(Craig Chapters 1–6)

1. Representations of 3D coordinate frames
2. Transformation of vector representation from frame to frame
3. Forward kinematic map construction
4. Inverse kinematics problem solution
5. Manipulator Jacobian construction
6. Identification of singular configurations
7. Manipulator dynamics (and equilibrium)

Fundamentals of linear algebra & linear inequalities

Given n vectors of length m,

1. Determine number of linearly independent vectors
2. Span of a set of vectors (a vector space)
3. Determine a basis for a vector space.

Given $A \in \mathbb{R}^{m \times n}$, $y \in \mathbb{R}^m$

1. Rank A
2. Null space of A
3. Row space of A
4. Transpose of A
5. Inverse of A
6. Pseudoinverse of A

$Ax = y$ represents a system of linear equations in the unknown $x \in \mathbb{R}^n$

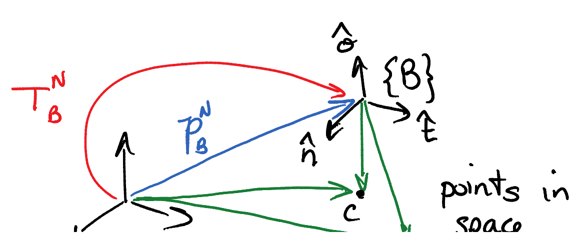
1. Solution existence
2. Solution uniqueness
3. Graphical interpretation

$Ax \geq y$ represents a system of linear inequalities

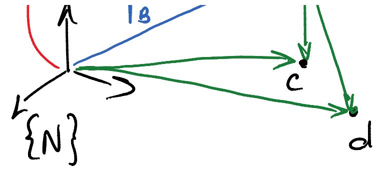
1. Solution existence
2. Solution uniqueness
3. Graphical interpretation

Homogeneous T'forms

$$T_B^N = \begin{bmatrix} R_B^N & p_B^N \\ 000 & 1 \end{bmatrix} \in SE(3)$$



$$T_B^N = \begin{bmatrix} R_B^N & p_B^N \\ 000 & 1 \end{bmatrix} \in SE(3)$$



where $R_B^N \in SO(3)$

$$\therefore \; (R_N^B)^{-1} = (R_N^B)^T = R_B^N = \left[\, \hat{n}_B^N \;\; \hat{t}_B^N \;\; \hat{o}_B^N \,\right]$$

$\hat{n}, \hat{t}, \hat{o}$ are orthonormal and

right-handed, i.e. $\mathrm{Det}(R) = +1$

R is parameterized with three or more parameters,

often a unit quaternion $[e_0 \; e_1 \; e_2 \; e_3]$, where

$$e_0^2 + e_1^2 + e_2^2 + e_3^2 = 1$$

$$(T_B^N)^{-1} = \begin{bmatrix} (R_B^N)^T & -(R_B^N)^T p_B^N \\ 000 & 1 \end{bmatrix} = T_N^B$$

Transforming a point c from frame {B} to frame {N}

$T_B^N \, c^B = c^N$, where c^B & c^N are in homogenous form,

i.e. $[c_x \; c_y \; c_z \; 1]^T$

Transforming a direction from frame {B} to frame {N}

unit vectors & vector differences represent directions.

$$T_B^N d^B - {}^N_B T c^B = T_B^N (d^B - c^B) = T_B^N \begin{bmatrix} d_x - c_x \\ d_y - c_y \\ d_z - c_z \\ 1 - 1 \end{bmatrix}$$

$$\begin{bmatrix} R_B^N & {}^N p_B \\ & \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix}^B = \begin{bmatrix} R_B^N \Delta^B \\ \text{----} \end{bmatrix}$$

$$\begin{bmatrix} R_B & p_B \\ 000 & 1 \end{bmatrix} \begin{bmatrix} \Delta y \\ \Delta z \\ \hline 0 \end{bmatrix} = \begin{bmatrix} R_B^N \Delta^B \\ \hline 0 \end{bmatrix}$$

So, transforming directions is simply vector rotation.

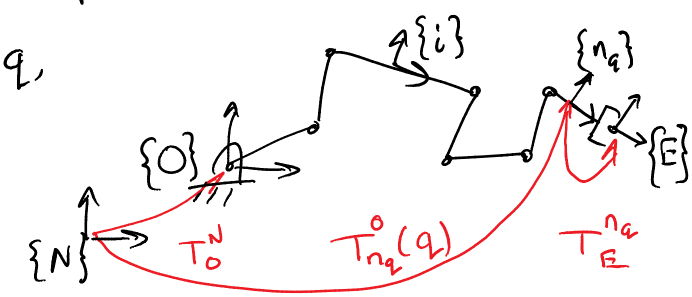
If you simply want to determine the components of a vel, ang. vel, force, or moment w.r.t. another frame, use R.

