On the Algebraic Geometry of Contact Formation Cells for Systems of Polygons

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Abstract

The efficient planning of contact tasks for intelligent robotic systems requires a thorough understanding of the kinematic constraints imposed on the system by rolling and sliding contacts. In this paper, we derive closed-form analytic solutions for the position and orientation of a passive polygon moving in contact with two or three active polygons whose positions and orientations are independently controlled. This is done by applying elimination techniques to solve the systems of appropriate contact constraint equations. We prove that the systems of contact constraint equations are smooth submanifolds of configuration space.

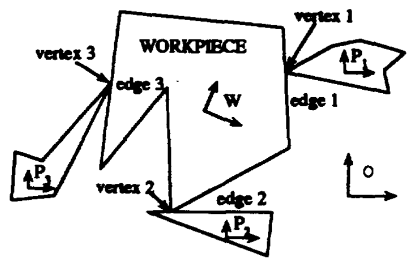

Figure 1: A System of Bodies in Contact.

1 Introduction

Consider a planar system of rigid polygonal bodies in contact (see Figure 1) and assume that the positions and orientations of all but one of the polygons are actively controlled. The polygon that is not actively controlled, but rather is "grasped," is referred to as the workpiece or the passive polygon, and those that are actively controlled through joint actuation are collectively referred to as the manipulator or the active polygons. The workpiece and the manipulator taken together are referred to as the manipulation system, or just the system. The process of reorienting the workpiece by controlling the manipulator is known as dexterous manipulation.

When the level of uncertainty is a significant factor in the planning and execution of a dexterous manipulation task, we refer to it as a fine motion or fine manipulation task. In contrast to the classical piano movers' problem [9] in which contact is avoided, every possible solution to any dexterous manipulation planning problem must include periods of contact.

Automatic fine manipulation planning is one of the most important unsolved problems in the field of robotics. Tasks in this class include mechanical assembly/disassembly and grasping operations. The development of a practical, reliable planner for this class of tasks would facilitate the automation of large portions of various manufacturing and service industries and would expand our ability to work in Space and other hazardous environments.

The major problem preventing the development of practical fine manipulation planners is that the best general-purpose planning algorithms have worst-case running times that are exponential in the dimension of the system's configuration space (C-space)[2]. An additional source of difficulty peculiar to fine manipulation planning is that a multibody contact model allowing for multiple sliding and rolling contacts must be used to predict the motion of the system.

Despite the difficulties, there are good reasons to believe that practical fine manipulation planning algorithms can be developed: first, to find a plan, the entire C-space need not be decomposed (for example, see[10]; second, quasistatic rigid body models of mechanics often capture enough of the relevant physical phenomena to derive reliable manipulation plans (for example, see [8, 10]) ; third, compliant control can facilitate the maintenance of large numbers of contacts, thereby helping to reduce the dimension of C-space; and fourth, cell decomposition techniques lend themselves to parallel and distributed computation.

A final practical issue is that a fine manipulation planner requires the representation of portions of C-space. Since C-space will typically have a farily high dimensionality, a representation that places minimal demands on computer memory resources is needed.

Our contribution to fine manipulation planning relates most directly to this final issue. In this paper, we present the geometric characterization and analytic representation of eight commonly occuring types of lower-dimensional subsets of C-space (called "contact formation cells" [11]. Using simple techniques from algebraic geometry, we derive analytic representations of the CF-cells which yield the following benefits: they provide an efficient means to represent large relevant portions of C-space (entire cells) by just the coefficients of the solutions; they obviate the need for the iterative solution of the nonlinear systems of contact constraints to determine the position and orientation of the workpiece given the configuration of manipulator; and they reveal configurations in which the motion of the system (under a quasistatic model of mechanics) can get stuck or become uncontrollable.

A contact formation cell (CF-cells) corresponds to a distinct combination of the contact constraint equations associated with sliding and/or rolling, type A and/or type B contacts [7] (see definitions below). Each sliding contact yields one constraint equation, while each rolling contact yields two. Under pure position control, the number of contact constraints is generally three, because all of the degrees of freedom of the system except for the three associated with the workpiece are directly controlled.

Even though we do not study three-dimensional systems, our analysis will be immediately useful in planning the manipulation of solid objects with constant cross-sectional geometry.

2 Problem Statement

We define type A and B elemental contacts as follows:
Type A Contact: an edge of the workpiece is in contact with a vertex of the manipulator.
Type B Contact: a vertex of the workpiece is in contact with an edge of the manipulator.

Figure 1 shows a workpiece and a manipulator with three elemental contacts: edge 1 of the workpiece contacts vertex 1 of the first manipulator polygon (a type A contact), vertex 2 of the workpiece contacts edge 2 of the second manipulator polygon (a type B contact), and edge 3 of the workpiece contacts vertex 3 of the third manipulator polygon (a type A contact).

