LaValle 3.2: Rigid Body Transformations

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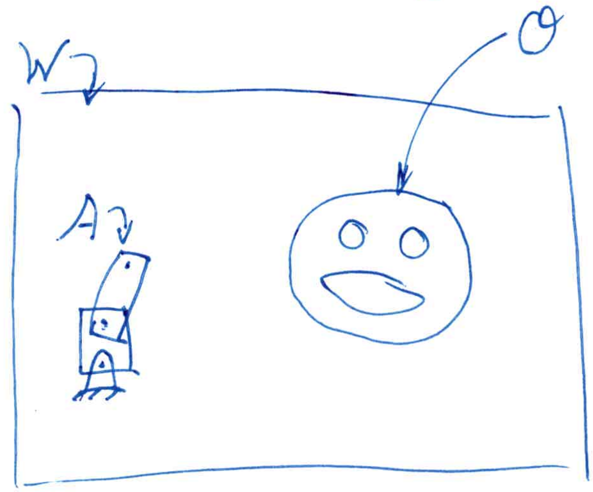
①

Recall $\mathcal{W}$, $\mathcal{O}$, $\mathcal{A}$.

$\mathcal{O}$ is fixed, but $\mathcal{A}$ moves.

∴ We need to t'form $\mathcal{A}$'s geometry to check for collision and plan grasps.



Let q denote the robot's config.

Let $a \in \mathcal{A}$ be a point on the robot.

Let $\mathcal{A}(q)$ represent all points in $\mathcal{W}$ that are in the robot when the robot's configuration is q.

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②

3.2.1: General Concepts

Rigid body transformation -

$h : A \to W$

In set notation:

$h(A) = \{ h(a) \in W \mid a \in A \}$

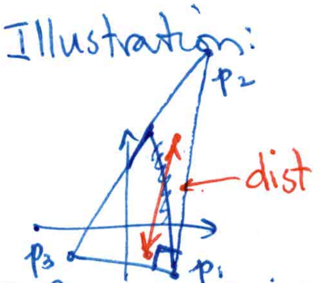
h maps every $a \in A$ to $w \in W$ such that

1) distance between any pair of points is unchanged.

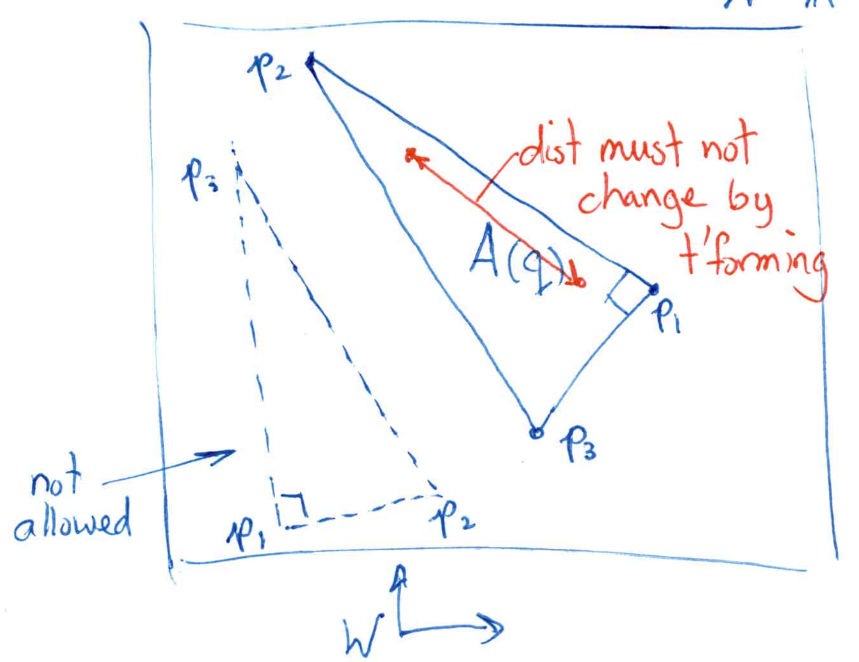
2) orientation of A must be preserved.

$W = \mathbb{R}^2$ or $\mathbb{R}^3$

Illustration:

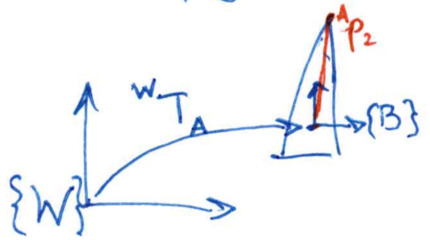


$\{B\}$ body-fixed frame in which geometry is defined



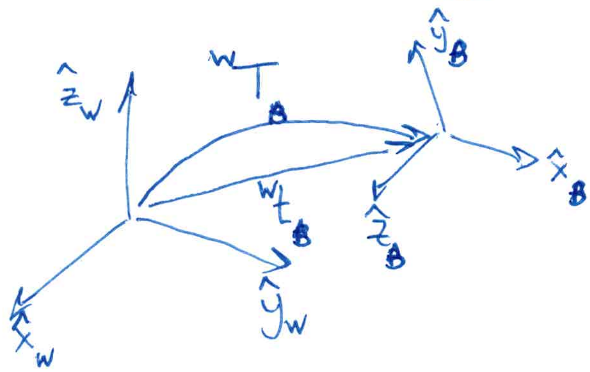
T'form boundard rep model

Simply transform all vertices (and edges).



$${}^W T_B \left[{}^B\tilde{p}_1 \; {}^B\tilde{p}_2 \cdots {}^B\tilde{p}_m \right] = \left[{}^W\tilde{p}_1 \; {}^W\tilde{p}_2 \cdots {}^W\tilde{p}_m \right]$$

where $${}^{W}T_{B} = \left[\begin{array}{c|c} {}^{W}R_{B} & {}^{W}t_{B} \\ \hline 0\;0\;0 & 1 \end{array}\right]$$



$${}^{W}R_{B} = \left[{}^{W}\hat{x}_{B} \;\vdots\; {}^{W}\hat{y}_{B} \;\vdots\; {}^{W}\hat{z}_{B} \right]_{(3\times 3)} = \text{Rotation Matrix}$$

$${}^{W}t_{B} = \left[{}^{B}x_t \;\; {}^{B}y_t \;\; {}^{B}z_t \right]^T = \text{Translation vector}$$

T'form solid model based on geometric primitives such as

$$H_i = \{ a \in \mathbb{R}^2 \mid f_i(a) \le 0 \}$$

robot defined in a space that is not necessarily W.

Using rigid body t'form defined above,

$$h(H_i) = \{ h(a) \in W \mid f_i(a) \le 0 \}$$

also using $h(a) \in W$, so let $w = h(a)$, and $a = h^{-1}(w)$, we get

$$h(H_i) = \{ w \in W \mid f_i(h^{-1}(w)) \le 0 \}$$

Key point: Don't t'form vertices and rebuild the solid rep. Instead $\boxed{\text{t'form points of interest in W into B}}$ and apply the predicates.

3.2.2.: 2D T'forms

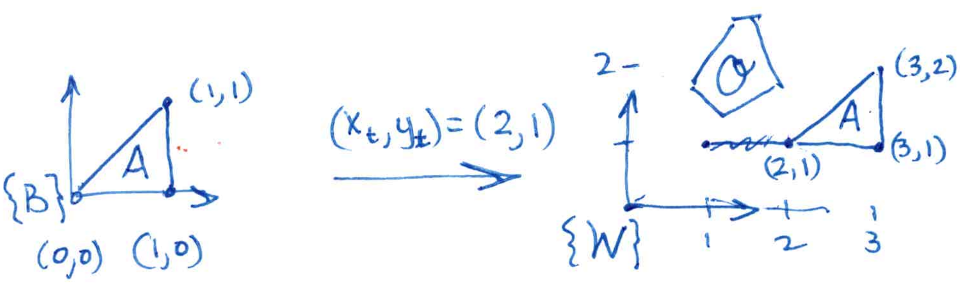
Translation of robot, A, by (x_t, y_t). Every pt $\overbrace{(x,y)}^{\text{in}\{B\}}$ in $\{B\}$ moves to $\{W\}$

$$h(x,y) = (x+x_t,\ y+y_t) = {}^{w}p$$

For bndry rep, replace every ~~point~~ vertex $\overbrace{(x,y)}^{\text{in}\{B\}}$ w/ $\overbrace{(x+x_t,\ y+y_t)}^{\text{in}\{W\}}$

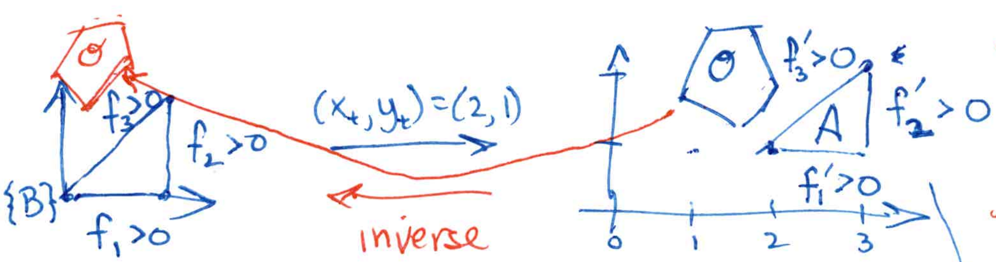
For solid rep, replace $\overbrace{(x,y)}^{\text{in}\{W\}}$ in functions with $(x-x_t,\ y-y_t)$. (Must use points in $\{B\}$).

Bndry Rep



Just move bndry rep to W and then do ~~comput~~ computations

Solid Rep



compute new bndry rep in W?

No!

