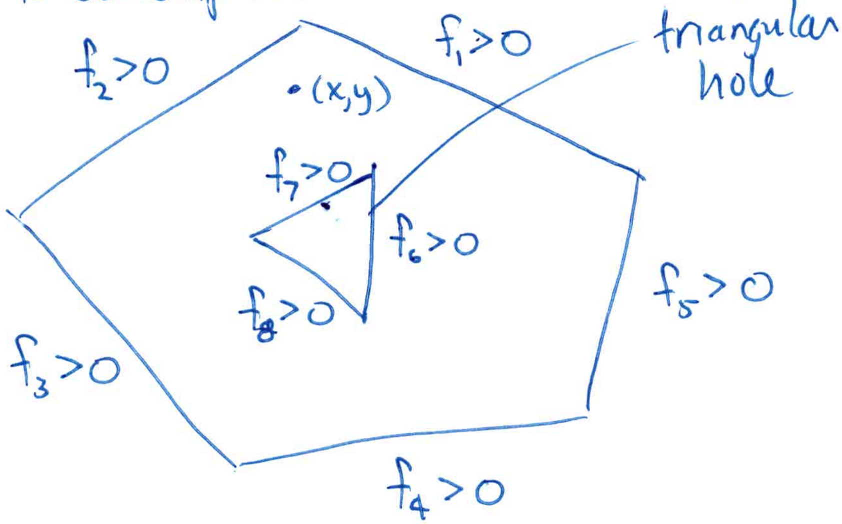
Then test point (x,y) in each trapezoid.



$$O = \bigcup_{i=1}^{8} O_i$$

Perhaps not necessary to decompose?

Construct a C.D. primitive. to test (x,y).



More efficient?

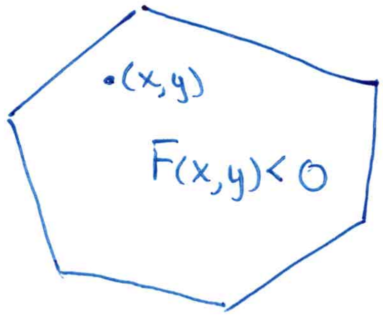
inside the pentagon → $(f_1<0 \wedge f_2<0 \wedge f_3<0 \wedge f_4<0 \wedge f_5<0)$

and → $\wedge$

outside the hole → $(f_6>0 \vee f_7>0 \vee f_8>0)$

Collision detection as a system of inequalities for a point and a polygon (convex):

$$F(x,y) = \begin{bmatrix} f_1(x,y) \\ f_2(x,y) \\ \vdots \\ f_m(x,y) \end{bmatrix} \leq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$



where vector inequality is defined to apply element-by-element.

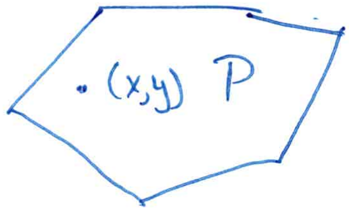
The point (x,y) is inside iff $f_i(x,y) < 0 \ \forall i$

Complexity of collision detection

Let E be the set of edges of a convex polygon.

$m = \quad = \#$ of edges $= |E|$

$m = \quad \overset{\text{also}}{=} \#$ of vertices $= |E|$



Determining if (x,y) is in polygon P

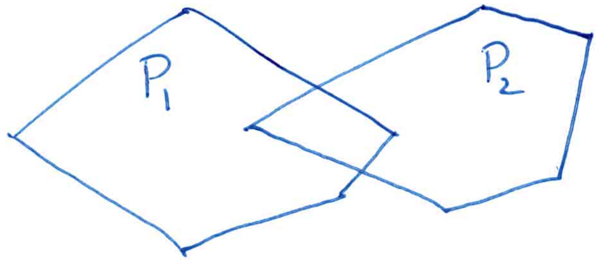
requires at most m evaluations of $f_i(x,y)$

(at most, because if any $f_i(x,y) > 0$, stop!)