$$R_B^N \left[v^B \,\vdots\, \omega^B \,\vdots\, f^B \,\vdots\, \tau^B \right] = \left[v^N \,\vdots\, \omega^N \,\vdots\, f^N \,\vdots\, \tau^N \right]$$

Forward Position Kinematic Map

Given joint angles, q,

determine T_E^0



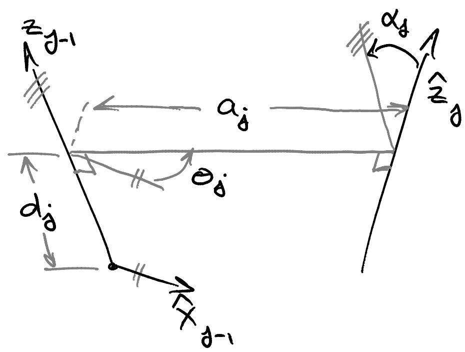
$$T_0^N \, T_1^0(q_1) \cdots T_{n_q}^{n_q-1}(q_{n_q}) \, T_E^{n_q} = T_E^N(q)$$

Methods:

Denavit-Hartenburg (std or mod)

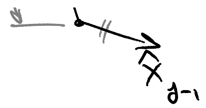
Product of Exponentials

Standard D-H parameters



$$T_i^{j-1} = \begin{bmatrix} c_j & -s_j c_j & s_j s_j & a_j c_j \\ s_j & c_j c_j & -c_j s_j & a_j s_j \end{bmatrix}$$

$$
{}^{j-1}T_j = \begin{bmatrix} c_j & -s_j c_j & s_j s_j & a_j c_j \\ s_j & c_j c_j & -c_j s_j & a_j s_j \\ 0 & s_j & c_j & d_j \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



Inverse Position Kinematic Map

Given $T^N_{E,\text{target}}$, determine q

Methods :

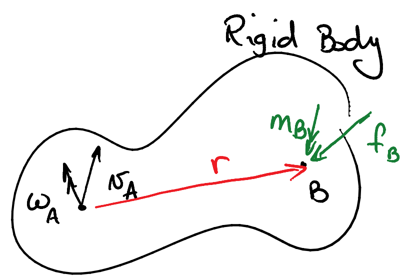
Closed form : expand and solve

$$
\underbrace{T^0_1(q_1) \cdots T^{n_q-1}_{n_q}(q_{n_q})}_{\substack{\text{multi-linear functions} \\ \text{of } \sin(q_i) \text{ \& } \cos(q_i)}} = \underbrace{T^0_N \, T^N_{E,\text{target}} \, T^E_{n_q}}_{\substack{\text{given fixed} \\ \text{values}}}
$$

Iterative :

Solve simultaneous nonlinear equations.

Transforming body wrenches & twists



$r = p_B - p_A$

Let $\nu = \begin{bmatrix} v \\ \omega \end{bmatrix}_{6\times 1}$ be the twist of a point on a rigid body.

If ν_A is known, what is ν_B ?

$$v_B = v_A + \omega_A \times r = v_A - r \times \omega_A$$

$$\omega_B = \omega_A$$

$$\begin{bmatrix} 0 & -r_z & r_y \end{bmatrix}$$

$\omega_B = \omega_A$

Let $P(r) = \begin{bmatrix} 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix}$

$$\nu_B = \begin{bmatrix} v_B \\ \omega_B \end{bmatrix} = \begin{bmatrix} I_{3\times3} & P^T(r) \\ O_{3\times3} & I_{3\times3} \end{bmatrix} \begin{bmatrix} v_A \\ \omega_A \end{bmatrix}$$

Let $g = \begin{bmatrix} f \\ m \end{bmatrix}$ be the wrench of a force f with a given line of action, l.

$g = \begin{bmatrix} f_B \\ m_B \end{bmatrix}$ where l contains B.

We want to determine g_A such that it has the same effect on the rigid body as g_B

$$\left.\begin{aligned} f_A &= f_B \\ \tau_A &= m_B + r \times f_B \end{aligned}\right\} \Rightarrow \quad g_A = \begin{bmatrix} I_{3\times3} & O_{3\times3} \\ P(r) & I_{3\times3} \end{bmatrix} g_B$$