Move points of interest in W into $\{B\}$

$$\begin{cases} f_1 = -y \\ f_2 = x - 1 \\ f_3 = -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y \end{cases}$$

$$\left.\begin{aligned} f_1' &= -y + 1 \\ f_2' &= x - 3 \\ f_3' &= -\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y + \frac{1}{\sqrt{2}} \end{aligned}\right\}$$

Example

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(5)

Assume we are interested in checking if the point ${}^W(5/2, 5/4)$ is in collision w/ A (which is translated)

If we use t'formed primitives, we get

$$f_1' = -\frac{1}{4} \qquad f_2' = -\frac{1}{2} \qquad f_3' = \frac{-1}{4\sqrt{2}}$$

If we t'form $\left(\frac{5}{2}, \frac{5}{4}\right)$ to frame of A, we get ${}^B\left(\frac{1}{2}, \frac{1}{4}\right)$

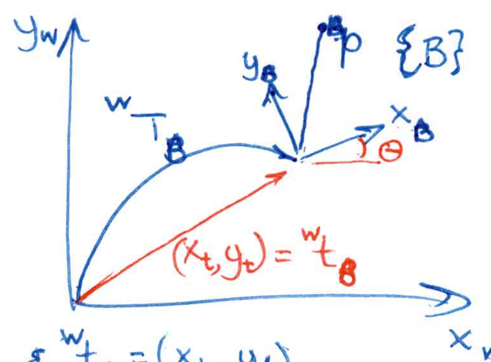
$$f_1 = -\frac{1}{4} \qquad f_2 = -\frac{1}{2} \qquad f_3 = \frac{-1}{4\sqrt{2}}$$

Same answers (of course), but one direction of t'form could be more work than the other.

Combine rotation & translation of A.

Let $\tilde{p}$ be homogeneous form of a point. $\tilde{p} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$, $p = \begin{bmatrix} x \\ y \end{bmatrix}$

Let ${}^W T_B$ be homogeneous t'form from $\{W\}$ to $\{B\}$



$$h(x,y) = {}^W T_B \, {}^B\tilde{p} = {}^W\tilde{p}$$

$${}^W T_B = \left[\begin{array}{c|c} {}^W R_B & {}^W t_B \\ \hline 0 & 1 \end{array}\right], \text{ where } {}^W R_B = \begin{bmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{bmatrix} \;\&\; {}^W t_B = (x_t, y_t)$$

The inverse t'form.

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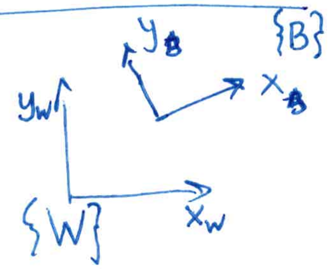
$$h^{-1}(x,y) = \left({}^{W}T_{B}\right)^{-1}\, {}^{W}\tilde{p} = {}^{B}\tilde{p}$$

where recall $\left({}^{W}T_{B}\right)^{-1} = \left[\begin{array}{c|c} {}^{W}R_{B}^{T} & -{}^{W}R_{B}^{T}\,{}^{W}t_{B} \\ \hline 0 & 1 \end{array}\right]$

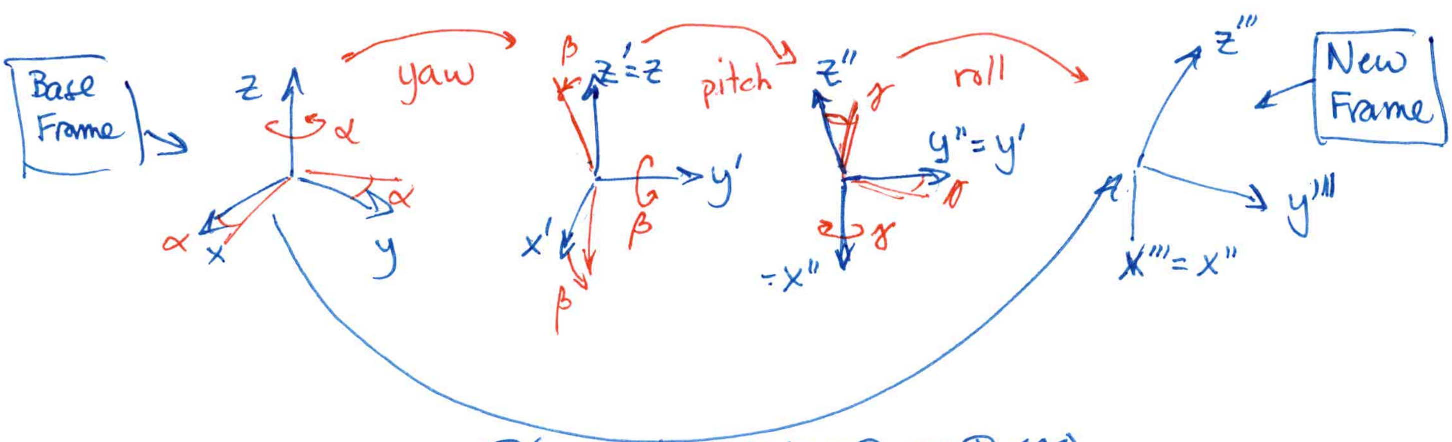
Also recall, ${}^{W}R_{B} = \left[\, {}^{W}\hat{x}_{B} \,\middle|\, {}^{W}\hat{y}_{B} \,\right]$



3.2.3. : 3D T'forms

The same ideas hold. But 3D rotation are more complicated.

One way to construct an arbitrary orientation is called roll-pitch-yaw angles. (LaValle gives expression for yaw-pitch-roll, similar.)



$$R(\alpha,\beta,\gamma) = R_z(\alpha)\,R_y(\beta)\,R_x(\gamma)$$

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where $R_z(\alpha) = \begin{bmatrix} c_\alpha & -s_\alpha & 0 \\ s_\alpha & c_\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$s_\alpha = \sin(\alpha) \qquad c_\alpha = \cos(\alpha)$

$R_y(\beta) = \begin{bmatrix} c_\beta & 0 & s_\beta \\ 0 & 1 & 0 \\ -s_\beta & 0 & c_\beta \end{bmatrix} \qquad R_x(\gamma) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\gamma & -s_\gamma \\ 0 & s_\gamma & c_\gamma \end{bmatrix}$

Note: $R(\alpha, \beta, \gamma)$ has elements which are polynomials (trilinear)

in $c_\alpha, s_\alpha, c_\beta, s_\beta, c_\gamma, s_\gamma$

Given 9 numbers in $R(\alpha, \beta, \gamma)$, one can recover

α, β, γ in almost all cases.

Singularity occurs if $r_{11} = 0$ or $r_{33} = 0$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

simplest expressions

$$\begin{cases} \alpha = \operatorname{atan2}(r_{21}, r_{11}) \\ \beta = \operatorname{atan2}\left(-r_{31}, \sqrt{r_{32}^2 + r_{33}^2}\right) \\ \gamma = \operatorname{atan2}(r_{32}, r_{33}) \end{cases}$$

$r_{11} = c_\alpha c_\beta$

$r_{21} = s_\alpha c_\beta$

$r_{31} = -s_\beta \qquad r_{32} = c_\beta s_\gamma \qquad r_{33} = c_\beta c_\gamma$

3D Combined Translation & Rotation

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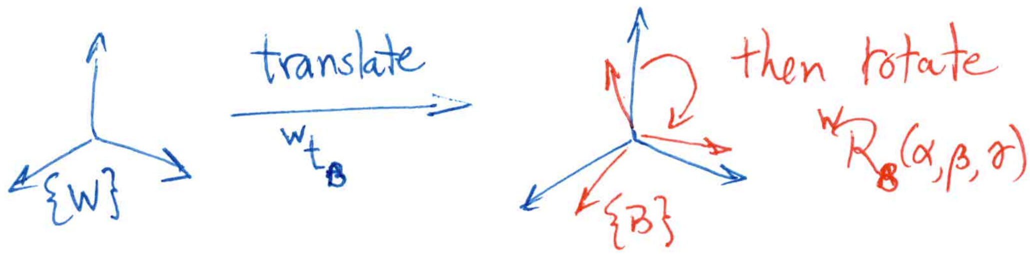

$${}^{W}T_{B} = \left[\begin{array}{c|c} {}^{W}_{B}R(\alpha,\beta,\gamma) & {}^{W}t_{B} \\ \hline 0\ \ 0\ \ 0 & 1 \end{array}\right]$$

Let $Transl_x(x_t)$ be the homogenous t'form that describes a frame translated x_t units along the ~~xx~~ x-axis of original frame.

Similarly define $Transl_y(y_t)$ and $Transl_z(z_t)$.

Then $${}^{W}T_{B} = Transl_x(x_t)\, Transl_y(y_t)\, Transl_z(z_t)\, R_z(\alpha)\, R_y(\beta)\, R_x(\gamma)$$

Prove transl then rotate $\neq$ rotate then transl

$$\left[\begin{array}{c|c} R & 0 \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c|c} I & t \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} R & Rt \\ \hline 0 & 1 \end{array}\right]$$

$\neq$

$$\left[\begin{array}{c|c} I & t \\ \hline 0 & 1 \end{array}\right]\left[\begin{array}{c|c} R & 0 \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} R & t \\ \hline 0 & 1 \end{array}\right]$$