$\therefore$ Collision checking of a point is $\mathcal{O}(m)$ where m is # of edges

Collision checking of two convex polygons

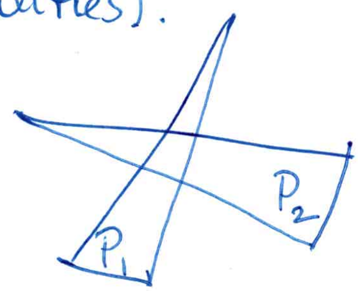
m_i = # edges of P_i



e-v: Check all vertices of P_1 inside P_2 requires checking $m_1 \cdot m_2$ inequalities

v-e: Check all vertices of P_2 inside P_1 requires $m_1 \cdot m_2$ ineqs.

e-e: Check all edges of P_1 against P_2 requires $m_1 \cdot m_2$ intersection tests (with inequalities).



Let $m = \max(m_1, m_2)$.

$\boxed{\text{P-P collision checking is } O(m^2)}$

Note: If you only need boolean result, then no need to do all $3m^2$ function calls when collision exists.

Must do all $3m^2$ to be sure they do not overlap.

Note: More efficient algorithms exist for boolean ~~result~~ check.

N. Amato, 1993: serial alg is $\Theta(n)$, where n is $2m$.

parallel on $O(n)$ processors is $O(\log n)$.

This is for ~~con~~ nonconvex, simple polygons.

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Recall some ideas from Asymptotic Analysis

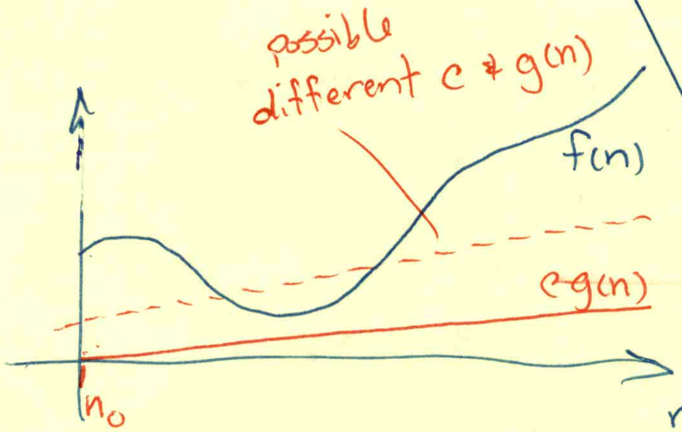
Big 'O'. O upper bound.

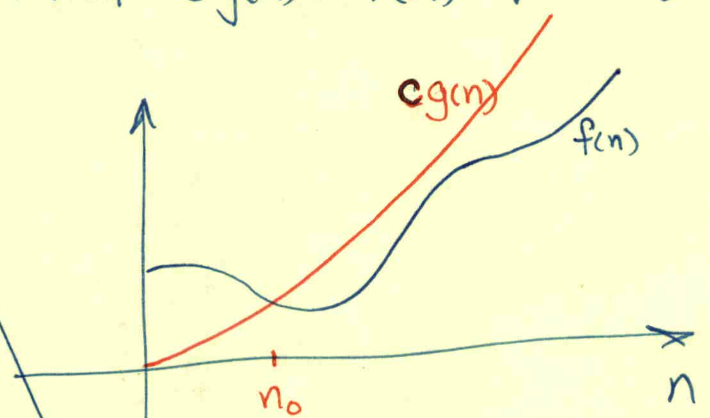
Big 'Ω'. Ω lower bound.

Big 'Θ'. Θ bound from above & below.

O: $f(n)$ is $O(g(n))$ if $\exists$ c & $n_0 > 0$

such that $c g(n) > f(n)$ $\forall$ $n > n_0$





Ω: $f(n)$ is $\Omega(g(n))$ if $\exists$ c & n_0 $\ni$ $0 < c g(n) < f(n)$ $\forall$ $n > n_0$

$c, n_0 > 0$

Θ: $f(n)$ is $\Theta(g(n))$ if $\exists$ $c_1, c_2,$ & $n_0 > 0$ $\in$

$0 < c_1 g(n) < f(n) < c_2 g(n)$ $\forall$ $n > n_0$

Simple example:

let $f(n) = 2n^2 + 14$

Is $f(n)$ $O(n)$?