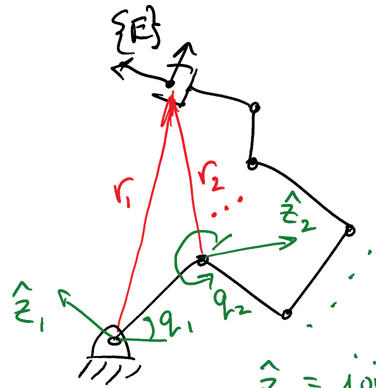
If there are coordinate frames $\{A\}$ and $\{B\}$ on the rigid body and we want g_A & ν_A expressed in $\{A\}$ and g_B & ν_B expressed in $\{B\}$, then we have:

$$\left.\begin{aligned} g_A^A &= \begin{bmatrix} R_B^A & O_{3\times3} \\ O_{3\times3} & R_B^A \end{bmatrix} \begin{bmatrix} I & O \\ P(r) & I \end{bmatrix} g_B^B \\ \nu_B^B &= \begin{bmatrix} R_A^B & O \\ O & R_A^B \end{bmatrix} \begin{bmatrix} I & P^T(r) \\ O & I \end{bmatrix} \nu_A^A \end{aligned}\right\} \text{where } R_B^A \text{ is } \begin{bmatrix} \hat{n}_B^A & \hat{t}_B^A & \hat{o}_B^A \end{bmatrix}$$

Manipulator Jacobians

$$J\dot{q} = \nu_E \quad \& \quad J^T \tau = g_E$$

where E denotes the end effector frame.



$\hat{z}_i$ = joint axis direction

q_i = joint displ.

$$J_E(q) = \begin{bmatrix} J_{p_E} \\ J_{\phi_E} \end{bmatrix}$$

$$J_{p_E} = \Big[\underbrace{P(r_1)\hat{z}_1}_{\text{revolute}} \;\vdots\; \underbrace{\hat{z}_2}_{\text{prism.}} \;\vdots\; \cdots \Big] \qquad J_{\phi_E} = \Big[\underbrace{\hat{z}_1}_{\text{rev.}} \;\vdots\; \underbrace{0}_{\text{prism.}} \;\vdots\; \cdots \Big]$$

Singular Configs

It is desirable for control, that $J_E(q)\dot{q} = \nu_E$ has a solution for every $\nu_E \in \mathbb{R}^6$.

If $\exists\ \nu_E \in \mathbb{R}^6 \ni \dot{q} \nexists$ satisfying $J_E(q)\dot{q} = \nu_E$, the q is a singular configuration.

If for some q, $\exists$ a set of q satisfying $J_E(q)\dot{q} = \nu_E$, then the manipulator is redundant.

Manipulator dynamics

$$\tau = M(q)\ddot{q} + V(q,\dot{q}) + \underbrace{G(q)}_{\text{gravity}} + F(q,\dot{q}) + J_E^T(q)\underbrace{g_E}_{\text{end effector wrench}}$$

$$\tau = M(q)\ddot{q} + V(q,\dot{q}) + G(q) + F(q,\dot{q}) + J_E^T(q)\, g_E$$

P.D. mass matrix — Coriolis centripetal — friction

Linear equations $\quad Ax = y \qquad A \in \mathbb{R}^{m\times n} \quad x \in \mathbb{R}^n \quad y \in \mathbb{R}^m$

$\text{Rank}(A) = \#$ of l.i. rows
$\qquad\qquad = \#$ of l.i. columns

always the same.

Null Space of $A = \mathcal{N}(A) = \{x \mid Ax = 0,\ x \in \mathbb{R}^n\}$

Left Null Space of A = Null Space of A^T

Column space of $A = \mathcal{R}(A) = \{y \mid Ax = y,\ x \in \mathbb{R}^n\}$

Row space of $A = \{x \mid A^T y = x,\ y \in \mathbb{R}^m\}$

Row space of $A = \mathcal{R}(A^T)$

$\text{Dimension}(\mathcal{R}(A^T)) = \text{Dimension}(\mathcal{R}(A))$ ALWAYS

Solution Existence

$Ax = y$ has a solution if $y \in \mathcal{R}(A)$

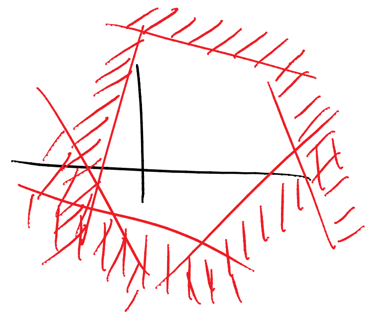
$$x = \underbrace{A^+}_{\text{pseudo-inverse}} y + \underbrace{N(A)}_{\text{basis for } \mathcal{N}(A)} \alpha$$

arbitrary vector of length = $\text{Dim}(\mathcal{N}(A))$

If $\mathcal{N}(A) = 0$, then solution is unique.

If A is square & full rank, then $A^+ = A^{-1}$ and $\mathcal{N}(A) = 0$.

Linear Inequalities



$Ax \geq y \Rightarrow$ a polytope

polytope may be:

empty

nonempty $\rightarrow$ compact convex set
$\searrow$ unbounded convex set

A necessary condition for the polytope to be empty is $m > n$.

Phase 1 of the Simplex method can determine if the polytope is empty, and if not, it will find a point in the polytope.