Coordinate Frame Assignment by Denavit-Hartenberg Method

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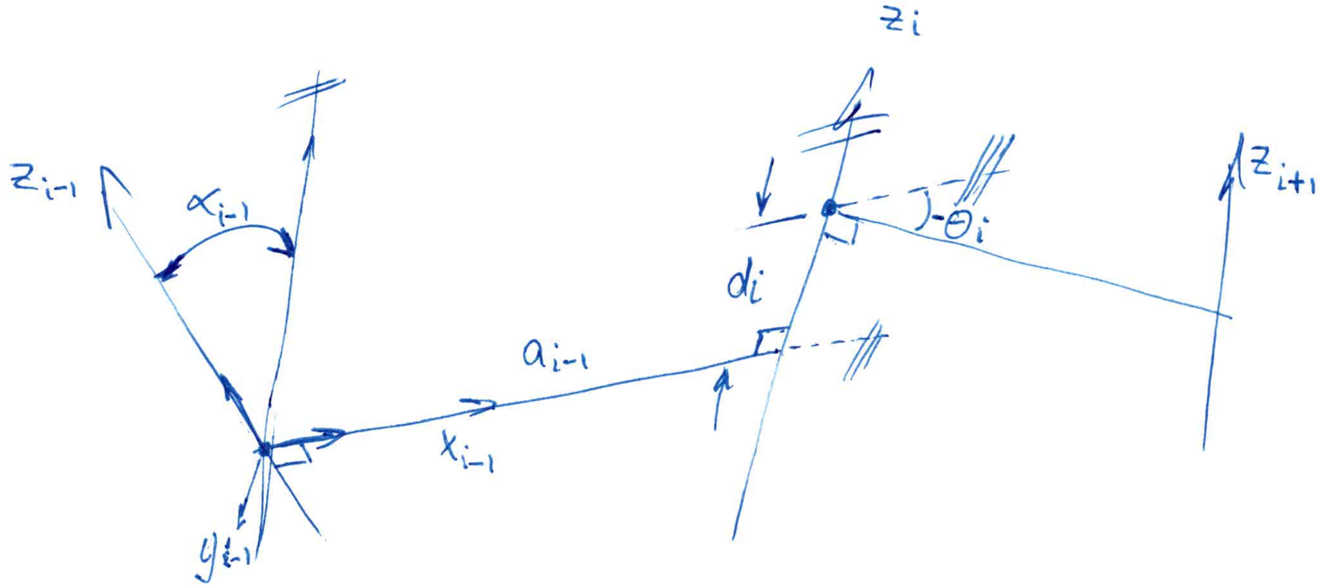
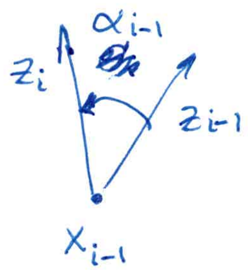
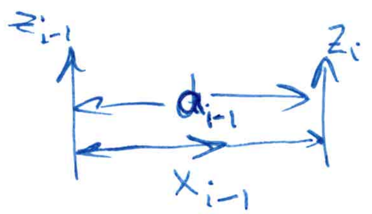
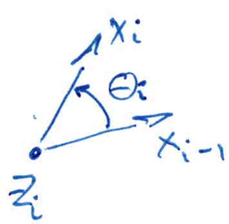
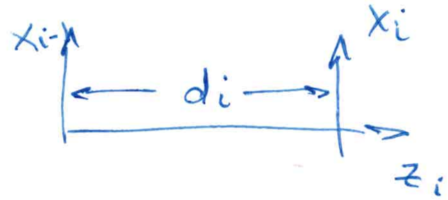
$$R_{x_{i-1}}(\alpha_{i-1})\ \mathrm{Transl}_{x_{i-1}}(a_{i-1})\ \mathrm{Transl}_{z_i}(d_i)\ R_{z_i}(\theta_i)$$











LaValle 3.3: Transforming Kinematic Chains

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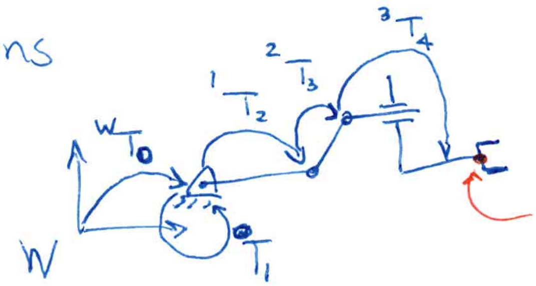
Planar Chains



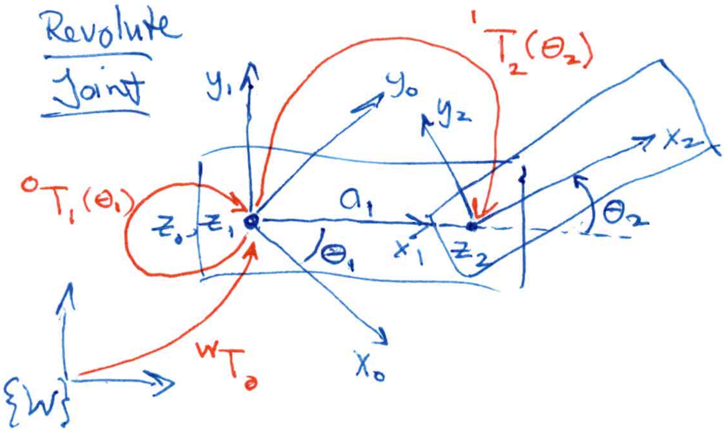
point known in $\{B_4\}$; ^4p

Express ^4p in frame {W}

$$^Wp = {}^WT_0\,{}^0T_1(q_1)\,{}^1T_2(q_2)\,{}^2T_3(q_3)\,{}^3T_4(q_4)\,{}^4p$$

Assigning Transforms in the plane

Revolute Joint

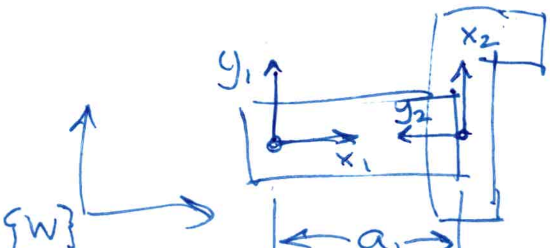


$$^1T_2(\theta_2) = \mathrm{Transl}_x(a_1)\,R_z(\theta_2) =$$

$$= \begin{bmatrix} c_2 & -s_2 & a_1 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where $c_2 \triangleq \cos(\theta_2)$, $s_2 \triangleq \sin(\theta_2)$

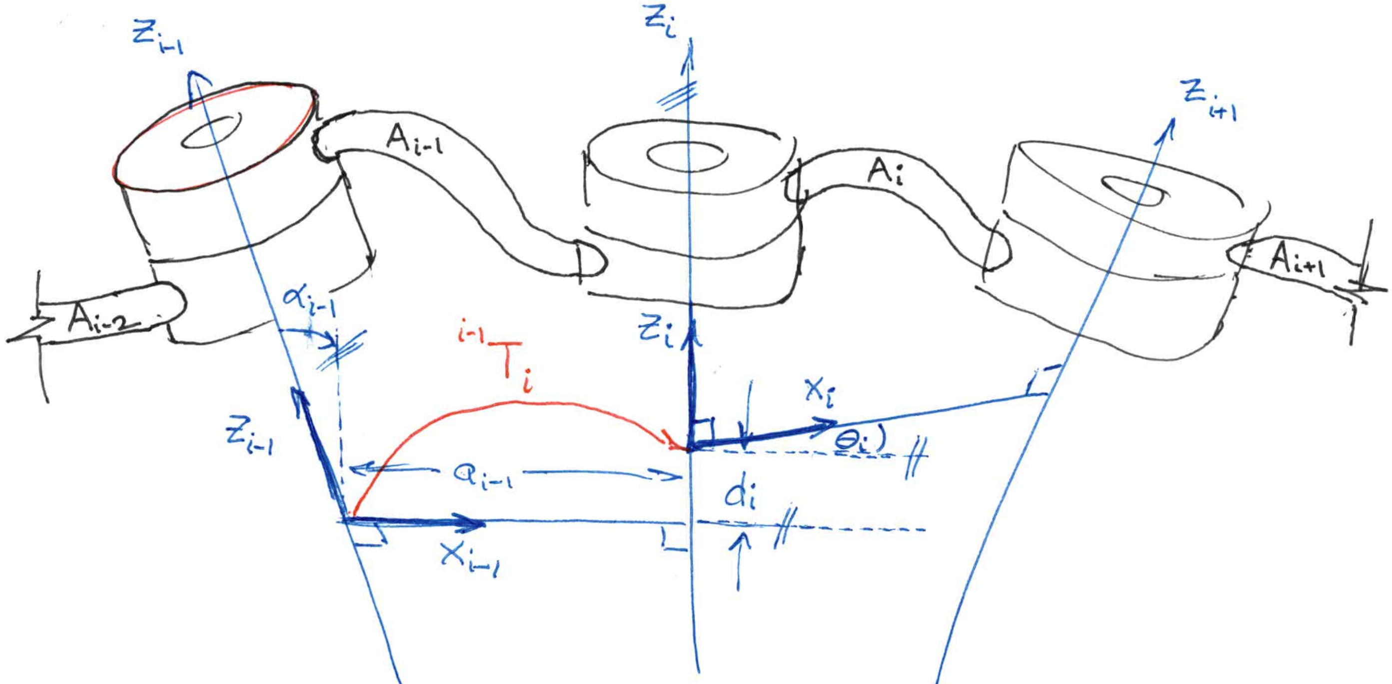
Prismatic Joint



$$^1T_2(a_1) = \mathrm{Transl}_x(a_1)\,R_z(\theta_2) = \text{same!}$$

Spatial Chains

Systematic, minimal frame assignment by Denavit-Hartenberg method



$$\underbrace{\overbrace{{}^{i-1}T_i = \mathrm{Transl}_{x_{i-1}}(a_{i-1})\, R_{x}(\alpha_{i-1})}^{\text{can reverse order}}}_{Q_{i-1}}\ \underbrace{\overbrace{\mathrm{Transl}_{z_i}(d_i)\, R_{z_i}(\theta_i)}^{\text{can reverse order}}}_{R_i}$$

LaVallé's notation → Q_{i-1} R_i

$$ {}^{i-1}T_i = \begin{bmatrix} \cos(\theta_i) & -\sin(\theta_i) & 0 & a_{i-1} \\ \sin(\theta_i)\cos(\alpha_{i-1}) & \cos(\theta_i)\cos(\alpha_{i-1}) & -\sin(\alpha_{i-1}) & -\sin(\alpha_{i-1})\, d_i \\ \sin(\theta_i)\sin(\alpha_{i-1}) & \cos(\theta_i)\sin(\alpha_{i-1}) & \cos(\alpha_{i-1}) & \cos(\alpha_{i-1})\, d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Defining D-H Frames

