On the Form-Closure of Polygonal Objects with Frictional and Frictionless Contact Models

Yu-Che Chen* and Jeffrey C. Trinkle**

*Department of Mechanical Engineering
The University of Tulsa
600 South College Avenue, Tulsa, OK. 74104-3189

**Department of Computer Science
Texas A&M University
College Station, TX 77843-3112

Abstract

This paper investigates the problem of "form-closure" for three-fingered grasps of polygonal objects under various combinations of contact models at the contact points. We show that equilibrium can be maintained even when some of the contacts between the fingers and the grasped object are frictionless and tangential contact forces do not exist. Feasible regions to place the fingers on and to safely manipulate the object under various combination of contact models are identified through our analysis.

1. Introduction

Recently, interest toward multi-fingered grasping has been rapidly increased. A particular problem studied by many researchers is the contact configurations for form-closure grasps[3,4,5,6,7]. For such grasps, fingers at the contact points are capable of generating any forces and moments to balance disturbances from the environment[8]. Previous studies on contact configurations assume that the contact models at the contacts are either all frictional[2] or all frictionless[9] contacts but not the combinations of both. Friction makes thing easier since the existence of the tangential contact force increase the range of finger force direction and thus help the generation of desired force and moment. However, the frictional constraint increase the computational complexity for the search of feasible contact force. On the other hand, under frictionless contact model, the contact force is restricted to the direction of contact inward normal and the complexity for the search of contact force is reduced. However, the domain of feasible contact forces for form-closure is restrictive for frictionless contact model and thus this model is over-conservative. The current analysis investigates the contact configurations of polygonal object under various combinations of these contact models. We show that form-closure can be achieved under proper combinations of frictional and frictionless contact models.

The structure of this paper is as follows. In section 2, we introduce the decomposition of contact forces and formulate the relationship between the desired force/moment and the contact forces. We start our analysis for three-fingered grasps of planar objects at points with non-parallel normals in section 3. Methods for blending frictionless and frictional contacts and the concept of ***normal force focus*** are introduced in this section. In sections 4 and 5, we extend our methods to special contact configurations with two or three parallel contact normals. Geometrical constraints for choosing contact points on the contact edges are investigated based on the principle that the fingers need to be in contact with the object and frictional constraints need to be satisfied for frictional contacts. In this analysis, we focus on the contact configurations problem for the object and the fingers are assumed to be able to reach any spot on the object.

2. Decompositions of Contact Forces

Generally, for grasps of planar objects, each contact force can be decomposed into components in the contact normal and tangential directions as :

$$f_i = f_{ni} + f_{ti} = a_i n_i + d_i t_i \qquad i=1,2,3 \tag{1}$$

where n_i and t_i are the unit vectors in the inward normal and the tangential directions of the i^{th} contact point and thus $n_i=[n_{ix}\ ,\ n_{iy}]^T$, $t_i=[n_{iy}\ ,\ -n_{ix}]^T$ (see Figure 1). The variables a_i and d_i are the normal and tangential contact force components to be found for a given set of desired force and moment. The squeezing and frictional constraints in our formulation can be expressed in terms of the magnitudes of the normal and tangential contact force components at each finger as follows:

$$a_i > 0 \text{ and } |d_i| < \mu a_i \quad i=1,2,3 \tag{2}$$

Since frictional constraints can only be satisfied if the fingers are squeezing (or in contact with) the object, the first inequality in equation (2) is a prior condition for the satisfaction of the second inequality in equation (2). Hence, we use $|a_i| = a_i$ in equation (2).

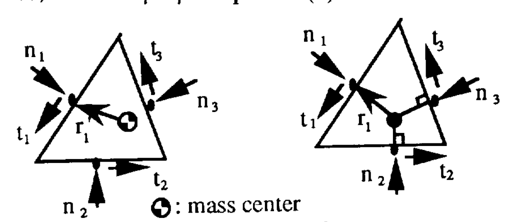

Figure 1. Grasps at non-parallel contact normals

3. Grasp at Non-Parallel Edges

For the grasp of planar object, we assume that the

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contact forces are planar forces and thus these contact forces will generate a resultant force in the same plane of the object and a moment perpendicular to it. The relationship between the desired force and moment (F'_x, F'_y, M'_z) at the mass center and the magnitudes of the normal and tangential contact forces at the contact points can be described as follows (for three fingers) :

$$[F_x', F_y', M_z']^T = [W]\ [d_1, d_2, d_3, a_1, a_2, a_3]^T \quad (3.a)$$

where

$$[W]=[[W_t],[W_n]]=\begin{bmatrix} t_1 & t_2 & t_3 & n_1 & n_2 & n_3 \\ m_{t1} & m_{t2} & m_{t3} & m_{n1} & m_{n2} & m_{n3} \end{bmatrix} = [w_1, w_2, w_3, w_4, w_5, w_6] \quad (3.b)$$

and

$$m_{ti}=r'_i \otimes t_i \text{ and } m_{ni}=r'_i \otimes n_i\ ,\ i=1,2,3 \quad (3.c)$$

The vector r_i' is the i^{th} contact position measured from the mass center and the scalars m_{ti} and m_{ni} are the moments induced at the mass center by unit forces in the i^{th} inward normal and tangential directions. The symbol $\otimes$ is an operator for two planar vectors that gives the magnitude of their cross product, i.e., $r'_i \otimes n_i = r'_{ix} n_{iy} - r'_{iy} n_{ix}$. The matrices $[W_n]$ and $[W_t]$, each of dimension 3x3, are termed the ***normal contact matrix*** and ***tangential contact matrix*** respectively. For contact points whose normal (tangential) vectors do not intersect at a point, $[W_n]$ and $([W_t])$ are non-singular.

