LaValle_Ch4-2

Wednesday, March 28, 2012
11:43 AM

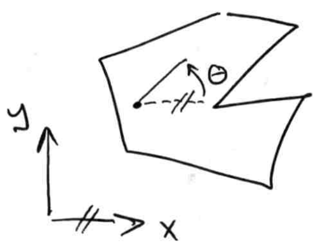
4.2: Defining the C-space

4.2.1: C-space of rigid bodies in the plane: SE(2)

__Goal:__ characterize all rigid body displacements

Valid config variables: If the variables are known, then every point of the rigid body is known.

One choice is (x, y, θ).



(x,y) = position of ref. pt.

θ = angle of ref. line

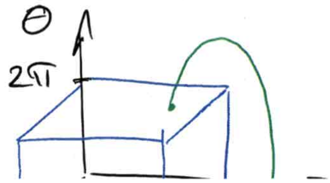
$$\text{elements of } SE(2) \Rightarrow \begin{bmatrix} c_\theta & -s_\theta & x \\ s_\theta & c_\theta & y \\ 0 & 0 & 1 \end{bmatrix}$$

C-space for (x,y) is $M_1 = \mathbb{R}^2$, where M_1 is a manifold

C-space for θ is $M_2 = S^1$ (the circle), also a manifold

$$\mathcal{C} = M_1 \times M_2 = \mathbb{R}^2 \times S^1.$$ product of manifolds is a manifold

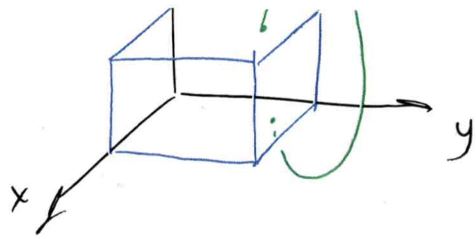
If $M_1 \subset \mathbb{R}^2$, then $\rightarrow$



If [illegible] = [illegible], then

$C = I^1 \times I^1 \times S^1$

C is a solid torus.



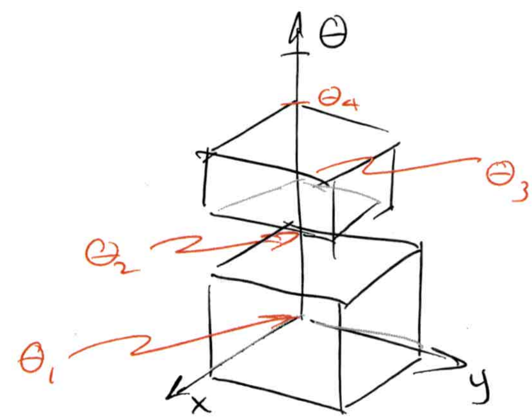
If θ is also bounded, then $C = I^1 \times I^1 \times I^1 = I^3$

which is a solid sphere or cube

If $\theta \in \{\underbrace{(\theta_1, \theta_2)}^{I^1}, \underbrace{(\theta_3, \theta_4)}^{I^1}\}$, then

$C = I^3 \sqcup I^3$

disjoint union

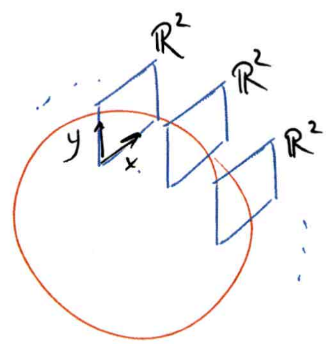


If x, y, θ are unbounded,

then we have a

"fibration" of a circle

where the fibers are $\mathbb{R}^2$



Matrix Groups:

Main interest is in $SO(n)$ = all $n \times n$ rotation matrices

General Linear Group: $GL(n)$

The space of all $n \times n$ real-valued nonsingular matricies

$$A \in GL(n) \text{ iff } A^{-1} \text{ exists}$$
$$\equiv \text{ iff } \det(A) \neq 0$$

$$\mathrm{Dim}(GL(n)) = n^2$$

Think of at $\mathbb{R}^{n^2} \setminus \{\det(A)=0\}$
An n^2-dim set w/ a 1-dimensional set removed.

Orthogonal Group: $O(n)$

$$A \in O(n) \text{ iff } A \in GL(n) \text{ and } A^T = A^{-1}$$
$$\text{ie., } A^T A = I$$

$\therefore$ all pairs of columns are orthogonal and all columns have unit length.

$$\mathrm{Dim}(O(n)) = n^2 - \binom{n}{2} - n = n^2 - \frac{(n-1)\,n}{2} - n$$
$$= \frac{n(n-1)}{2}$$

# of orthogonality relationships

# of normality relationships

$\mathrm{Dim}(O(2)) = 1$ = # dof of orientation in plane

$\mathrm{Dim}(O(3)) = 3$ = # dof " " " $\mathbb{R}^3$.