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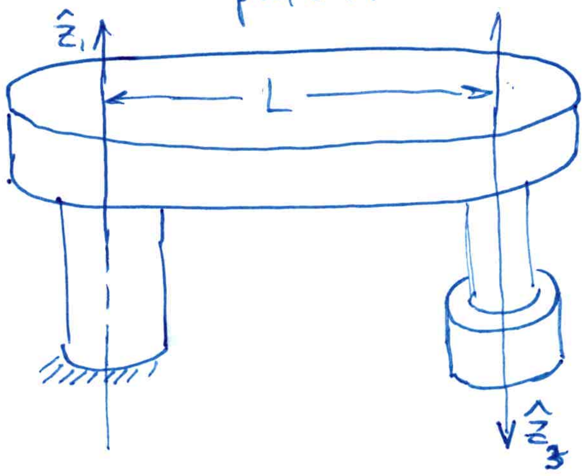
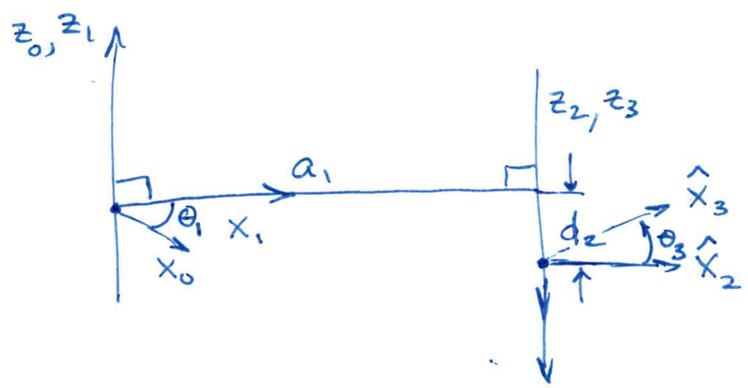
1. Identify joint axes. Assign $\hat{z}_i$ to joint i; $\forall\, i \in \{1, \dots, N\}$
2. Identify common normals. Assign $\hat{x}_i$ from $\hat{z}_i$ to $\hat{z}_{i+1}$.
3. Assign $\hat{y}_i = \hat{z}_i \times \hat{x}_i$.
4. Assign frame $\{0\}$ to equal $\{1\}$ when joint variable is zero.
5. Assign frame $\{N\}$ such that many DH parameters are zero.

Example: PRP Manipulator





i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0	var
2	π	L	var	0
3	0	0	0	var

LaValle 3.4: Kinematic Trees

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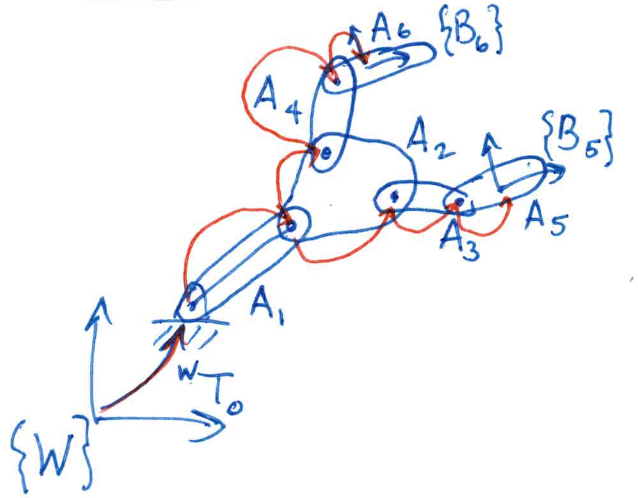


Frame assignment is the same (e.g., use DH approach), but branch on links with more than 2 joints.

Finger 1: ${}^{W}T_0\,{}^{0}T_1\,{}^{1}T_2\,{}^{2}T_3\,{}^{3}T_5 = {}^{W}T_5$

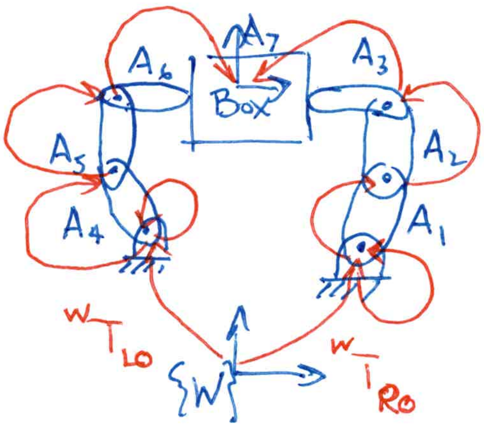
Finger 2: ${}^{W}T_0\,{}^{0}T_1\,{}^{1}T_2\,{}^{2}T_4\,{}^{4}T_6 = {}^{W}T_6$

Closed Kinematic Chains

${}^{W}T_{R0}\,{}^{R0}T_1\,{}^{1}T_2\,{}^{2}T_3\,{}^{3}T_7 = {}^{W}T_7$

AND

${}^{W}T_{L0}\,{}^{L0}T_4\,{}^{4}T_5\,{}^{5}T_6\,{}^{6}T_7 = {}^{W}T_7$

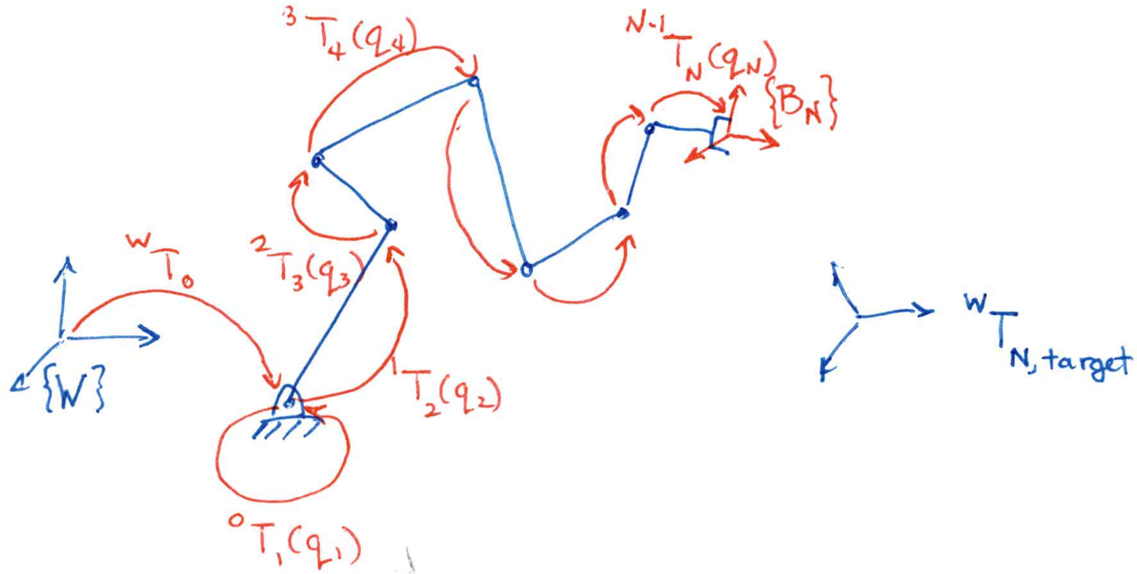


constraints remove degrees of freedom

6 joints + 3dof of box = 9

if fingers stick to box, the 4 dof removed!

Fwd. & Inv. Kinematics



Fwd. Kin. Problem: $\quad {}^WT_0\,{}^0T_1(q_1)\cdots{}^{N-1}T_N(q_N) = {}^WT_N$

given $q_1, q_2, \ldots, q_N$, compute elements of unique solution!

Inv. Kin. Problem:

given elements of ${}^WT_N$, compute $q_1, q_2, \ldots, q_N$.

3D: $N = 6 \Longrightarrow$ typically 8 solutions (up to 16 possible)

$N > 6 \Longrightarrow \infty$ solutions

$N < 6 \Longrightarrow$ no solutions

2D: $N = 3 \Rightarrow$ 2 solutions

$N > 3 \Rightarrow$ infinity solutions

$N < 3 \Rightarrow$ no solutions

Inverse Kinematics Examples

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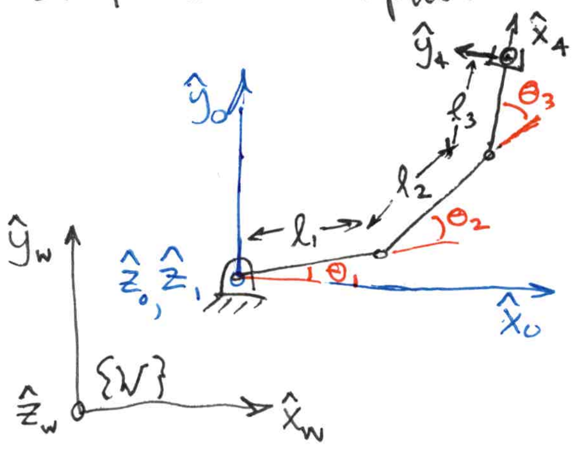
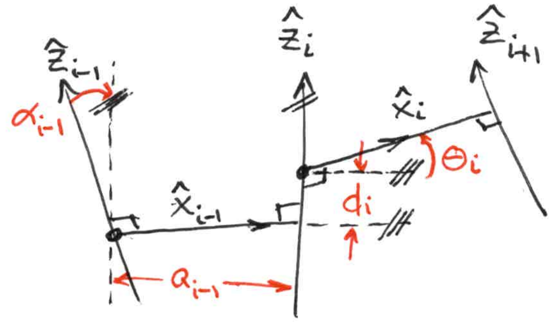
3R planar manipulator.