A general form for the magnitudes of the tangential and normal contact forces can be expressed as follows:

$$[d_1, d_2, d_3, a_1, a_2, a_3]^T = [d_{p1}, d_{p2}, d_{p3}, a_{p1}, a_{p2}, a_{p3}]^T + k\ [d_{h1}, d_{h2}, d_{h3}, a_{h1}, a_{h2}, a_{h3}]^T \quad (4)$$

where the subscripts p and h stand for the particular and homogeneous solutions of equation (3.a). Form closure requires that there exist a solution to equation (4) for any possible applied force and moment (F_x', F_y', M_z'). Sufficient conditions for form-closure are $a_{hi}>0$ and $\mu a_{hi} > |d_{hi}|$; $i=1,2,3$. If these conditions are satisfied, then the inequalities (2) can be satisfied by appropriately scaling the homogeneous components in equation (4).

Lemma: Form-closure can be achieved under frictional contact model if $a_{hi}>0$,$k>0$ and $\mu a_{hi} > |d_{hi}|$, $i=1,2,3$.

Proof :

A set of d_{pi} and a_{pi} for any (F_x', F_y', M_z') can be found by setting three of d_i and a_i to zero and solve for equation (3.a). The homogeneous solution (d_{hi} and a_{hi}) of equation (3.a) can be used to adjust the a_i and d_i to satisfy the frictional constraint in equation (2) as shown :

$$|d_i| = |d_{pi} + k d_{hi}| < \mu(a_{pi} + k d_{hi}) = \mu a_i\ , \quad (P0)$$

Since $a_{hi}>0$ and $\mu a_{hi} > |d_{hi}|$, we define the following:

$$\delta_i = \mu a_{hi} - |d_{hi}|\ ,\quad \delta_i > 0$$

From Cauchy-Schwarz inequality ($k>0$), we have

$$|d_{pi} + k d_{hi}| \le |d_{pi}| + k|d_{hi}| = |d_{pi}| + k(\mu a_{hi} - \delta_i) = (|d_{pi}| - k\delta_i) + \mu k a_{hi}. \quad (P1)$$

If the variable k is chosen to satisfy the condition : $k > (|d_{pi}| - \mu a_{pi})/\delta_i$, then we have

$$(|d_{pi}| - k\delta_i) + \mu k a_{hi} < \mu(a_{pi} + k d_{hi}). \quad (P2)$$

From equations (P1) and (P2), we reach (P0) :

$$|d_i| = |d_{pi} + k d_{hi}| < \mu(a_{pi} + k d_{hi}) = \mu a_i$$

Thus, form-closure can be achieved using equation (4) if $a_{hi}>0$, $\mu a_{hi} > |d_{hi}|$ and k satisfies the following inequality :

$$k > \max\{(|d_{pi}| - \mu a_{pi})/\delta_i\} \quad i=1,2,3$$

This lemma is also true in general grasps of solid objects. Realizing the significance of the homogeneous solution of equation (3.a), we start our investigation of the structure of the null space of the contact matrix $[W]$. The magnitudes of the internal forces can be found using equation (A.4) as :

$$\begin{bmatrix} d_{h1} \\ d_{h2} \\ d_{h3} \\ a_{h1} \\ a_{h2} \\ a_{h3} \end{bmatrix} = k_1 \begin{bmatrix} (w_4 \times w_5) \cdot w_6 \\ 0 \\ 0 \\ (w_6 \times w_5) \cdot w_1 \\ (w_4 \times w_6) \cdot w_1 \\ (w_5 \times w_4) \cdot w_1 \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ (w_4 \times w_5) \cdot w_6 \\ 0 \\ (w_6 \times w_5) \cdot w_2 \\ (w_4 \times w_6) \cdot w_2 \\ (w_5 \times w_4) \cdot w_2 \end{bmatrix} + k_3 \begin{bmatrix} 0 \\ 0 \\ (w_4 \times w_5) \cdot w_6 \\ (w_6 \times w_5) \cdot w_3 \\ (w_4 \times w_6) \cdot w_3 \\ (w_5 \times w_4) \cdot w_3 \end{bmatrix} \quad (5)$$

The three internal force magnitudes given in equation (5) are similar. Each of them have two of the tangential forces equalling zero. This special structure allows us to conveniently consider form-closure with frictionless contacts. By setting $k_i=0$ and letting the other elements of k grow without bound, the tangential component of the i^{th} contact becomes insignificant.

Two Frictionless and One Frictional Contacts

By setting k_2 and k_3 to zero, we achieve the equilibrium of the object using two frictionless contacts and one frictional contact. To maintain the contacts between fingers and the object while still keep the object in equilibrium, the squeezing constraints must be satisfied ($a_{hi}>0$). Thus, for all contact forces to be compressive, the following condition must be met :

$$sgn(w_6 \times w_5 \bullet w_1)=sgn(w_4 \times w_6 \bullet w_1)=sgn(w_5 \times w_4 \bullet w_1)=sgn(k_1) \quad (6)$$

To ensure that the frictional contact (i.e., contact 1) does not slip, we must have:

$$|(w_5 \times w_6)\bullet w_4| \leq \mu\ |(w_6 \times w_5)\bullet w_1| \quad (7)$$

Two Frictional and One Frictionless Contacts

By setting one of $\{k_1,k_2,k_3\}$ in equation (6) to zero, we model two frictional contacts (fixed contacts) and one frictionless contact. There are three combinations of such contact models and we demonstrate the case for $k_3=0$. For such grasps, the squeezing constraints, lead to the following inequalities:

$$\begin{aligned}(w_6 \times w_5)\bullet(k_1 w_1 + k_2 w_2)>0\\ (w_4 \times w_6)\bullet(k_1 w_1 + k_2 w_2)>0\\ (w_5 \times w_4)\bullet(k_1 w_1 + k_2 w_2)>0\end{aligned} \quad (8)$$