If $A \in O(n)$, then $\mathrm{Det}(A) = 1$ or -1.

half have 1, half have -1.

Special Orthogonal Group: $SO(n)$

$A \in SO(n)$ iff $A \in O(n)$ and $\underbrace{\det(A) = +1}$.

constraint apparently wipes out half of the space of $SO(n)$, so dimension is unchanged.

$\mathrm{Dim}(SO(2)) = 1$

$\mathrm{Dim}(SO(3)) = 3$

If space was 4D ... $\mathrm{Dim}(SO(4)) = 6$

Example: Matrix groups.

$\left\{ \begin{bmatrix} a & c \\ b & d \end{bmatrix} \;\middle|\; (a,b,c,d) \in \mathbb{R}^4 \right\}$ is homeomorphic to $\mathbb{R}^4$

$GL(2) = \left\{ \begin{bmatrix} a & c \\ b & d \end{bmatrix} \;\middle|\; (a,b,c,d) \in \mathbb{R}^4,\ ad - bc \neq 0 \right\}$

$\mathrm{Dim}(GL(2)) = 4$, but is not homeo. to $\mathbb{R}^4$

because we've removed a one-dimension set (which has no ~~area~~ volume!).

Next enforce $A^T A = I$

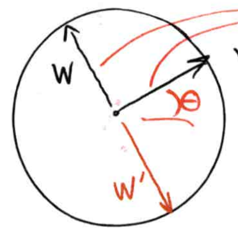
$O(2) \supset GL(2) \ni \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3 eqs. $\left\{ \begin{array}{l} a^2 + c^2 = 1 \\ \left.\begin{array}{l} ab + cd = 0 \\ ba + dc = 0 \end{array}\right\} \text{same} \\ b^2 + d^2 = 1 \end{array} \right\} \Rightarrow \mathrm{Dim}(O(2)) = 1$

What is the structure of $O(2)$?

Let $\begin{bmatrix} a \\ b \end{bmatrix} = w$, $\|w\| = 1$; $\begin{bmatrix} c \\ d \end{bmatrix} = v$, $\|v\| = 1$

$w \perp v$



(This comment is better for $SO(2)$) these can be though of as moving as a rigid unit to trace out a circle.

Clearly v can move on a circle, S^1, possible (a,b) pairs

Let S^1 be the base manifold.

For each pt. on S^1 there are 2 distinct w's

Only $\theta \in [0, 2\pi)$ needed to parametrize

So the space $O(2)$ is $S^1 \sqcup S^1$, i.e.

disjoint union

union operation that retains index to set of original membership.

two circles that never touch ($w \neq w'$ for any v).

SO(2) requires $ad - bc = 1$ ← This eq. selects one of the two circles of O(2).

$\therefore$ SO(2) is one circle, SO(2) is homeo. to S^1

Interpret SO(2) as a circle in the complex plane

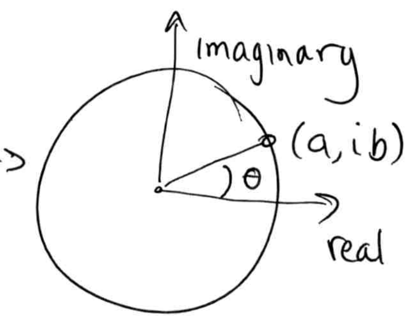
$$a + ib \equiv re^{i\theta}$$

$$i^2 = -1$$

$$r = \sqrt{a^2 + b^2}$$

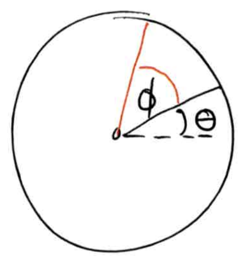
$$\theta = \text{atan2}(b, a)$$

$a^2 + b^2 = 1 \longrightarrow$



Note that multiplying a complex # by another (of unit length) has effect of rotating the first.

$$re^{i\theta} \cdot \cancel{r}e^{i\phi} = re^{i(\theta + \phi)}$$



Unit quaternions work similarly in SO(3)!