~~Given~~

Given Target frame, compute

$\theta_1, \theta_2, \theta_3$ such that $\{4\} = \{\text{Target}\}$

Assign frame via DH

1. Assign $\hat{z}_i$; $i = 1, 2, 3$ ~~along~~ along joint axes (out of plane)
2. Assign $\hat{x}_i$ along common normal from $\hat{z}_i$ to $\hat{z}_{i+1}$
3. $\hat{y}_i = \hat{z}_i \times \hat{x}_i$
4. Assign {0} and {4} (not necessarily DH method)

	i	α_{i-1}	a_{i-1}	d_i	θ_i
{0}→{1}	1	0	0	0	var
{1}→{2}:	2	0	l_1	0	var
{2}→{3}:	3	0	l_2	0	var
{3}→{4}:	4	0	l_3	0	0

not constrained to DH.

option with most zeros

The IK problem: solve the following nonlinear equations

$$ {}^{W}T_0 \, {}^{0}T_1(\theta_1) \, {}^{1}T_2(\theta_2) \, {}^{2}T_3(\theta) \, {}^{3}T_4 = {}^{0}T_{4,target} $$

elements are known values.

$$ \equiv \underbrace{{}^{0}T_1(\theta_1) \, {}^{1}T_2(\theta_2) \, {}^{2}T_3(\theta)}_{\text{variables}} = \underbrace{\left({}^{W}T_0\right)^{-1} \, {}^{W}T_{4,target} \left({}^{3}T_4\right)^{-1}}_{{}^{0}T_{3,target}} $$

known values

Substitute DH values into DH T'forms and expand

$$ \underbrace{\begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{{}^{0}T_1(\theta_1)} \underbrace{\begin{bmatrix} c_2 & -s_2 & 0 & l_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{{}^{1}T_2(\theta_2)} = \underbrace{\begin{bmatrix} c_{12} & -s_{12} & 0 & c_1 l_1 \\ s_{12} & c_{12} & 0 & s_1 l_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}}_{{}^{0}T_2(\theta_1, \theta_2)} $$

used trig identities

$$ c_1 c_2 - s_1 s_2 = c_{12} \triangleq \cos(\theta_1 + \theta_2) $$

$$ s_1 c_2 + c_1 s_2 = s_{12} \triangleq \sin(\theta_1 + \theta_2) $$

$$ {}^{0}T_1(\theta_1) \, {}^{1}T_2(\theta_2) \, {}^{2}T_3(\theta_3) = \begin{bmatrix} c_{123} & -s_{123} & 0 & c_{12} l_2 + c_1 l_1 \\ s_{123} & c_{123} & 0 & s_{12} l_2 + s_1 l_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

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Algebraic Solution

$$
{}^{0}T_{3,\text{target}} = \begin{bmatrix} t_{11} & t_{12} & 0 & t_{14} \\ t_{21} & t_{22} & 0 & t_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_{123} & -s_{123} & 0 & c_{12}l_2 + c_1 l_1 \\ s_{123} & c_{123} & 0 & s_{12}l_2 + s_1 l_1 \\ & & & \\ & & & \end{bmatrix}
$$

Look for simplest equations:

$$
\left.\begin{aligned} t_{14} &= c_{12}l_2 + c_1 l_1 \\ t_{24} &= s_{12}l_2 + s_1 l_1 \end{aligned}\right\}
$$

square and add.
also use trig identity
(law of cosines)

$$
t_{14}^2 + t_{24}^2 = l_2^2 + 2l_1 l_2 c_2 + l_1^2
$$

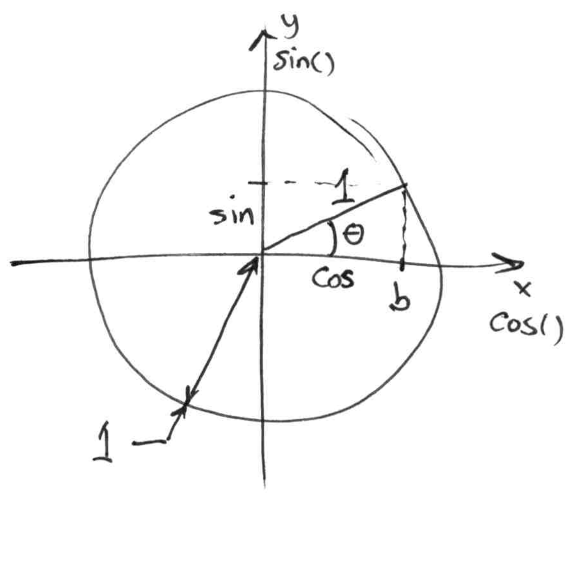
$$
c_2 = \frac{t_{14}^2 + t_{24}^2 - l_1^2 - l_2^2}{2 l_1 l_2}
$$

equation of the form $\cos(\theta_2) = b$, where b is given

$\therefore$ Two solutions

$$
\cancel{\theta_2 = \text{Atan2}(b, \pm\sqrt{1-b^2})}
$$

$$
\boxed{\theta_2 = \text{Atan2}(\pm\sqrt{1-b^2}, b)}
$$



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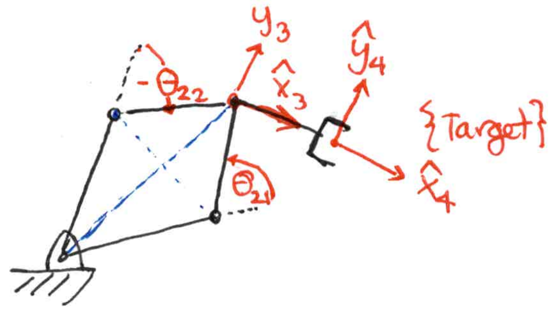
16.1

Geometrical Interpretation



Disconnect end effector & match end effector frame to target. This fixes {3}.

Now that $\left\{ \begin{array}{l} \theta_{21} \\ \theta_{22} \end{array} \right\}$ ~~is~~ are known,

~~Compute~~ solve for the corresponding θ_1's & θ_3's

Now that θ_2 is known, the next simplest equations are the same ones we just used.

(We can reuse them, since some info was lost when they were squared and added.)

Expand c_{12} and s_{12}

$$(c_1c_2 - s_1s_2)\ell_2 + c_1\ell_1 = t_{14}$$
$$(s_1c_2 + c_1s_2)\ell_2 + s_1\ell_1 = t_{24}$$

Manipulate into the form:

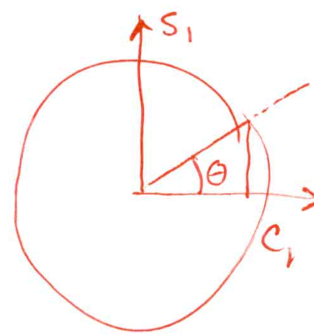
$$\left.\begin{aligned} ac_1 - bs_1 &= e \\ as_1 + bc_1 &= d \end{aligned}\right\} \quad \text{where} \quad \begin{aligned} a &= c_2\ell_2 + \ell_1 \qquad & e &= t_{14} \\ b &= s_2\ell_2 & d &= t_{24} \end{aligned}$$

Solve

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} c_1 \\ s_1 \end{bmatrix} = \begin{bmatrix} e \\ d \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ s_1 \end{bmatrix} = \frac{1}{\underline{a^2+b^2}}\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\begin{bmatrix} e \\ d \end{bmatrix}$$

no need to compute a^2+b^2



Unique Solution $\quad \theta_1 = \text{Atan2}(ad - be,\ ae + bd)$

Finally solve for θ_3.

$${}^{0}T_2(\theta_1,\theta_2)\,{}^{2}T_3(\theta_3) = {}^{0}T_{3,target}$$

$$\boxed{\text{OR USE}}\ \boxed{{}^{2}T_3(\theta_3) = {}^{0}T_2^{-1}(\theta_1,\theta_2)\,{}^{0}T_{3,target}}$$

We get the same form

$$\left.\begin{aligned} a c_3 - b s_3 &= e \\ b c_3 + a s_3 &= d \end{aligned}\right\} \quad \text{where} \quad \begin{aligned} a &= c_{12} \quad & e &= t_{11} \\ b &= s_{12} \quad & d &= t_{21} \end{aligned}$$

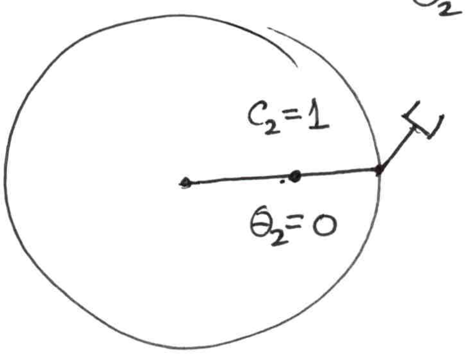
So, $\boxed{\theta_3 = \mathrm{Atan2}(ad - be,\ ae + bd)}$

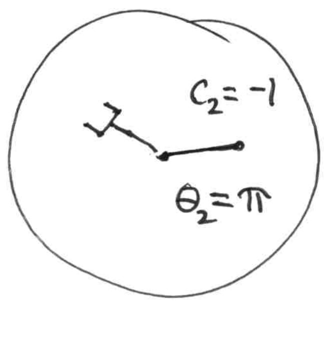
Overall there are two solutions; elbow up & elbow down, since for each choice of θ_2, θ_1 & θ_3 are unique.

But wait! There are not always 2 solutions.

When $c_2 = b = 1$ or -1, then the solution for θ_2 is unique

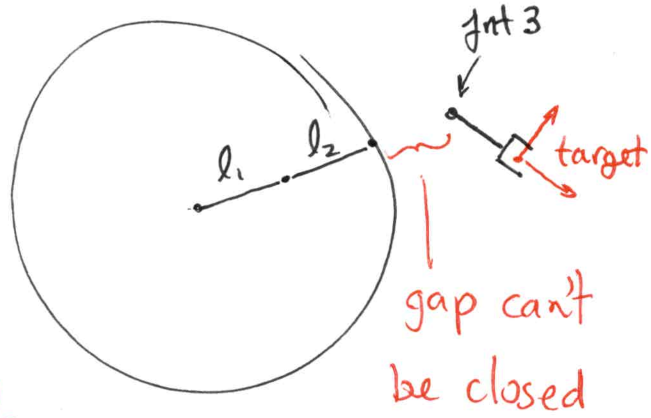
$$\theta_2 = \mathrm{Atan2}(b, 0) \quad \text{if } b \in \{1, -1\}$$



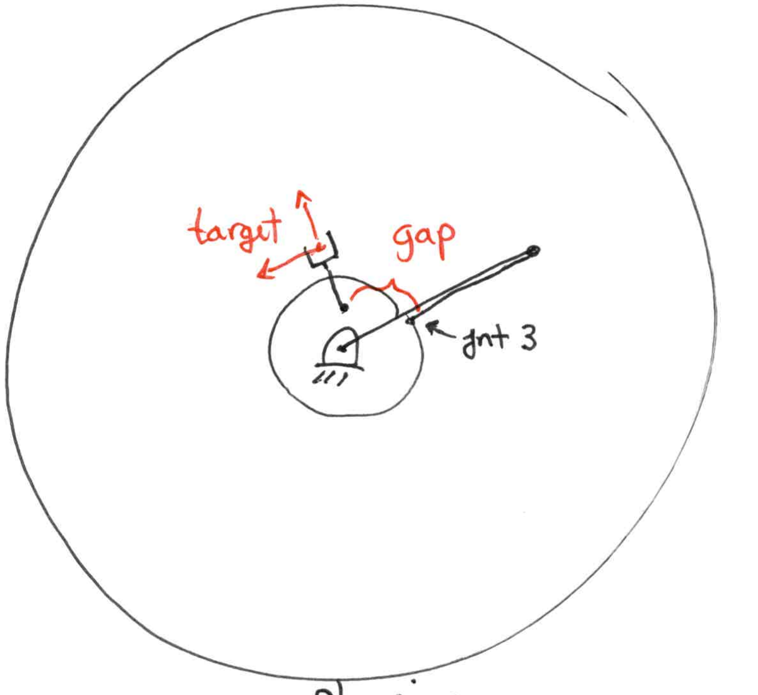


It's possible that no solution exists.