To maintain the two frictional contacts, the frictional constraints give :

$$\begin{aligned}|(w_4 \times w_5)\bullet w_6| \leq \mu(w_6 \times w_5)\bullet (k_1 w_1 + k_2 w_2)\\ |(w_4 \times w_5)\bullet w_6| \leq \mu(w_4 \times w_6)\bullet(k_1 w_1 + k_2 w_2)\end{aligned} \quad (9)$$

Three Frictional Contacts

Another expression of the internal forces magnitudes can be found using equation (A.5) :

$$\begin{bmatrix} d_{h1}\\ d_{h2}\\ d_{h3}\\ a_{h1}\\ a_{h2}\\ a_{h3}\end{bmatrix}=\begin{bmatrix}(w_3 \times w_2)\cdot(k_1 w_4+k_2 w_5+k_3 w_6)\\ (w_1 \times w_3)\cdot(k_1 w_4+k_2 w_5+k_3 w_6)\\ (w_2 \times w_1)\cdot(k_1 w_4+k_2 w_5+k_3 w_6)\\ k_1(w_1 \times w_2)\cdot w_3\\ k_2(w_1 \times w_2)\cdot w_3\\ k_3(w_1 \times w_2)\cdot w_3\end{bmatrix} \quad (10)$$

To maintain the contacts between the objects and the fingers, a_{hi} needs to be positive. Thus the following relationship must holds :

$$sgn(k_1)=sgn(k_2)=sgn(k_3)=sgn((w_1 \times w_2)\bullet w_3). \quad (11)$$

The frictional constraints ($|d_{hi}| \leq \mu a_{hi}$) gives:

$$\begin{aligned}|(w_3 \times w_2)\bullet (k_1 w_4+k_2 w_5+k_3 w_6)| \leq \mu k_1(w_1 \times w_2)\bullet w_3\\ |(w_1 \times w_3)\bullet (k_1 w_4+k_2 w_5+k_3 w_6)| \leq \mu k_2(w_1 \times w_2)\bullet w_3\\ |(w_2 \times w_1)\bullet (k_1 w_4+k_2 w_5+k_3 w_6)| \leq \mu k_3(w_1 \times w_2)\bullet w_3\end{aligned} \quad (12)$$

Equations (11) and (12) bound a feasible region for k_1,k_2,k_3 such that the homogeneous solution in equation (10) will satisfy the squeezing and frictional constraints. If the feasible region of k_1,k_2,k_3 is a null region, the fingers at the contact points can not hold the object firmly so the contact configuration does not have form-closure.

The Normal Force Focus

A special situation occurs when the contact normals intersect at a point called the ***normal force focus*** (see Figure 1). In our work, this is a key concept which will be discussed at length and adapted to various geometry. A pair of force and moment (F_x', F_y', M_z') at the mass center of an object is equivalent to another force and moment (F_x, F_y, M_z) at the ***normal force focus*** and the normal and tangential contact forces are related to the equivalent force and moment as follows:

$$\begin{bmatrix}F_x\\ F_y\\ M_z\end{bmatrix}=[\ [W_t]\ ,\ [W_n]\ ]\begin{bmatrix}D\\ A\end{bmatrix}=\begin{bmatrix}t_1 & t_2 & t_3 & n_1 & n_2 & n_3\\ |r_1| & |r_2| & |r_3| & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}D\\ A\end{bmatrix} \quad (13)$$

where $[D^T\ A^T]=[d_1,d_2,d_3,a_1,a_2,a_3]$. The zeros in the bottom row of $[W_n]$ appear due to the fact that the contact normal intersect at the ***normal force focus***. A set of internal contact forces can be found as[2] :

$$[d_{h1},d_{h2},d_{h3},a_{h1},a_{h2},a_{h3}]^T=k[0,0,0,n_2 \otimes n_3\ ,n_3 \otimes n_1,n_1 \otimes n_2]^T \quad (14)$$

where $n_i \otimes n_j = sin\theta_{ij}$ with θ_{ij} the angle between n_i and n_j (see Figure 1). Using the internal forces given in equation (14) in the general solution of contact forces in equation (4), we see that any force and moment at the ***normal force focus*** (and therefore also at the mass center) can be generated ***(form-closure)*** under squeezing and frictional constraints at the contact points if θ_{12}, θ_{23} and θ_{31} are less than 180^0 and at least one contact is frictional.

4. Three Contacts with Two Parallel Contact Normals

We now discuss the contact configurations where two of the contact normals are parallel or anti-parallel.

4.1. Two Contact Normals Parallel

A special contact configuration occurs when two of the contact normals are parallel and the tangential directions of these contacts are along the same line (see Figure 2.a). The angle between the contact edges equals ϕ. In this case, we define the ***modified normal force focus*** at the point where the line along the third contact normal intersects the line connecting the first two contact points (Figure 2.a). Currently, we assume that the ***modified normal force focus*** falls between the first two contact points. We will show this assumption is a necessary condition for form-closure in our later discussion. A coordinate system is assigned at the ***modified normal force focus*** and the positions of the contact points can be described as : $r_1=(0,r_{1y}), r_2=(0,r_{2y}), r_3=(|r_3|cos\phi,|r_3|sin\phi)$. The relation between the force and moment at the ***normal force focus*** and the normal and tangential contact force magnitudes can be expressed as :

$$\begin{bmatrix} F_x \\ F_y \\ M_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\sin\phi & 1 & 1 & -\cos\phi \\ -1 & -1 & \cos\phi & 0 & 0 & -\sin\phi \\ 0 & 0 & |r_3| & -r_{1y} & r_{2y} & 0 \end{bmatrix} \begin{bmatrix} D \\ A \end{bmatrix} = [W] \begin{bmatrix} D \\ A \end{bmatrix} \quad (15)$$