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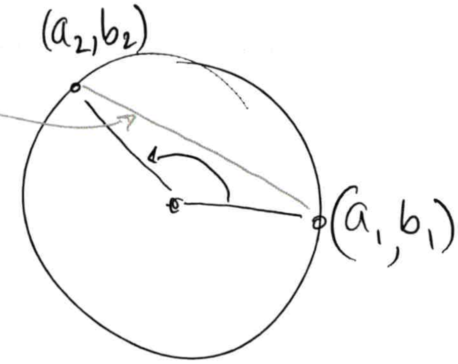
Linear interpolation of angles?

$$\begin{bmatrix} a \\ b \end{bmatrix} = \alpha \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} + (1-\alpha) \begin{bmatrix} a_2 \\ b_2 \end{bmatrix}$$

Not valid rotation.



Use SLERP (Spherical Linear Interpolation) to stay on circle in SO(2) & 3-sphere in SO(3).

$$v = \text{slerp}(\alpha, \hat{v}_1, \hat{v}_2) = \frac{\sin((1-\alpha)\phi)}{\sin(\phi)} \hat{v}_1 + \frac{\sin(\alpha\phi)}{\sin(\phi)} \hat{v}_2$$

where $\phi = \cos(\hat{v}_1 \cdot \hat{v}_2)$

$\alpha = 0 \Rightarrow v = \hat{v}_1$

$\alpha = 1 \Rightarrow v = \hat{v}_2$

This works in SO(2) & SO(3).

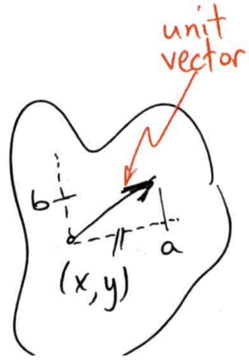
There is no analogous simple formula for Euler angles (or other 3-param. representations of SO(3).

Special Euclidean Group: SE(n)

$SE(2) = \mathbb{R}^2 \times SO(2) \longrightarrow$ 2 parameters

one parametrization w/ $a^2+b^2=1$

$$\begin{bmatrix} a & -b & x \\ b & a & y \\ 0 & 0 & 1 \end{bmatrix}$$



When orientation is updated incrementally, $a^2+b^2 \neq 1$.

Make sure to normalize after rotation change.

w/ $a^2+b^2=1$ $\quad [0 \quad 0 \quad 1]$

another (θ, x, y)

1-param. rep. of $SO(2)$

$SE(3) = ?$

parameterization: $\overbrace{(\underbrace{x,y,z}_{\mathbb{R}^3}, \underbrace{roll, pitch, yaw}_{S^3 = S^1 \times S^1 \times S^1 \ ?})}^{6 \text{ params}}$

another param. : $\overbrace{(x,y,z,Q)}^{7 \text{ params}}$

where Q = unit quaternion

LaValle notation.
unit quat $\triangleq h$.

$Q = (a,b,c,d)$ with $a^2+b^2+c^2+d^2=1$

$SE(3) = \mathbb{R}^3 \times SO(3)$ NOT! $\mathbb{R}^3 \times \underbrace{S^1 \times S^1 \times S^1}_{\cancel{S^3} = T^3 \text{ (the 3-torus)}}$

Problem with Euler Angle (all 3-angle) parameterizations!

singularities: $(\alpha, \beta, \gamma) = (0,0,0) \Rightarrow I_{3\times3} =$ no rotation

However, $(\alpha, \beta, \gamma) \neq (0,0,0)$ can also correspond to no rotation

This is a problem in motion planning, since we need to use the inverse orientation mapping a lot.

~~For any singularity~~, a continuum of (α,β,γ) exists.
For for any ~~reachable~~ singular orientation

$Rot_x(\alpha)\, Rot_y(\beta)\, Rot_z(\gamma)$

e.g. for X-Y-X Euler angles, we have these problems whenever β is near 0 or π.

Every 3 angle rep has singularities
Euler, RPY, ...
in some order

Also, when near a singular config, numerical solution loses accuracy.

$$\begin{bmatrix} & & s_\beta s_\alpha \\ & & s_\beta c_\alpha \\ s_\beta s_\gamma & s_\beta c_\gamma & c_\beta \end{bmatrix}$$

c_β = const ~~gives~~ becomes singular when $c_\beta = \pm 1$.

$(\alpha, \beta, \gamma) \Rightarrow R$ (rotation matrix) is easy, but

$R \Rightarrow (\alpha, \beta, \gamma)$ can be difficult ← we did this for spherical wrist IK solution of Puma

THE GROUP OF UNIT QUATERNIONS, $\mathbb{H}$.

Unit Quaternions (a.k.a. Euler Parameters)

Lavalle uses h.