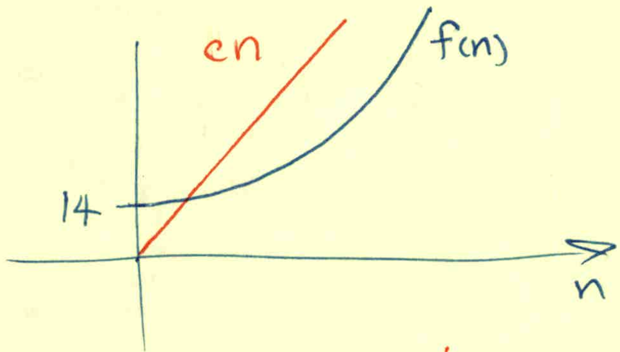
Does $\exists$ c and $n_0 \ni cn > 2n^2 + 14 \quad \forall n > n_0$?

Asked another way, for a fixed c, is

$-2n^2 + cn - 14 > 0$ as $n \to \infty$?

Since $-2n^2 < 0$ dominates, the answer is "No!"

(But $f(n)$ is $\Omega(n)$.)



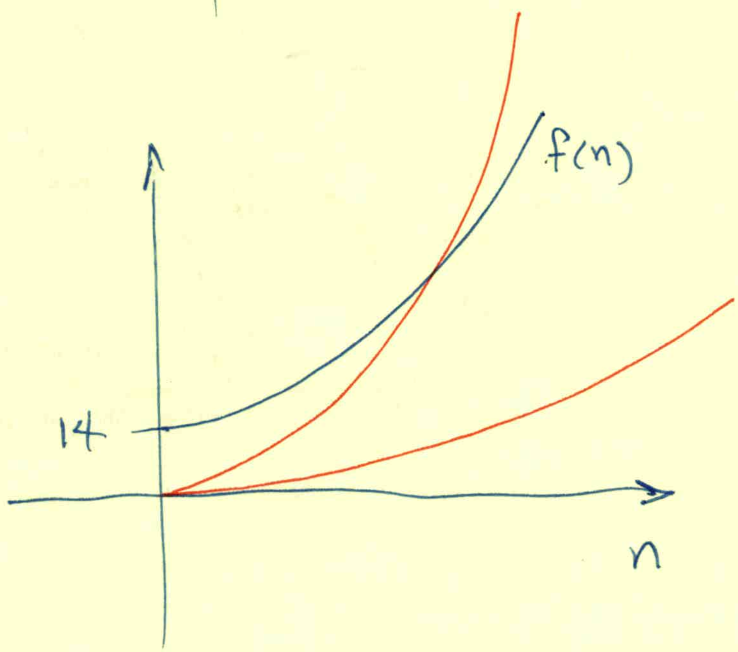
Is $f(n)$ $O(n^2)$?

~~Is $f(n)$ $O(n^3)$?~~

Yes, Choose $c > 2$,

then $(c-2)n^2 - 14 > 0$

for large enough values of n.



Is $f(n)$ $O(n^3)$?

Yes, because n^3 dominates

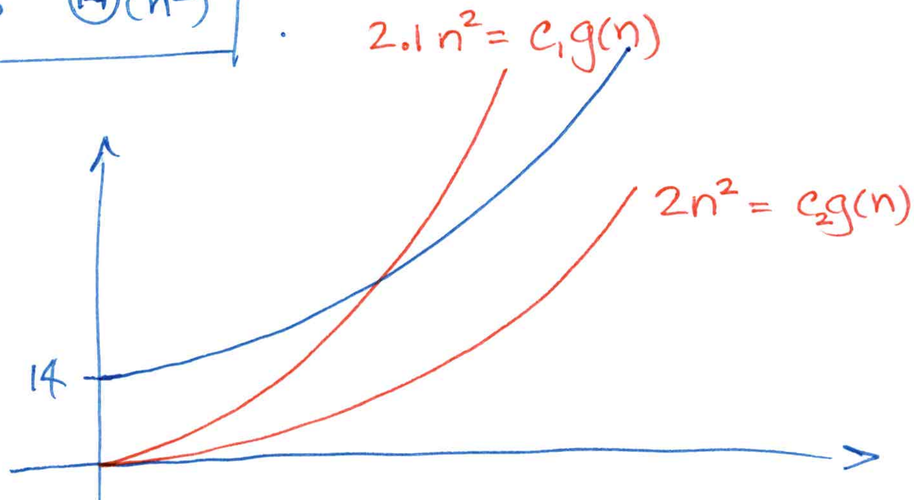
$c_{\ast}n^3 - 2n^2 - 14 > 0 \quad \forall n$ large enough

$f(n)$ is also $O(n^4)$, but it gives much less info about complexity than $O(n^2)$, which is the tightest bound.