The first (last) three columns of $[W]$ consist of the unit vectors for the tangential (normal) contact forces and the moment generated at the ***modified normal force focus*** by the unit tangential (normal) contact forces. A set of internal contact force components can be found as (equation (A.4)):

$$\begin{bmatrix} d_{h1} \\ d_{h2} \\ d_{h3} \\ a_{h1} \\ a_{h2} \\ a_{h3} \end{bmatrix} = k_1 \begin{bmatrix} (r_{2y}-r_{1y})\sin\phi \\ 0 \\ 0 \\ -r_{2y}\cos\phi \\ r_{1y}\cos\phi \\ r_{1y}-r_{2y} \end{bmatrix} + k_2 \begin{bmatrix} 0 \\ (r_{2y}-r_{1y})\sin\phi \\ 0 \\ -r_{2y}\cos\phi \\ r_{1y}\cos\phi \\ r_{1y}-r_{2y} \end{bmatrix} + k_3 \begin{bmatrix} 0 \\ 0 \\ (r_{1y}-r_{2y})\sin\phi \\ |r_3|\sin\phi-r_{2y} \\ r_{1y}-|r_3|\sin\phi \\ (r_{1y}-r_{2y})\cos\phi \end{bmatrix} \quad (16)$$

The following discussions demonstrate the physical significance of each homogeneous solution and the geometrical restrictions for deciding the contact points.

Two frictionless contacts and one frictional contact

In equation (16), each of the homogeneous solution has two of its tangential contact forces equalling zero. The corresponding contact models at these contact points are frictionless contacts. Additionally, the first and second components of homogeneous solutions of the contact forces have the same normal contact forces and the nonzero tangential contact forces are of the same magnitude. Thus, we will discuss these two components of homogeneous solution in an unified manner. The third components of the homogeneous solution of the contact forces is quite different from the first two and will be discussed separately.

For the first two components of the internal contact force magnitudes to satisfy the squeezing constraint, we need $a_{hi}>0$ which leads to the following conditions :

$$0° < \phi < 90°,\ r_{1y} > 0 \text{ and } r_{2y} < 0. \quad (17)$$

For the first component of the internal force magnitudes ($d_{h1}\neq 0$, $d_{h2}=d_{h3}=0$) to satisfy the frictional constraint,we substitute d_{h1} and a_{h1} given in the first term of equation (16) into the frictional constraint ($|d_{h1}|<\mu a_{h1}$) to reach the following necessary conditions :

$$\frac{r_{1y}-r_{2y}}{-r_{2y}} < \frac{\mu}{\tan\phi}\ . \quad (18)$$

Similarly, for the second component of the internal force magnitude to be feasible, we have the following :

$$\frac{r_{1y}-r_{2y}}{r_{1y}} < \frac{\mu}{\tan\phi}\ ,\ r_{1y}>0\ ,r_{2y}<0. \quad (19)$$

Geometrically, inequality (17) requires that the line along the third contact normal intersect the line segment connecting the first two contact points,i.e., the ***modified normal force focus*** must fall between the first two contact points. In other words, the third contact point must be chosen to lie in the projection of the line segment connecting the first and second contact points onto the third edge of the triangle (the segment AC in Figure 2.b).

Inequality (18) can now be visualized as follows. Notice that the left-hand side of inequality (18) is the ratio of the lengths of two line segments : the segment connecting the first two contact points and the length connecting the ***modified normal force focus*** and the second contact points. Geometrically (see Figure 2.b), we have the following condition for this ratio :

$$1 < (r_{1y}-r_{2y})/(-r_{2y}).$$

Thus, for inequality (18) to be true, we have the following condition:

$$1 < (\mu/\tan\phi) \text{ or } \tan\phi < \mu.$$

Suppose that the above condition is satisfied,i.e.,$\tan\phi<\mu$, and the third finger is initially placed on the projection of the first contact point onto the third edge (point A in Figure 2.b). Thus, initially $r_{1y}=0$ and the left-hand side of the inequality (18) equal to 1 since the ***modified normal force focus*** is defined at the same spot as the first contact. As the third finger slides downward along the third edge (the contact at this point is frictionless), the ***modified normal force focus*** move downward and the ratio $(r_{1y}-r_{2y})/(-r_{2y})$ increases (so does the ratio of the projections of these line segments onto the third edge). In the incipient condition where the first component of equation (16) is about to be invalid, we have $|AC|/|EC|=\mu/\tan\phi$ and the distance on which the third contact can safely slide can be expressed as follows according to Figure 2.b. :

$$|\overline{AE}| = (1-\frac{\tan\phi}{\mu})\ |\overline{AC}|\ . \quad (20)$$

Similar explanations can be applied for the second component of internal contact force magnitude from inequality (19). As the third finger slides upward along the third edge from the projection of the second contact onto the third edge (point C in Figure 2.b), $(r_{1y}-r_{2y})/(r_{1y})$ increases. For the incipient condition, we have $|AC|/|E'C|=\mu/\tan\phi$. The distance on which the third contact can safely slide can be expressed as (Figure 2.b):

$$|\overline{E'C}| = (1-\frac{\tan\phi}{\mu})\ |\overline{AC}| \quad (21)$$

For the third component of the internal force magnitudes to satisfy the squeezing constraint, the following conditions must be satisfied.