$Q \rightarrow R$ easy

$R \rightarrow Q$ easy

Q can be thought of as a complex #

$Q = a + bi + cj + dk$ $\qquad i^2 = j^2 = k^2 = -1$

$ij = k,\ jk = i,\ ki = j$

Complex conjugate: $Q^* = a - bi - cj - dk$

$Q_1 \cdot Q_2 = Q_3$

$= (a_1 + b_1 i + c_1 j + d_1 k) \cdot (a_2 + b_2 i + c_2 j + d_2 k)$

$$\left.\begin{array}{l} a_3 = a_1 a_2 - b_1 b_2 - c_1 c_2 - d_1 d_2 \\ b_3 = a_1 b_2 + a_2 b_1 + c_1 d_2 - c_2 d_1 \\ c_3 = \quad \text{similar} \\ d_3 = \quad \text{similar} \end{array}\right\} \text{eq 4.19, pg 150 of Lavalle}$$

Note that (unit quat)$\cdot$(unit quat) = (unit quat)

There is no equivalent way to multiply Euler angles (or elements of other 3-param reps).

(Unless you view $R(\alpha,\beta,\gamma)\, R(\alpha_2,\beta_2,\gamma_2)$ as equivalent!

Alternative view

$Q = (a, v)$

a — scalar part

v — vector part

$Q_1 \cdot Q_2 = (a_1, v_1) \cdot (a_2, v_2)$

Think of multiplying these in all possible pairs. Some give scalars, some give vectors.

$Q_1 \cdot Q_2 = \big((a_1 a_2 - v_1 \cdot v_2),\ (a_1 v_2 + a_2 v_1 + v_1 \times v_2)\big)$

usual dot prod.

Mapping Q to R

$$\begin{bmatrix} 2(a^2+b^2)-1 & 2(bc-ad) & 2(bd+ac) \end{bmatrix}$$

$$R(Q) = \begin{bmatrix} 2(a^2+b^2)-1 & 2(bc-ad) & 2(bd+ac) \\ 2(bc+ad) & 2(a^2+c^2)-1 & 2(cd-ab) \\ 2(bd-ac) & 2(cd+ab) & 2(a^2+d^2)-1 \end{bmatrix}$$

Mapping $R \rightarrow Q$

Let $R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$ and

replacing 1 in $R(Q)$ with $a^2+b^2+c^2+d^2$ yields:

$$a^2 = \tfrac{1}{4}(1 + r_{11} + r_{22} + r_{33}) \qquad (1)$$

$$b^2 = \tfrac{1}{4}(1 + r_{11} - r_{22} - r_{33}) \qquad (2)$$

$$c^2 = \tfrac{1}{4}(1 - r_{11} + r_{22} - r_{33}) \qquad (3)$$

$$d^2 = \tfrac{1}{4}(1 - r_{11} - r_{22} + r_{33}) \qquad (4)$$

These equations imply 16 solutions, but only 2 exist.

Off-diagonal terms give:

(5) $ab = \tfrac{1}{4}(r_{32} - r_{23})$ $\qquad$ $bc = \tfrac{1}{4}(r_{12} + r_{21})$ (8)

(6) $ac = \tfrac{1}{4}(r_{13} - r_{31})$ $\qquad$ $bd = \tfrac{1}{4}(r_{13} + r_{31})$ (9)

(7) $ad = \tfrac{1}{4}(r_{21} - r_{12})$ $\qquad$ $cd = \tfrac{1}{4}(r_{23} + r_{32})$ (10)

Soln approach (to minimize numerical error):

① Use equation (1), (2), (3), or (4) with largest value to get 2 values of a, b, c, or d.

② Use 3 eqs from (5-10) to solve for other 3 values

③ Result is 2 solutions satisfying $Q_1 = -Q_2$

Alg.

If trace(R) > 0

solve eq (1) for $a = \frac{1}{2}\sqrt{tr(R)+1}$

solve for b, c, d using equations (5-7)

eq (5) yields $\Rightarrow$ $b = \frac{r_{32} - r_{23}}{4a}$

eq(6) $\Rightarrow$ similar

eq(7) $\Rightarrow$ similar

else if $r_{11} = \max(r_{11}, r_{22}, r_{33})$

Solve for b using eq (2)

Solve for a, c, d using eqs (5, 8, 9)

else if $r_{22} = \max(r_{11}, r_{22}, r_{33})$

Solve for c using eq (3)

solve for a, b, d using eqs (6, 8, 10)

else

else

    solve for d using eq (4)

    Solve for a, b, c using eqs (7, 9, 10)

end

finally $Q_1 = (a, b, c, d)$ & $Q_2 = -Q_1$

are the two solutions.