Similarly $f(n)$ is $\Omega(n^2)$, $\Omega(n \log n)$, $\Omega(n)$, ...
but $\Omega(n^2)$ is tight.

Since $f(n)$ is $O(n^2)$ and $\Omega(n^2)$, then

$\boxed{f(n) \text{ is } \Theta(n^2)}$.



As long as $c_1 > 2$ we get $O(n^2)$

If $c_2 \leq 2$, we get $\Omega(n^2)$

Polyhedral Models

Replace half-spaces in $\mathbb{R}^2$ w/ half-spaces in $\mathbb{R}^3$

Replace vertex/edge boundary with vertex/edge/face boundary.

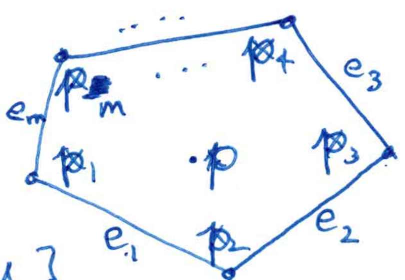
Boundary representation

vertices, edges, & faces.

faces are "oriented" (to define outside & inside)

view each face from outside, then vertices are ordered ccw.

for convex face,



$$\text{Face} \triangleq \{p \mid p = \sum_{i=1}^{m} \alpha_i p_i ; \; \alpha_i \geq 0, \; \sum_{i=1}^{m} \alpha_i = 1\}$$

Face as given, is the convex hull of $p_i \; \forall i$.

Note: given a point p, ~~if x is inside the face,~~ ~~then~~ $\alpha_1, \ldots, \alpha_m$ are not (in general) unique.

Why? $\sum_i \alpha_i p_{ix} = p_x$, $\sum_i \alpha_i p_{iy} = p_y$, $\sum_i \alpha_i = 1$ ← 3 eqs.

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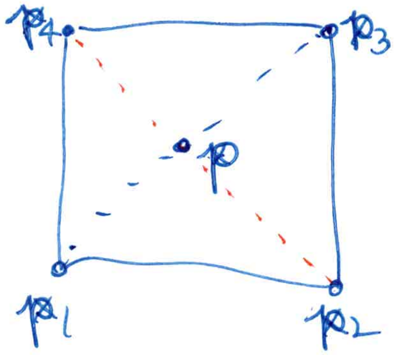
Examples of non uniqueness

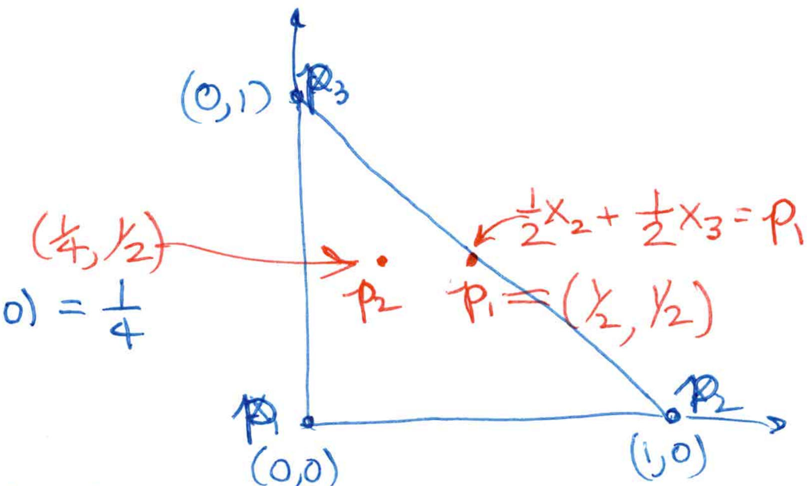
p is in center →



$$p = \tfrac{1}{2}p_1 + \tfrac{1}{2}p_3$$
$$= \tfrac{1}{2}p_2 + \tfrac{1}{2}p_4$$

If the face is a triangle, then the alphas, the barycentric coordinates, are unique.