$$0°<\phi<90°,\ r_{2y}< |r_3|sin\phi=r_{3y} < r_{1y}. \quad (22)$$

In this component of internal forces, the frictional constraint needs to be satisfied at the third contact point ($d_{h1}=d_{h2}=0, d_{h3}\neq 0$) and the inequality $|d_{h3}|<\mu a_{h3}$ gives the following inequality :

$$tan\phi < \mu \quad \text{or} \quad 1<\mu/\,tan\phi\,. \quad (23)$$

Inequality (22) requires the third contact point be chosen from the segment of the third edge bounded by the parallel strip perpendicular to the first contact edge and passing through the first two contact points (B and D in Figure 2.c). Inequality (23) requires the angle between the first and third edges to be less than $tan^{-1}\mu$. The first two fingers are allowed to slide along the first edge once inequalities (22) and (23) are satisfied. Apparently, the third component of internal contact force is superior than the first two components of internal contact forces since constraints (22) and (23) can be easily met.

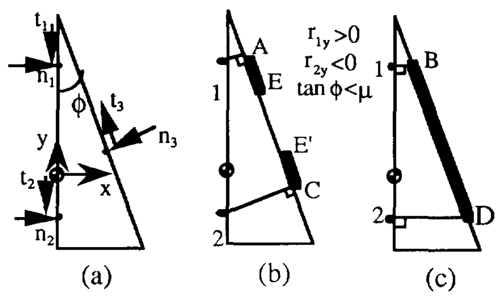


Figure 2. Feasible regions for placing the third finger for two frictionless and one frictional contacts

Two frictional contacts and one frictionless contact

Physically, we can have two frictional contacts and one frictionless contact by assigning one of $\{k_1,k_2,k_3\}$ in equation (16) to zero. Previously, we found that the object can be most efficiently held if the third contact is frictional while the other two contacts are frictionless. We now investigate the complement situation where the third contact is frictionless and is allowed to slide on the third edge while the first two contact is frictional and need to be fixed. Let $k_1=kr_{2y}/(r_{2y}-r_{1y}), k_2=-kr_{1y}/(r_{2y}-r_{1y})$ and $k_3=0$ in equation (16) to reach the following expression :

$$[d_{h1},d_{h2},d_{h3},a_{h1},a_{h2},a_{h3}]=k[r_{2y}sin\phi,-r_{1y}sin\phi,0,\; -r_{2y}cos\phi,r_{1y}cos\phi,r_{1y}-r_{2y}] \quad (24)$$

Notice that the magnitudes of the normal contact forces (the last three elements of equation (24)) remain the same as the corresponding elements in the second and third terms of equation (16). Thus, the squeezing constraints can be described using inequality (17). The frictional constraints to be satisfied at the first two contacts gives ($|d_{hi}|<\mu a_{hi}$, i=1,2):

$$tan\phi<\mu\;.$$

The above condition is identical to inequality (23). Thus, if the angle between the contact edges ϕ, is smaller than $tan^{-1}\mu$, then equilibrium can be maintained even when the third finger slides along the portion third edge bounded the projection of the first two contact points (A and C in Figure 3.a).

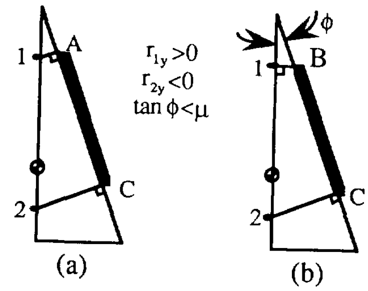


Figure 3. Feasible regions for placing the third finger
(a) contacts 1 and 2 are frictional and contact 3 is frictionless
(b) Three frictional contacts

Three Frictional Contacts

The case for three frictional contacts can be found by combining the third term of equations (16) and equation (24) as:

$$[D^T A^T]_H= k[r_{2y}sin\phi,-r_{1y}sin\phi,0,-r_{2y}cos\phi,r_{1y}cos\phi,r_{1y}-r_{2y}] + k_3[0,0,(r_{1y}-r_{2y})sin\phi,\; |r_3|sin\phi-r_{2y}, r_{1y}-|r_3|sin\phi,(r_{1y}-r_{2y})cos\phi]^T\;. \quad (25)$$

The following conditions guarantee the squeezing effects:

$$0°<\phi<90°,\ r_{1y}>0,\ r_{2y}<0, \text{ and } \ r_{2y}< |r_3|sin\phi=r_{3y}< r_{1y}\,. \quad (26)$$

The feasible region for placing the third contact point will be the intersection regions shown in Figure 3.b (segment BC). Substituting d_{hi} and a_{hi} given in equation (25) into $|d_{hi}|<\mu a_{hi}$ to reach the following condition :

$$-\,kr_{2y}sin\phi \;<\mu\;[-kr_{2y}\cos\phi+ k_3(|r_3|\sin\phi -r_{2y})]$$
$$kr_{1y}sin\phi \;<\mu\;[kr_{1y}\cos\phi+ k_3(r_{1y}-|r_3|\sin\phi)]$$
$$k\,(r_{1y}-r_{2y})\,sin\phi <\mu\;[k\,(r_{1y}-r_{2y})+ k_3\,(r_{1y}-r_{2y})\cos\phi\;]\,.$$

Dividing the first, second, and third inequalities shown above by $-kr_{2y}\cos\phi$, $kr_{1y}\cos\phi$, and $k_3(r_{1y}-r_{2y})\cos\phi$ respectively (k and k_3 are positive) gives

$$\begin{aligned} &\tan\phi < \mu(1+\gamma_1),\ \gamma_1=[k_3(|r_3|\cos\phi-r_{2y})]/(-kr_{2y}\cos\phi) > 0 \\ &\tan\phi < \mu(1+\gamma_2),\ \gamma_2=[k_3(r_{1y}-|r_3|\sin\phi)]/(k_1|r_1|\cos\phi) > 0 \\ &\tan\phi < \mu(1+\gamma_3),\quad \gamma_3 = k\sec\phi/k_3 > 0 . \end{aligned} \qquad (27)$$

The values of γ_1,γ_2 and γ_3 are positive since the contact points are chosen according to inequality (26). Inequality (27) can be summarized as follows :

$$\tan\phi < \mu \min\{1+\gamma_1, 1+\gamma_2, 1+\gamma_3\}. \quad (28)$$

We see that inequality (23) is a conservative bound for the case of three frictional contacts since the satisfaction of inequality (23) guarantees the satisfaction of (28).