Note: The group of unit quaternions covers $SO(3)$ twice w/ no point identified between the covers, i.e. Q is never equal to $-Q$, since $Q = 0 \notin \underbrace{H}_{\text{space of unit Quats}}$

Slerp $\frac{\sin((1-\alpha)\phi)}{\sin(\alpha)} Q_1 + \frac{\sin(\alpha\phi)}{\sin(\alpha)} Q_2 = Q$ (rotation about fixed axis)

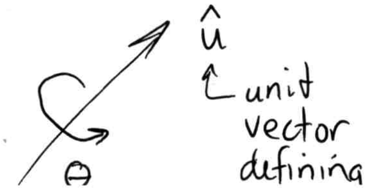
Rotating a point w/ Q

$$R(Q)\begin{bmatrix} x \\ y \\ z \end{bmatrix} \equiv (h * p * h^*)_{\text{vector part}} \quad \text{where } p = \begin{bmatrix} x \\ y \\ z \\ \hdashline 0 \end{bmatrix}$$

$$h^* = a - bi - cj - dk$$

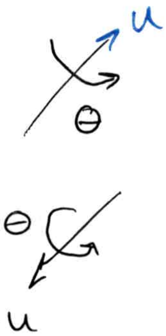
Do not confuse axis/angle with unit quaternion!

axis/angle $\rightarrow$ Q



$$Q = \left(\cos\left(\tfrac{\theta}{2}\right), \hat{u}\sin\left(\tfrac{\theta}{2}\right)\right)$$

$Q = \left(\cos\left(\frac{\theta}{2}\right), \hat{u}\sin\left(\frac{\theta}{2}\right)\right)$



vector defining axis of rotation.

Easy to see that $Q = -Q$

$Q \rightarrow$ axis/angle

$$\theta = 2 \cdot \operatorname{atan2}(\|v\|, a)$$

$$u = v / \|v\|$$

A unit quaternion is NOT the same as an axis/angle

$$R(\theta, u) = \begin{bmatrix} u_x^2 + C_\theta(1-u_x^2) & u_x u_y(1-C_\theta) + u_z S_\theta & u_x u_z(1-C_\theta) - u_y S_\theta \\ u_x u_y(1-C_\theta) - u_z S_\theta & u_y^2 + C_\theta(1-u_y^2) & u_y u_z(1-C_\theta) + u_x S_\theta \\ u_x u_z(1-C_\theta) + u_y S_\theta & u_y u_z(1-C_\theta) - u_x S_\theta & u_z^2 + C_\theta(1-u_z^2) \end{bmatrix}$$

Note that u is the eigenvector of R corresponding to the eigenvalue $\underline{1}$

Unit quaternions have a natural metric defining distance between Q_1 & Q_2. It is the length of the arc of great circle they define on the 3-sphere S^3.

$$\alpha = \cos^{-1}(Q_1 \cdot Q_2)$$

usual dot product of vectors

A metric allows us to construct algorithms to perform uniform sampling of $SO(3)$.

Velocity Kinematics of Unit Quaternions

$$\begin{bmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \\ \dot{d} \end{bmatrix} = \frac{1}{2} \underbrace{\begin{bmatrix} -b & -c & -d \\ a & -d & c \\ d & a & -b \\ -c & b & a \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}}_{\omega}$$

where ω is angular velocity of body expressed in the body-fixed frame.

$$\boxed{{}^{e}\dot{Q} = B(Q)\,{}^{B}\omega}$$

After incremental rotation ($Q + \dot{Q}\Delta t = Q_{new}$), $\|Q_{new}\| \neq 1$, so normalize

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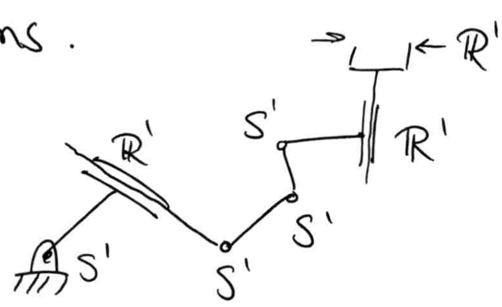
Finally apply to robot systems.

1 dof joints w/o limits $\Rightarrow$



$\mathbb{R}^1$ or S^1

1 dof joints w/ limits

$\Rightarrow I^1$

$\mathcal{C} = S^1 \times \mathbb{R}^1 \times (S^1)^3 \times \mathbb{R}^2$

$\mathcal{C} = S^4 \times \mathbb{R}^3$

with limits

$\mathcal{C} = I^7$

For every free rigid body, cross $\mathcal{C}$ with $\mathbb{R}^3 \times SO(3)$.

For every spherical joint, cross $\mathcal{C}$ with $SO(3)$.

Remember: Product of manifolds is a manifold, so C-spaces & State spaces robots are manifolds (unless they have closed kinematic chains)