If matching the gripper frame to the target, the distance to jnt 3 is greater than $l_1 + l_2$



If $l_1 \neq l_2$, another non-existence situation arises.



Closed form solutions allows planning algorithms to know existence, uniqueness, and which solution branch is being used.

A few IK sub-problems to look for

1. $\sin(\theta) = a \longrightarrow \theta = \text{atan2}(a, \pm\sqrt{1-a^2})$

   2 solutions

2. $\cos(\theta) = b \longrightarrow \theta = \text{atan2}(\pm\sqrt{1-b^2}, b)$

   2 solutions

3. $\left.\begin{array}{l}\sin(\theta) = d \\ \cos(\theta) = b\end{array}\right\} \longrightarrow \theta = \text{atan2}(a, b)$

   unique solution

4. $a\cos(\theta) + b\sin(\theta) = 0$

   $\longrightarrow \text{atan2}(-a, b)$ AND $\text{atan2}(a, -b)$

   two solutions (differ by π)

5. $a\cos(\theta) + b\sin(\theta) = c$

   $\longrightarrow \text{atan2}(b, a) \pm \text{atan2}(\sqrt{a^2+b^2-c^2}, c)$

   two solutions

6. $\left.\begin{array}{l}a\cos(\theta) - b\sin(\theta) = c \\ a\sin(\theta) + b\cos(\theta) = d\end{array}\right\} \longrightarrow \text{atan2}(ad-bc, ac+bd)$

   unique solution

I.K. for Spatial Arms w/ Spherical Wrists

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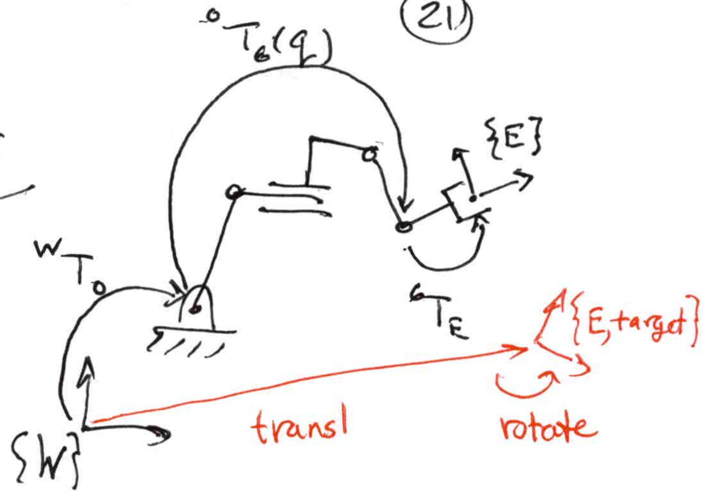
$$\underbrace{{}^{W}T_0 \, {}^{0}T_6(q) \, {}^{6}T_E = {}^{W}T_{E,target}}$$

where q is a vector of joint displacements



$$\rightarrow \; {}^{0}T_6(q) = {}^{0}T_{6,target}$$

where ${}^{0}T_{6,target} =$

$$\underbrace{\left({}^{W}T_0\right)^{-1} \, {}^{W}T_{E,target} \left({}^{6}T_E\right)^{-1}}_{\text{numeric values are given.}}$$

* Joints 1, 2, 3 maybe revolute or prismatic

  Joints 4, 5, 6 must be revolute and ~~t~~their axes must intersect at a point.

$$\text{Let } {}^{0}T_{6,target} \triangleq \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Position of center of wrist; known

$${}^{0}T_6(q) = \left[\begin{array}{c|c} R(q) & p(q_1, q_2, q_3) \\ \hline 0\ 0\ 0 & 1 \end{array}\right]$$

3 eqs in 3 unknowns

Two steps: ① Solve for q_1, q_2, q_3, then ② solve for q_4, q_5, q_6.

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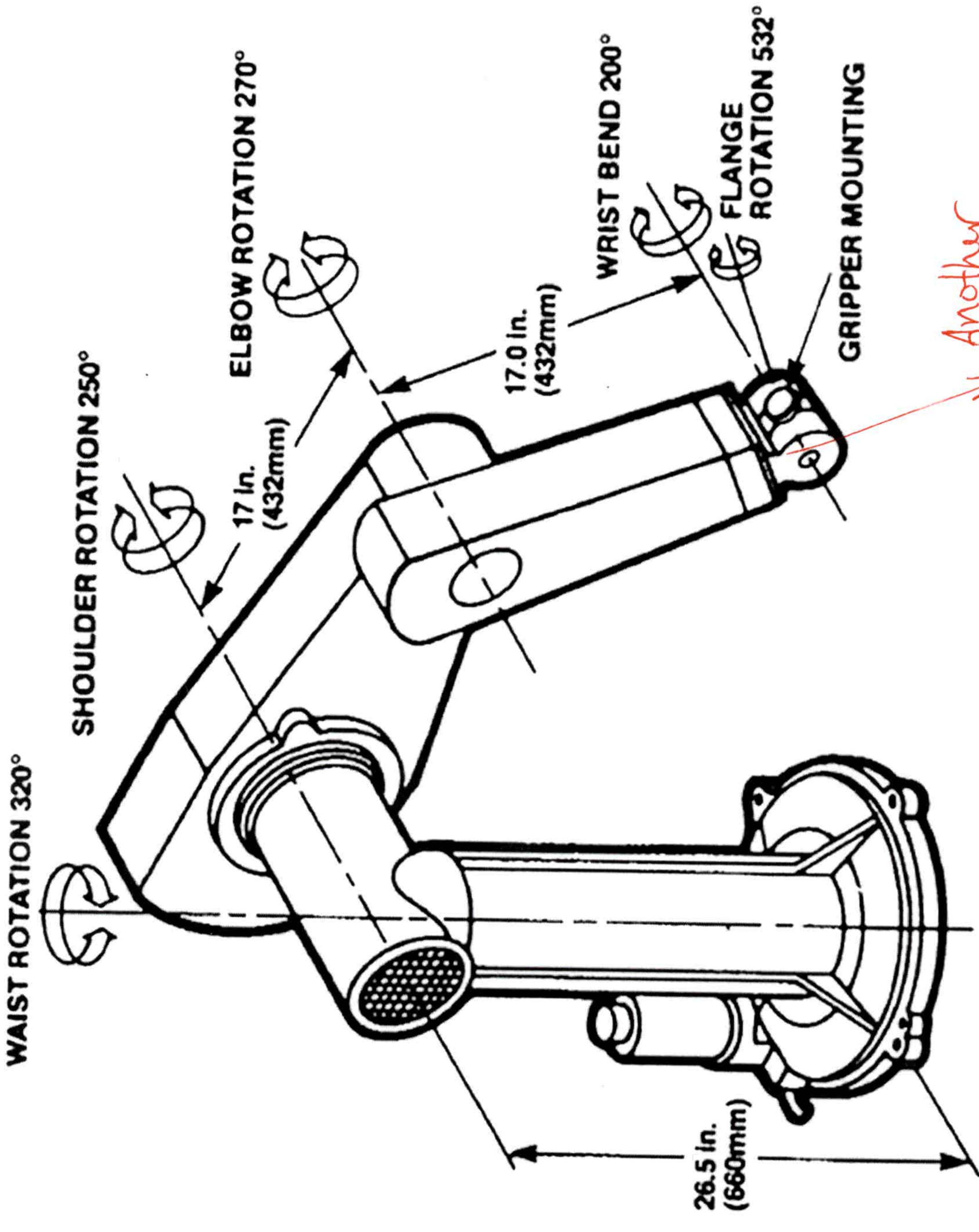
21.1