4.2. Two Oppositely Directed Contact Normals

Finally, we discuss the contact configurations with two of the contact normals direct oppositely (Figure 4). The ***modified normal force focus*** is assigned at the intersection of lines along the non-parallel contact normals (Figure 4). Assigning a coordinate system at the ***modified normal force focus*** with the x- axis parallel to two of contact normals (Figure 4). The relation between a desired force/moment at the ***modified normal force focus*** and the contact forces can be described as :

$$\begin{bmatrix} F_x \\ F_y \\ M_z \end{bmatrix} = [W]\begin{bmatrix} D \\ A \end{bmatrix},\ [W]=\begin{bmatrix} 0 & 0 & \cos\phi & -1 & 1 & -\sin\phi \\ 1 & -1 & \sin\phi & 0 & 0 & \cos\phi \\ r_{1x} & -r_{2x} & |r_3| & r_{1y} & 0 & 0 \end{bmatrix} \quad (29)$$

where ϕ is the angle between those contact edges whose contact normals define the ***modified normal force focus***. The homogenous solution of $[D^T\ A^T]^T$ can be found as :

$$\begin{bmatrix} d_{h1} \\ d_{h2} \\ d_{h3} \\ a_{h1} \\ a_{2h} \\ a_{3h} \end{bmatrix} = k_1\begin{bmatrix} -r_{1y}\cos\phi \\ 0 \\ 0 \\ r_{1x}\cos\phi \\ r_{1x}\cos\phi+r_{1y}\sin\phi \\ r_{1y} \end{bmatrix} + k_2\begin{bmatrix} 0 \\ -r_{1y}\cos\phi \\ 0 \\ -r_{2x}\cos\phi \\ -r_{2x}\cos\phi-r_{1y}\sin\phi \\ -r_{1y} \end{bmatrix} + k_3\begin{bmatrix} 0 \\ 0 \\ -r_{1y}\cos\phi \\ |r_3|\cos\phi \\ |r_3|\cos\phi+r_{1y} \\ r_{1y}\sin\phi \end{bmatrix} \quad (30)$$

Two frictionless contacts and one frictional contact

By setting two of $\{k_1,k_2,k_3\}$ in equation (30) to zero, a set of internal contact force magnitudes can be found for the case of two frictionless and one frictional contacts (two of d_{h1},d_{h2},d_{h3} equal zero). The geometrical restrictions for placing the contact points, drawn from the squeezing and frictional constraints with the scheme used previously, are summarized in the follows.

case 1 : The first contact is assumed to be frictional while the second and third contacts are frictionless ($k_2=k_3=0$):

$$\phi<90^0,\ r_{1x}>0,\ r_{1y}>0,\ r_{1y}/r_{1x}<\mu . \quad (31)$$

Geometrically, the first finger must be placed above the second fingers ($r_{1y}>0$ and $r_{2y}=0$). Suppose that the second finger is placed in a spot such that $r_{1y}/r_{1x}=\mu$ (Figure 4.a). As the second finger moves upward the value of r_{1y}/r_{1x} decreases and the third inequality of (31) is satisfied.

case 2 : The second contact is frictional while the first and third contacts are frictionless ($k_1=k_3=0$) :

$$\phi<90^0,\ r_{1y}<0,\ r_{2x}<0,\ (-r_{1y}\cos\phi)/(-r_{2x}\cos\phi-r_{1y}\sin\phi)<\mu. \quad (32)$$

Geometrically, the first finger must be placed below the second fingers ($r_{1y}<0$ and $r_{2y}=0$). Thus the first and second components of the internal force magnitudes can't be valid at the same time. Assume that the first and third contact points are so placed such that the incipient condition $(-r_{1y}\cos\phi)/(-r_{2x}\cos\phi-r_{1y}\sin\phi)=\mu$ in equation (32) holds. By decreasing $-r_{1y}$ or increasing $-r_{2x}$ (both positive number from inequality (32)), the inequality $(-r_{1y}\cos\phi)/(-r_{2x}\cos\phi-r_{1y}\sin\phi)<\mu$ can be achieved. Geometrically, these scenarios correspond to moving the first finger upward along its contact edge or moving the third finger in the up-right direction along its contact edge (Figure 4.b).

case 3 : The third contact is frictional while the first and second contacts are frictionless ($k_1=k_2=0$):

$$\phi<90^0,\quad r_{1y}>0 \quad \text{and} \quad 1/\mu<\tan\phi . \quad (33)$$

Again,the first finger must be placed above the second fingers ($r_{1y}>0$ and $r_{2y}=0$, see Figure 4.a). Also, the occasion to use this set of internal force magnitudes is in odd with the occasion to the second set of internal forces.