Example



p_2 solution.

x coord: $\alpha_1(0) + \alpha_2(1) + \alpha_3(0) = \frac{1}{4}$

$\Rightarrow \boxed{\alpha_2 = \frac{1}{4}}$

y coord: $\alpha_1(0) + \alpha_2(0) + \alpha_3(1) = \frac{1}{2}$

$\Rightarrow \boxed{\alpha_3 = 1/2}$

$\sum_{i=1}^{3} \alpha_i = 1 \longrightarrow \alpha_1 + \frac{1}{4} + \frac{1}{2} = 1 \longrightarrow \boxed{\alpha_1 = \frac{1}{4}}$

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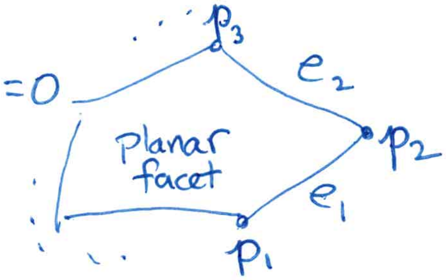
(14)

Solid representation

Same as 2D, but use planes "supporting" faces rather than lines "supporting" edges.

plane equation:

$f(x,y,z) = ax + by + cz + d = 0$



Given a planar face, how can we compute $a, b, c, \& \; d$?

Recall $e_1 \times e_2$ = normal vector to facet $= \vec{n}$

if $p_1, p_2, \ldots$ ccw and face is convex,

then $e_i \times e_{i+1 \,(\mathrm{mod}\; m)}$ is outward normal.

Unit normal $\hat{n} = \dfrac{(p_2 - p_1) \times (p_3 - p_2)}{\|(p_2 - p_1) \times (p_3 - p_2)\|} = \dfrac{e_1 \times e_2}{\|e_1 \times e_2\|}$

Recall gradient of $f = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. $\therefore$ $\boxed{\begin{bmatrix} a \\ b \\ c \end{bmatrix} \propto \hat{n}}$

Divide by $\sqrt{a^2 + b^2 + c^2}$ to get $\hat{n}$

Plug p_i into $ax_i + by_i + cz_i + d = 0$, so

$\boxed{d = -(ax_i + by_i + cz_i)}$, where $p_i = (x_i, y_i, z_i)$.

Formulas for a,b,c. from cross product definition.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ (e_1)_x & (e_1)_y & (e_1)_z \\ (e_2)_x & (e_2)_y & (e_2)_z \end{vmatrix} = \frac{\begin{bmatrix} (e_1)_y(e_2)_z - (e_1)_z(e_2)_y \\ (e_1)_z(e_2)_x - (e_1)_x(e_2)_z \\ (e_1)_x(e_2)_y - (e_1)_y(e_2)_x \end{bmatrix}}{(\text{magnitude of numerator})} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \hat{n}$$

Note: to make normal vector pointing inward, negate a,b,c. The corresponding equation for the plane requires negating a,b,c,d!

Solid rep of convex polyhedron

$$f_i(x,y,z) = a_i x + b_i y + c_i z + d_i \; ; \; i = \{1, 2, \dots |F|\}$$

number of faces

The solid is: $f_1 \le 0 \wedge f_2 \le 0 \wedge \dots \wedge f_{|F|} \le 0$

Solid rep of non convex polyhedron requires disjunctions:

$$(f_1 \le 0 \wedge f_2 \le 0 \wedge \dots \wedge f_m \le 0) \vee (f_{m+1} \le 0 \wedge \dots)$$

(Example in disjunctive normal form, but other forms are possible.)