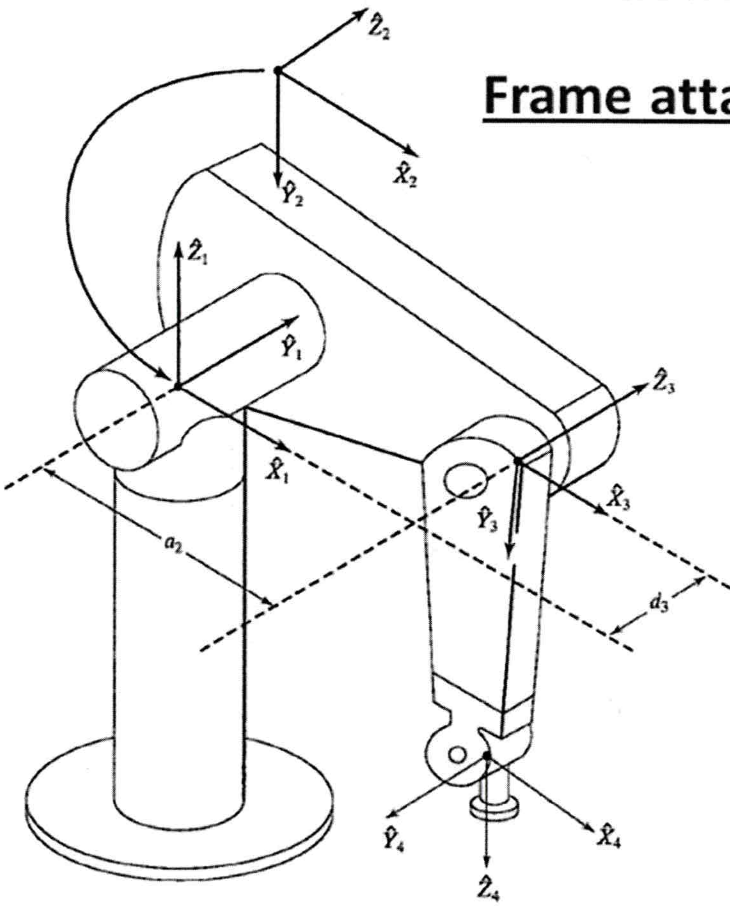
Example: Puma 560

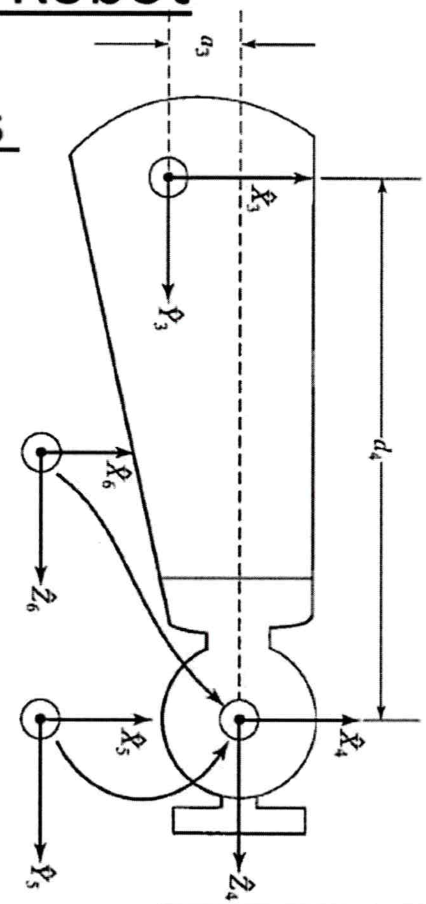


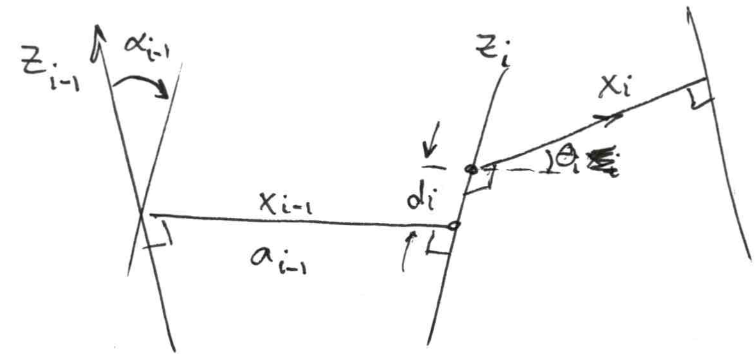
Another rotation axis!

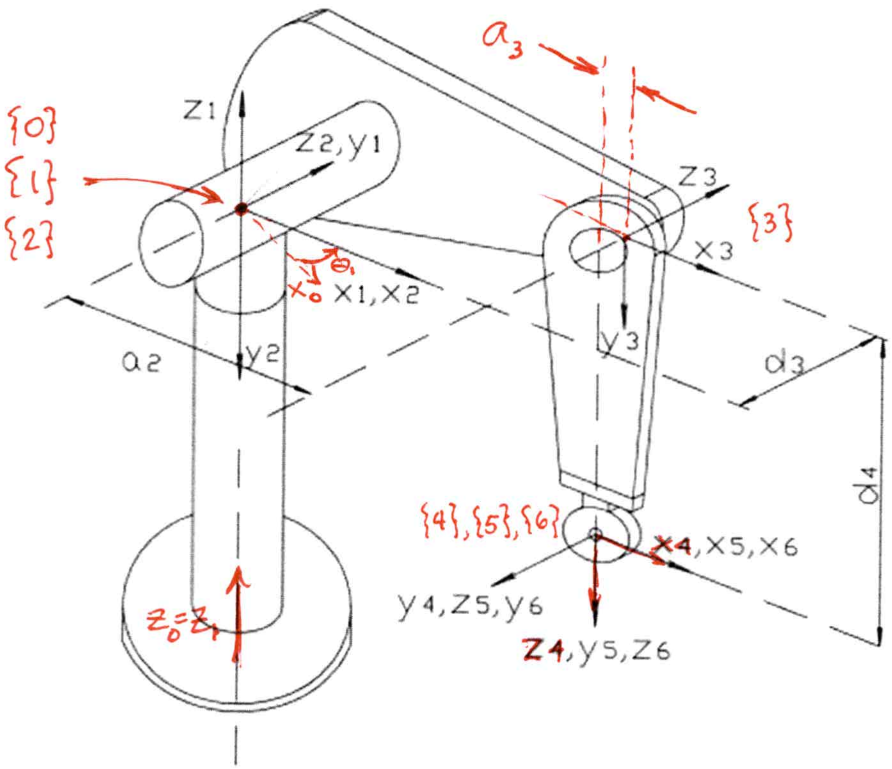
Example: Kinematics of PUMA Robot

Frame attachments









Picture 1: Robotic manipulator PUMA560 with assigned link parameters according to J.J. Craig

		about x_{i-1}	along x_{i-1}	along z_i	about z_i
	i	α_{i-1}	a_{i-1}	d_i	θ_i
{0} →	1	0°	0	0	θ_1
{1} →	2	-90°	0	0	θ_2
{2} →	3	0°	a_2	d_3	θ_3
{3} →	4	-90°	a_3	d_4	θ_4
{4} →	5	90°	0	0	θ_5
{5} →	6	-90°	0	0	θ_6

Table 1: Link parameters for PUMA 560 robotic manipulator

looks like wrong sign in figure.
Pictures in Craig's book ~~does~~ right.

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21.2

Substitute DH params into T'forms

$$ {}^0T_1(\theta_1) = \begin{bmatrix} c_\theta & -s_\theta & 0 & 0 \\ s_\theta & c_\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

etc. for all 6.

Expand product and simplify

$$ {}^0T_6(\theta) = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$p_x = c_1[a_2c_2 + a_3c_{23} - d_4s_{23}] - d_3s_1$

$p_y = s_1[\quad " \quad] + d_3c_1$

$p_z = a_2s_2 - a_3s_{23} - d_4c_{23}$

The p_x & p_y eqs. are in form 6, so one could solve for θ_1 as a function of θ_2 and θ_3, BUT there's a better way!

Write equations in a different frame.

$$ {}^0T_6(\theta) = {}^0T_1(\theta_1)\,{}^1T_6(\theta_2, \dots \theta_6) = {}^0T_{6,target} $$

$$ ({}^0T_1(\theta_1))^{-1}\,{}^0T_{6,target} = {}^1T_6(\theta_2, \dots, \theta_6) $$

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$$\underbrace{\left({}^{0}T_1(\theta_1)\right)^{-1}}\ \underbrace{{}^{0}T_{6,\text{target}}}$$

$$\begin{bmatrix} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} r_{11} & \cdots & & p_x \\ \cdots & & & p_y \\ \cdots & & & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^{1}T_6(\theta_2, \ldots, \theta_5)$$

Equate the (2,4) elements (since they are simple):

$$\left.\underbrace{c_1}_{a} p_y \underbrace{- p_x}_{b} s_1 = \underbrace{d_3}_{c}\ \right\} \leftarrow \text{special form } 5$$

$$\boxed{\theta_1 = \operatorname{atan2}(b,a) \pm \operatorname{atan2}\left(\sqrt{a^2+b^2-c^2}, c\right)}$$

Two solutions

Known as "Left" & "Right"

Equate the (1,4) & (3,4) elements to get:

$$\left.\begin{aligned} c_1 p_x + s_1 p_y &= a_3 c_{23} - d_4 s_{23} + a_2 c_2 \\ -p_z &= a_3 s_{23} + d_4 c_{23} + a_2 s_2 \end{aligned}\right\} \text{ square and add}$$

After squaring and adding:

$$\left.\underbrace{a_3}_{a} c_3 \underbrace{- d_4}_{b} s_3 = \underbrace{K}_{c}\ \right\} \leftarrow \text{special form } 5$$

where $K = \left(p_x^2 + p_y^2 + p_z^2 - a_2^2 - a_3^2 - d_3^2 - d_4^2\right) / 2a_2$

$$\boxed{\theta_3 = \operatorname{atan2}(b,a) \pm \operatorname{atan2}\left(\sqrt{a^2+b^2-c^2}, c\right)}$$

Two solns. Known as "elbow-up" & "elbow-dn"

Next solve for θ_2.

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There are multiple ways to get 2 eqs in the form 6.

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For example expand p_x & p_y equations on page 22 using $C_{23} = C_2C_3 - S_2S_3$ & $S_{23} = S_2C_3 + C_2S_3$.

Then manipulate algebraically to get into form 6.

Another way...

$$\underbrace{{}^{0}T_3(\theta_1, \theta_2, \theta_3)^{-1}}_{\theta_1,\ \theta_3\ \text{known}}\ {}^{0}T_{6,target} = {}^{3}T_6(\theta_4, \theta_5, \theta_6)$$

Equating elements (1,4) & (2,4)

$$\left.\begin{aligned} \underline{(c_1 p_x + s_1 p_y)}\, c_{23} - p_z s_{23} &= a_3 + a_2 c_2 \\ (+p_z)\, c_{23} + (c_1 p_x + s_1 p_y) s_{23} &= -(d_4 - a_2 s_3) \end{aligned}\right\} \leftarrow \text{special form 6}$$

<u>Unique</u> solution for choice of θ_1 & θ_3.

$$\theta_{23} = \theta_2 + \theta_3 = \text{atan2}(ad - bc,\ ac + bd)$$

$$\boxed{\theta_2 = \text{atan2}(ad - bc,\ ac + bd) - \theta_3}$$

4 solutions to get wrist center to correct position!