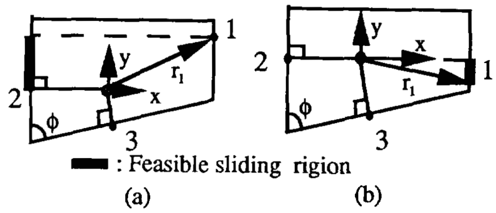


Figure 4. One frictional and two frictionless contacts
(a) Finger 1 or 3 is frictional
(b) Finger 2 is frictionl

Two Frictional and one Frictionless contacts

We learned that the second component of the internal force magnitudes can't be used along with the first and third components of internal force magnitudes. We combine the first and third components of internal force magnitudes in equation (30) for the case of two frictional and one frictionless contacts. The second finger is allowed to slide along its contact edge (frictionless). By letting $k_1=k(r_{1y}cos\phi+|r_3|)/r_1$, $k_2=0$ and $k_3=-kr_{1x}/r_{1y}$ in equation (30), we have the following expression :

$$\begin{bmatrix} d_{h1} \\ d_{h2} \\ d_{h3} \\ a_{h1} \\ a_{2h} \\ a_{3h} \end{bmatrix} = k \begin{bmatrix} (-r_{1y}\cos\phi+|r_3|)\cos\phi \\ 0 \\ r_{1x}\cos\phi \\ r_{1x}\cos^2\phi \\ (-r_{1x}\sin\phi+r_{1y}\cos\phi+|r_3|)\sin\phi \\ -r_{1x}\sin\phi+r_{1y}\cos\phi+|r_3| \end{bmatrix}. \quad (34)$$

The squeezing constraints lead to the following conditions:

$$r_{1x} > 0 \quad \text{and} \quad r_{1y}cos\phi + |r_3| > r_{1x}\, sin\phi \quad . \quad (35)$$

To satisfy the frictional constraints at the first and third contacts, we have the following conditions :

$$r_{1y}cos\phi+|r_3| < \mu\, r_{1x}\, cos\phi$$
$$r_{1x}\, cos\phi < \mu\, (|r_3|+r_{1y}cos\phi- r_{1x}\, sin\phi\,). \quad (36)$$

Two-fingered Grasp with Opposite Contact Normals

Finally, assigning $k_1=k_2=k$ and $k_3=0$ in equation (30), we obtain a special component of internal force magnitudes as follows :

$$[d_{h1},d_{h2},d_{h3},a_{h1},a_{h2},a_{h3}]= k[-r_{1y}cos\phi,-r_{1y}cos\phi,0,(r_{1x}-r_{2x})cos\phi,(r_{1x}-r_{2x})cos\phi,0]. \quad (37)$$

Physically, this component of internal forces corresponds to a two-fingered grasp with oppositely directed contact normals and the third finger is released from its contact edge ($d_{h3}=0$, $a_{h3}=0$). The squeezing constraints lead to the inequality $r_{1x}-r_{2x} > 0$ and the frictional constraints lead to the inequality $|-r_{1y}|/(r_{1x}-r_{2x}) < \mu$. Using the fact that $r_{2y}=0$, this inequality can be expressed as $|r_{2y}-r_{1y}|/(r_{1x}-r_{2x}) < \mu$. Additionally, we learn from equation (37) that the tangential forces are not required in the case of $r_{1y}=0$ which implies the alignment of the first and second contact normals. Equilibrium can be achieved by simply squeezing the object using normal contact forces.

5. Grasp Points with All Parallel Normals

For three-fingered grasps with all parallel contact normals, one of the inward normal must direct oppositely to the other two to achieve force equilibrium in the normal direction (Figure 5). The ***modified normal force focus*** can be defined on any point along the line containing the opposite directed inward normal. Assign a coordinate system at the ***modified normal force focus*** with the directions of the x and y axes parallel to the tangential and normal directions of the contacts. The normal contact forces are capable of generating any force in the y-direction and a moment in the directions normal to the planar object. For the normal forces to balance, the following equation holds

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} n_1 & n_2 & n_3 \\ m_{n1} & m_{n2} & m_{n3} \end{bmatrix} [A_h] = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -1 & -1 \\ 0 & r_{2x} & r_{3x} \end{bmatrix} \begin{bmatrix} a_{h1} \\ a_{h2} \\ a_{h3} \end{bmatrix} \quad (38)$$

A set of feasible $[A_h]$ can be immediately found as

$$[A_h]=[a_{h1},a_{h2},a_{h3}]^T=k_1[1,\frac{-r_{3x}}{r_{2x}-r_{3x}},\frac{r_{2x}}{r_{2x}-r_{3x}}]^T \quad (39)$$

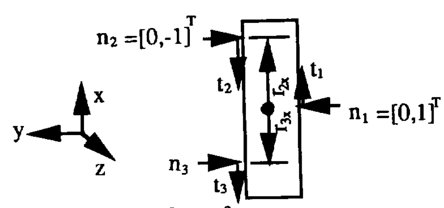
To achieve squeezing ($a_{hi}>0$) for the internal forces given inequation (39), we need $r_{2x}>0$ and $r_{3x}<0$. Physically, $r_{2x}>0$ and $r_{3x}<0$ imply that the opposite directed inward normal to lie between the other two inward normals (Figure 5). A set of the magnitudes of the normal contact forces that generate a force F_y at the ***normal force focus*** without producing any moment M_z can be verified as $[A_p]=[F_y,0,0]^T$ for $F_y>0$ and $[A_p]=F_y[0,r_{3x}/(r_{2x}-r_{3x}),-r_{2x}/(r_{2x}-r_{3x})]^T$ for $F_y<0$. The rest of the desired force and moment (F_x,M_z) can be generated using the tangential contact forces

$$\begin{bmatrix} F_x \\ 0 \\ M_z \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ -r_{1y} & r_{2y} & r_{3y} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \; or \; \begin{bmatrix} F_x \\ M_z \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ -r_{1y} & r_{2y} & r_{3y} \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \quad (40)$$

The homegeneous solution of equation (40) can be found using equation (A7) as:

$$[d_{h1},d_{h2},d_{h3}]=k_2[r_{2y}-r_{3y},r_{1y}-r_{3y},r_{2y}-r_{1y}]. \quad (41)$$

We can form a completely set of the internal contact forces by combining equations (39) and (41). Inequalities that constraint the geometry of the contact points can be found by using this complete set of d_{hi} and a_{hi} in the inequality $|d_{hi}|<\mu a_{hi}$. Also, contact points satisfying these geometrical constraints will also assure form-closure grasp by properly choosing k_1 and k_2 in equations (39) and (41).