A set of elemental contacts constitutes a contact formation, CF, [3]. Based on the numbers of type A and B contacts, we classify CF's into various types. For example, the CF shown in Figure 1 is of
A CF of this type can be maintained *only* [illegible]
contacts slide as the manipulator polygons move. The same is true for every CF of type 2BA, 3A, or 3B; all of which involve three elemental contacts. We will be studying the type 3A in this paper. The other CF types of interest will be denoted by: A_RA, A_RB, B_RA, and B_RB. Every CF of each of these types has only two elemental contacts, one of which rolls as indicated by the subscript "R." Note that these CF types have, in addition to the constraints specifying the two elemental contacts, a third constraint specifying the location of the rolling contact point.

In what follows, we will assume that as many as three rigid manipulator polygons are in contact with a rigid workpiece polygon; that the positions and orientations of the manipulator polygons can be controlled directly and independently; and that the position and orientation of the workpiece is controlled purely as a byproduct of maintaining (if possible) the specified CF.

3 The 3A CF-Cell

Referring again to Figure 1, let x_l, y_l, and θ_l denote the position and orientation of a frame $\mathcal{P}_l$, attached to the l^{th} manipulator polygon, measured with respect to the world frame, $\mathcal{O}$. Similarly, let x, y, and θ, without any subscripts, denote the position and orientation of a workpiece-fixed frame $\mathcal{W}$ with respect to $\mathcal{O}$. Next define the workpiece configuration vector $\mathbf{q}$, the manipulator configuration vector $\mathbf{r}$, and the system configuration vector $\mathbf{p}$ as follows:

$$\mathbf{q} = [x, y, c, s] \quad (1)$$
$$\mathbf{r} = [x_1, x_2, x_3, y_1, y_2, y_3, \theta_1, \theta_2, \theta_3] \quad (2)$$
$$\mathbf{p} = [x, y, c, s, x_1, x_2, x_3, y_1, y_2, y_3, \theta_1, \theta_2, \theta_3] \quad (3)$$

where c and s "represent" $cos(\theta)$ and $sin(\theta)$, respectively. The variables c and s are to be thought of as independent variables and are used to represent the orientation of the workpiece in place of θ, so that the contact constraints may be written as algebraic, rather than trigonometric, equations. This, however, will require the introduction of an additional algebraic constraint, namely, $c^2 + s^2 - 1 = 0$.

In this spirit, we define the system's *modified C-space*, $\mathcal{Z}$, to be the set of all possible vectors $\mathbf{p}$, and denote the three contact constraints by $C_l(\mathbf{p}) = 0$ for $l = 1, 2, 3$. We then define the resulting CF-cell, $\mathcal{CF}$, as follows:

$$\mathcal{CF} = \{\mathbf{p} \in \mathcal{Z} \,|c^2 + s^2 - 1 = 0 \text{ and } C_l(\mathbf{p}) = 0 \quad (4)$$

Consider Figure 2. Let ϕ_l denote the angle between the outward normal to edge l and the positive x-axis of $\mathcal{W}$. The signed distance between vertex l and the line supporting edge l, the so called C-function [6], is given by:

$$C_l(\mathbf{p}) = \mathbf{v}_l \cdot (\mathbf{a}_l - \mathbf{b}_l) \quad (5)$$

be rewritten as:

$$D_l(\mathbf{r},c,s) + E_l(\mathbf{r},c,s)x + F_l(\mathbf{r},c,s)y = 0\,; \quad (8)$$
$$l = 1,2,3$$
$$c^2 + s^2 - 1 = 0 \quad (9)$$

The first two of the three equations represented by equation (9) can be solved for x and y yielding:

$$x = \frac{(F_1D_2 - F_2D_1)}{E_1F_2 - F_1E_2} \quad (10)$$

$$y = \frac{(E_2D_1 - E_1D_2)}{E_1F_2 - F_1E_2} \quad (11)$$

where the denominator $E_1F_2 - F_1E_2$ simplifies to $e_1f_2 - e_2f_1 = sin(\phi_2 - \phi_1)$. Note that if $sin(\phi_2 - \phi_1)$ is zero, a different pair of equations in (9) must be used.

Proposition 1: *For the 3A CF, elimination fails, i.e., the rank of* $\begin{bmatrix} e_1 & f_1 \\ e_2 & f_2 \\ e_3 & f_3 \end{bmatrix}$ *is less than two, if and only if the lines supporting the edges designated for contact are parallel (This includes the possibility that two or three of the lines are coincident.).*

Proof: The proof is given in [4]. .... □

When elimination fails, we will say that the particular manipulator configuration vector $\mathbf{r}$, is *nongeneric.* Brost identified this nongeneric situation with three contacts on parallel edges and noted that either there is zero or an infinite number of geometrically admissible workpiece configurations [1]. When the number is infinite, the workpiece can be translated along the contacted edges while maintaining the specified CF. Brost also stated that the situation with three contacts on parallel edges was the *only* situation in which the workpiece could attain the specified CF in an infinite number of configurations. However, we will show later that nongeneric situations also exist when the edges are not parallel. First, we need to make clear precisely what we mean by "nongeneric."