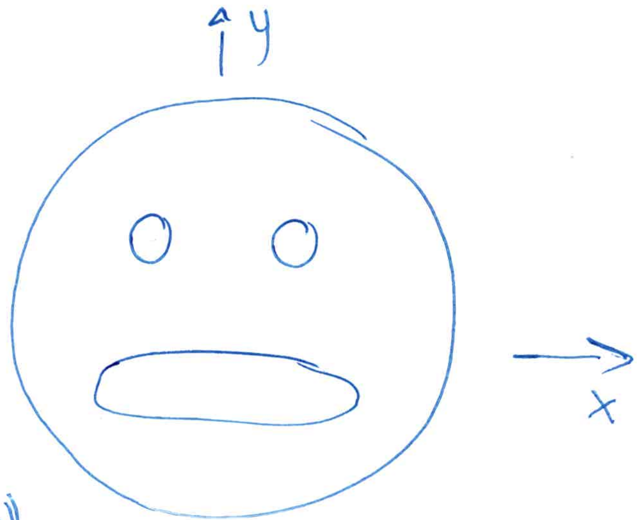
Other geometric models

$$H_i = \{(x,y,z) \mid f_i(x,y,z) \leq 0\}$$

If f_i is a polynomial, then H_i is an ~~semi~~ algebraic set

Sets formed by boolean operations on algebraic sets are known as semi-algebraic sets.

Example.
Fig 3.4



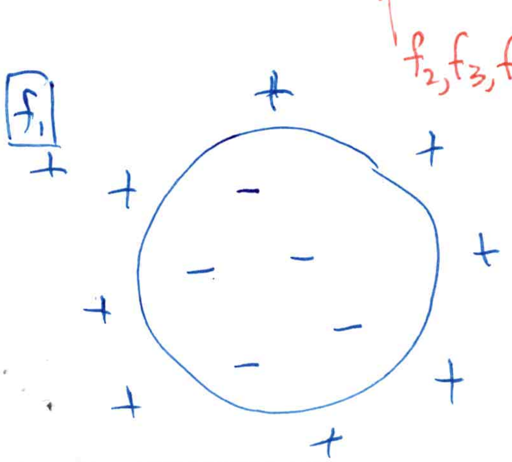
Head: $f_1 = x^2 + y^2 - r_1^2$

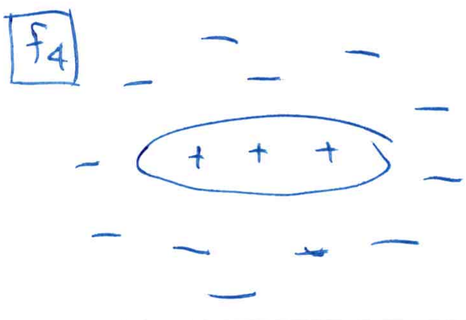
Eyes: $f_2 = -\left((x-x_2)^2 + (y-y_2)^2 - r_2^2\right)$

$f_3 = -\left((x-x_3)^2 + (y-y_2)^2 - r_2^2\right)$

Mouth: $f_4 = -\left(x^2/a^2 + (y-y_4)^2/b^2 - 1\right)$

f_2, f_3, f_4: leading minus signs & turn those inside out.





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Semi-algebraic sets can represent curved shapes more efficiently than polygons & polyhedra.

fewer primitives are necessary.

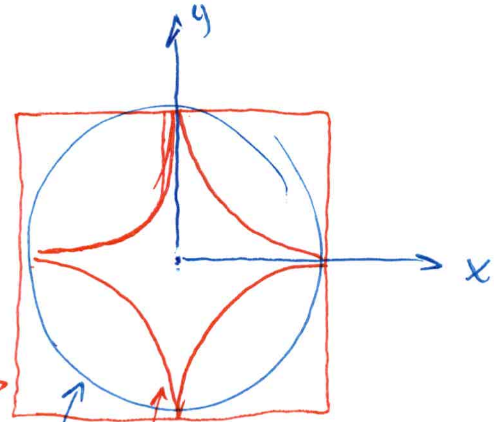
Extension to 3D shapes is trivial. Instead of polynomials in (x, y), use polynomials in (x, y, z). Everything else is the same.

Other models

Superquadrics $|x|^r + |y|^s + |z|^t - 1 \leq 0$; $s, r, t \in \mathbb{R}^+$

Subclass of superquadrics: superellipsoid $\left(\left|\frac{x}{a}\right|^{n_1} + \left|\frac{y}{b}\right|^{n_2} \right)^{n_1/n_2} + \left|\frac{z}{c}\right|^{n_1} - 1 \leq 0$; a, b, c, $n_1 n_2 \in \mathbb{R}^+$

Look at (x, y) cross-section with $n_1 = n_2$



square $\lim_{n_1 \to \infty}$

circle: $n_1 = 2$

$\lim_{n_1 \to 0}$

Check out wikipedia page for more.