●: modified normal force focus

Figure 5. Grasps at contact points with parallel normals

6. Conclusions

This paper provides an unified approach to the analysis of three-fingered form-closure grasps of polygonal objects. Equilibrium and form-closure are shown to be achievable under various combinations of frictionless and frictional contacts models at the fingers. Regions for safely manipulating the polygonal object can be easily identified using our method. Special contact configurations are taken into account. The results of this research will be useful for planning the dexterous manipulation of objects since our method allows the adjustment of contact positions (or regrasping) with the equilibrium of the object maintained. We will extend this approach to the case of solid objects.

7. Acknowledgement

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Appendix A. Null Space of a rectangular matrix

Consider the following algebraic equation

$$[v_1,v_2,v_3,v_4]_{3\times4}[x]_{4\times1}=[U]_{3\times1}\ ,\ [x]^T=[x_1,x_2,x_3,x_4] \qquad (A.1)$$

where v_i is a 3×1 column matrix and $[V]=[v_1,v_2,v_3,v_4]$. The solution of $[x]$ consists of a particular and homogeneous solutions. Since this is a system with only one extra variable, the homogeneous solution of $[x]$ can be found using Laplace's Theorem as :

$$[x]^T_{null}=k[(v_3\times v_2)\bullet v_4,(v_1\times v_3)\bullet v_4,(v_2\times v_1)\bullet v_4,(v_1\times v_2)\bullet v_3] \qquad (A.2)$$

This technique has been applied to find the joint velocities of redundant arms with one extra degree of freedom (Bailleul 1985). Now, Consider the following equation

$$[v_1,v_2,v_3,v_4,v_5,v_6]_{3\times6}[x]_{6\times1}=[U]_{3\times1} \qquad (A.3)$$

where $[x]^T=[x_1,x_2,x_3,x_4,x_5,x_6]$. Three sets of $[x]_{null}$ can be found by setting two of $\{x_1,x_2,x_3\}$ to zero and taking out the corresponding v_i's in equation (A.3) to make $[V]$ a 3×4 matrix. A general form of $[x]_{null}$ can be found using equation (A.2).

$$\begin{aligned}[x]^T_{null}=&k_1[(v_4\times v_5)\bullet v_6,0,0,(v_6\times v_5)\bullet v_1,(v_4\times v_6)\bullet v_1,(v_5\times v_4)\bullet v_1]\\ &+k_2[0,(v_4\times v_5)\bullet v_6,0,(v_6\times v_5)\bullet v_2,(v_4\times v_6)\bullet v_2,(v_5\times v_4)\bullet v_2]\\ &+k_3[0,0,(v_4\times v_5)\bullet v_6,(v_6\times v_5)\bullet v_3,(v_4\times v_6)\bullet v_3,(v_5\times v_4)\bullet v_3]\end{aligned} \qquad (A.4)$$

Similarly, three sets of $[x]_{null}$ can be found by setting two of $\{x_4,x_5,x_6\}$ to zero

$$\begin{aligned}[x]^T_{null}=&k_1[(v_3\times v_2)\bullet v_4,(v_1\times v_3)\bullet v_4,(v_2\times v_1)\bullet v_4,(v_1\times v_2)\bullet v_3,0,0]\\ &+k_2[(v_3\times v_2)\bullet v_5,(v_1\times v_3)\bullet v_5,(v_2\times v_1)\bullet v_5,0,(v_1\times v_2)\bullet v_3,0]\\ &+k_3[(v_3\times v_2)\bullet v_6,(v_1\times v_3)\bullet v_6,(v_2\times v_1)\bullet v_6,0,0,(v_1\times v_2)\bullet v_3]\end{aligned}$$

$$\begin{aligned}=&[(v_3\times v_2)\bullet(k_1v_4+k_2v_5+k_3v_6),\quad (v_1\times v_3)\bullet(k_1v_4+k_2v_5+k_3v_6),\\ &(v_2\times v_1)\bullet(k_1v_4+k_2v_5+k_3v_6),\\ &k_1(v_1\times v_2)\bullet v_3,\ k_2(v_1\times v_2)\bullet v_3,\ k_3(v_1\times v_2)\bullet v_3]\end{aligned} \qquad (A.5)$$

Equation (A.4) is used in equation (5),(16), (30) of the main text. Equation (A.5) is used in equation (10). Similarly, consider the following equation

$$[V]_{2\times3}[x]_{3\times1}=[U]_{2\times1}\ ,\ [v]^T=[v_1,v_2,v_3] \qquad (A.6)$$

where $v_i=[v_{ix},v_{iy}]^T$. The homogeneous solution of $[x]$ is similar to equation (A.2) :

$$[x]^T_{null}=k[v_2\otimes v_3,v_3\otimes v_1,v_1\otimes v_2] \qquad (A.7)$$

where $v_i\otimes v_j=v_{ix}v_{jy}-v_{jx}v_{iy}$.Equation (A.7) is used to find the homogeneous solutions of the normal and tangential contact force magnitudes (equations (38) and (40)) for grasps at points with all parallel contact normals.