Now for the wrist solutions

$$\underbrace{({}^0T_3)^{-1}\,{}^0T_{6,target}}_{\text{known values}} = {}^3T_6(\theta_4, \theta_5, \theta_6)$$

$$= \begin{bmatrix} \text{complex fcn}(\theta_4,\theta_5,\theta_6) & \text{"} & -c_4 s_5 & a_3 \\ s_5 c_6 & -s_5 s_6 & c_5 & d_4 \\ \text{complex fcn}(\theta_4,\theta_5,\theta_6) & \text{"} & s_4 s_5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

focus on the simple eqs.

$$\text{Let } ({}^0T_3)^{-1}\,{}^0T_{6,target} = {}^3T_{6,target} = \begin{bmatrix} \cdot & \cdot & {}^3(r_{13})_6 & \cdot \\ {}^3(r_{21})_6 & {}^3(r_{22})_6 & {}^3(r_{23})_6 & \cdot \\ \cdot & \cdot & {}^3(r_{33})_6 & \cdot \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Equate the (2,3) elements

$$c_5 = \underbrace{{}^3(r_{23})_6}_{b} \quad \} \leftarrow \text{special form ②}$$

$$\boxed{\theta_5 = \text{atan2}(\pm\sqrt{1-b^2},\ b)}$$

$\boxed{\text{Two solutions. "Wrist flip"}}$

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Solve for θ_4 & θ_6 now.

$$\left.\begin{aligned} s_4 &= \underbrace{{}^3(r_{33})_6}_{a} / s_5 \\ c_4 &= \underbrace{-{}^3(r_{13})_6}_{b} / s_5 \end{aligned}\right\} \leftarrow \text{special form } 3$$

~~and~~

$$\boxed{\theta_4 = \text{atan2}(a, b)}$$

unique

$$\left.\begin{aligned} s_6 &= \underbrace{-{}^3(r_{22})_6}_{a} / s_5 \\ c_6 &= \underbrace{{}^3(r_{21})_6}_{b} / s_5 \end{aligned}\right\} \leftarrow \text{form } 3$$

$$\boxed{\theta_6 = \text{atan2}(a, b)}$$

unique

In total, there are 8 generic solutions:

Given ${}^0T_{6,target}$, one gets 2^3 solns

corresponding to

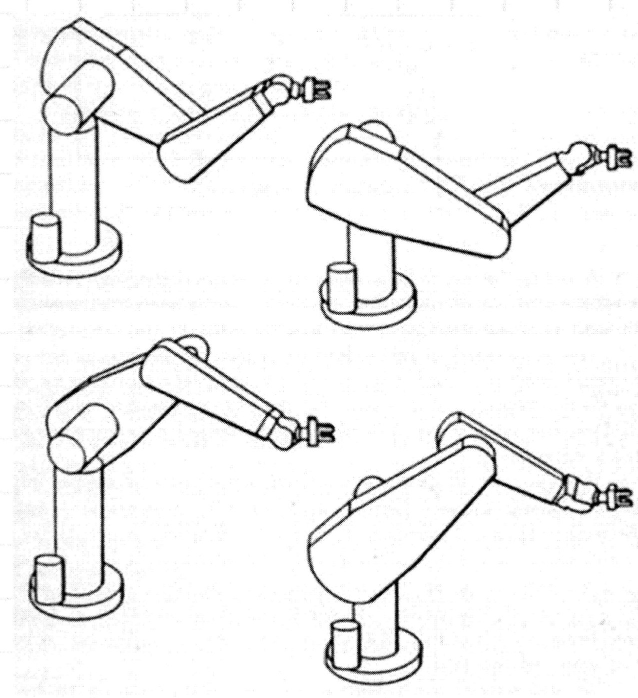
- left or right
- elbow up or down
- wrist flip or not

Show Puma IK slide

Number of Solutions

Depends upon the number and range of joints and also is a function of link parameters α,a,d

8 solutions exits



4 Solutions of the PUMA 560

<u>Another 4 solution</u>

$$\theta_4' = \theta_4 + 180^\circ,$$

$$\theta_5' = -\theta_5,$$

$$\theta_6' = \theta_6 + 180^\circ.$$

IK Fast (from OpenRave)

Solves many useful variations on the standard IK problem.

For example, robot walking or climbing, robot grasping, robot assembly....

Whenever a kinematic loop closes; ie, tree → graph, then you have an inverse kinematics problem.

Claims

IK is first solved analytically, then optimized code is generated.

All degenerate cases are handled.

IK solution requires only about 4 microseconds.

All possible discrete solutions may be calculated.

Detects singularities where 2 or more axes align (cause ∞ solutions)

Detects ~~no~~ solution non-existence (out side workspace)

All divide-by-zeros handled.

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IK Types

Still some failures e.g. Boston Dynamics Atlas

Transform6D – usual general case for 6-jnt manip.

Rotation3D – just match orientation of end effector w/ desired orientat.

Translation3D – just " origin ~~position~~ of e.e. w/ desired point.

Diff?

Direction3D – vector in end-effector frame points in desired direction.

Lookat3D – direction on end eff. points to desired 3D point

TranslationDirection5D – vector on end eff lies on vector in world with tails points coinciding

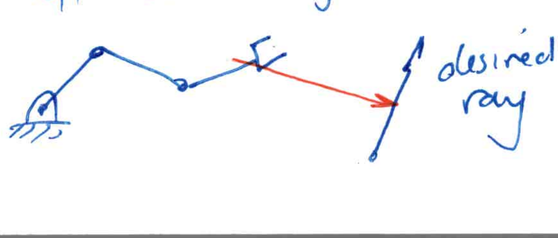
TranslationXY2D – end effector origin reaches desired (x,y) position in world.

TranslationLocalGlobal6D – end effector point reaches desired world point.

TranslationXAxis4D – e.e. origin matches desired pt AND manip(e.e.) direction make chosen angle with world x-axis. (Also for Y & Z).

needs a lot of work

Ray4D – ray on end eff coord system reaches desired ray? → ?



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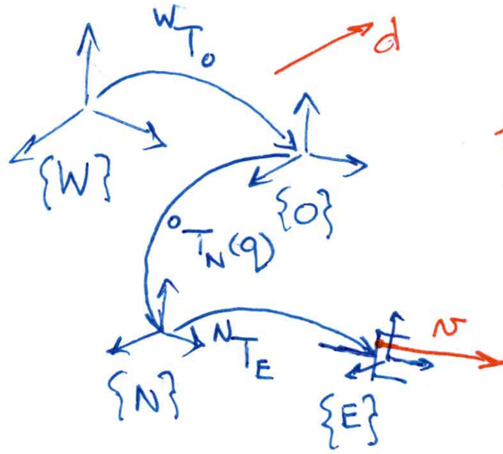
(29)

Variations on IK Problems

$${}^{W}T_{0}\,{}^{0}T_{N}(q_{1:N})\,{}^{N}T_{E} = {}^{W}T_{E,target}$$



d - a desired direction

p - a point of interest

v - a vector of interest attached to {E} (e.g., a laser).

Transform6D - this is the problem we've been studying

How to handle N > 6 joints? Fix N-6 and then follow the approach already used.

Rotation3D - only match orientation to target t'form

$${}^{0}T_{N}(q) = {}^{0}T_{N,target} \qquad {}^{0}R_{N}(q) = {}^{0}R_{N,target}$$

equate

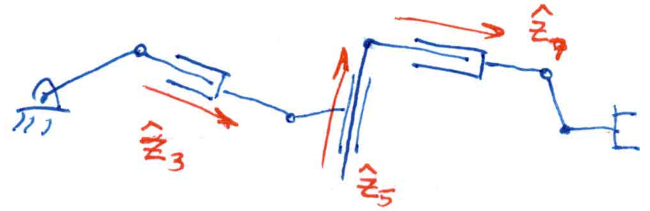
$$\left[\begin{array}{c|c} \boxed{{}^{0}R_{N}(q)} & {}^{0}p_{N}(q) \\ \hline 0\ 0\ 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \boxed{\begin{array}{cc} ({}^{0}r_{11})_{N} & \cdots \\ \cdots & \cdots \end{array}} & \vdots \\ \hline 0\ 0\ 0 & 1 \end{array}\right]$$

Must be at least 3 revolute joints to match arbitrary orientation.

Translation3D - only match origin of {E}

$$^{0}p_N(q) = {}^{0}p_{N,target}.$$

3 equations. Easiest to solve if there are three prismatic joints whose directions are linearly independent.



$$\left| \hat{z}_3 \ \hat{z}_5 \ \hat{z}_7 \right| \neq 0$$

determinant

Direction3D - ~~direc~~ vector in {E} points in desired direction.

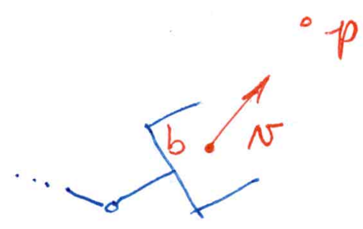
$$\underbrace{\left({}^{W}R_0 \, {}^{0}R_N(q) \, {}^{N}R_E \, {}^{E}v \right)}_{{}^{W}v = v \text{ with coords in } \{W\}} \cdot {}^{W}d - \|{}^{E}v\| \, \|{}^{W}d\| = 0$$

d, v parallel

$\boxed{\text{Note: } \|Rv\| = \|v\| \quad \forall R \in SO(3)}$

Lookat3D - direction on end effector points at a desired point.

Let b be the point at the base of direction vector v.



$v \times (p - b) = 0$ ← Not quite. v could point in opposite direction.

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Possibly better choice

$$b + n\lambda = p \; ; \; \lambda > 0$$

Starting equations

$$^{W}\tilde{b} = {}^{W}T_E(q) \begin{bmatrix} {}^{E}b \\ 1 \end{bmatrix} \quad , \text{ where } {}^{W}\tilde{b} = \begin{bmatrix} {}^{W}b \\ 1 \end{bmatrix}$$

$$^{W}b + \lambda \, {}^{W}R_E(q) \, {}^{E}n = {}^{W}p$$

$\vdots$