Our use of the term "nongeneric" is similar to Brost's [1]: the system of contact constraint equations does not satisfy "general position." Loosely speaking, a loss of general position implies that either a system of n equation in m unknowns ($m > n$) has more than the expected $m - n$ degrees of freedom or that solutions occur with multiplicities, or that the equations are inconsistent. In the context of this work, a system of three contact constraint equations for a fixed $\mathbf{r}$ usually constrains the workpiece to a finite number of configurations. However, if general position is lost, then the contact constraints could allow an infinite number of workpiece configurations.

By contrast, a *generic* situation is the typical situation in which each constraint equation reduces the

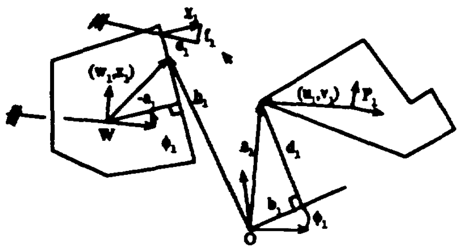

Figure 2: Definitions for A-type Contacts.

where the vectors $\mathbf{a}_l$, $\mathbf{b}_l$, and $\mathbf{v}_l$ are given in [4]. Here $\mathbf{a}_l$ is the position of vertex l, $\mathbf{b}_l$ is the position of the selected point on edge l, and $\mathbf{v}_l$ is the outward unit normal of edge l; all of these quantities are expressed with respect to the frame $\mathcal{O}$.

Notice that we have labeled the edges of the workpiece and the polygons so that contact l is between polygon l of the manipulator and the line supporting edge l of the workpiece. Note also that we handle the cases in which an edge of the workpiece is to be contacted by more than one vertex of the manipulator by labeling the edge more than once.

The C-function corresponding to the l_{th} type A contact can be written as a function of the configuration of the workpiece to yield the following system of contact constraint equations:

$$C_l(\mathbf{p}) = a_l + b_lc + d_ls - e_lxc + f_lxs - f_lyc - e_lys = 0 \quad (6)$$
$$c^2 + s^2 - 1 = 0 \quad (7)$$

where $l = 1,2,3$. The coefficients a_l, b_l, d_l, e_l, and f_l, illustrated in Figure 2, are functions of the geometry of the bodies in the system (including the workpiece) and the configuration of the manipulator and are given in [4].

Each C-function, $C_l(\mathbf{p})$, as well as the unit circle equation (equation (7)), defines a quadric hypersurface in C-space. The intersection of these four hyper surfaces defines the set of *geometrically admissible* 3A configurations of the workpiece as a function of the manipulator configuration vector, (*i.e.*, the intersection defines the CF-cell, $\mathcal{CF}$, in $\mathcal{Z}$). The reader should be cautioned that "geometrically admissible" means that contact is maintained along the line supporting the edge, not the actual physical edge of the workpiece.

To obtain an analytic solution which describes the 3A CF-cell, notice that each equation in (6) is linear in x and y so that the system of equations (6-7) can number of degrees of freedom of the system in question by one and no solution has multiplicity greater than one. A characteristic of genericity is that for every (suitably small) perturbation of the coefficients of the system of polynomials, the number of distinct solutions will not change.

In the 3A case, we will find that the generic situations are those in which there are two distinct real geometrically admissible workpiece configurations or two distinct complex solutions leading to no geometrically admissible workpiece configurations. The nongeneric situations are those for which there is an infinite number of solutions or just one solution (of multiplicity two), or inconsistent equations leading to no solutions. In these nongeneric cases there will be infinitesimal perturbations of $\mathbf{r}$ that change the number of solutions. The one exception to this definition occurs in the case when elimination fails and there are no solutions. Strictly speaking such a case is generic, *i.e.*, small perturbations of $\mathbf{r}$ will not yield solutions. However, the geometry itself is nongeneric and for that reason we do not consider this case as generic.

Apart from the nongeneric case with all contacts on parallel edges, the contacts can always be relabeled so that $sin(\phi_2 - \phi_1)$ is not zero and equations (10) and (11) are valid. Substituting x and y into the third equation of the system (9) gives a polynomial in c and s of the form:

$$G(\mathbf{r})c + H(\mathbf{r})s + I(\mathbf{r}) = 0 \quad (12)$$

$$s^2 + c^2 - 1 = 0 \quad (13)$$

where G, H, and I are given in [4].

To determine the geometrically admissible workpiece configurations $\mathbf{q}$, given a specific 3A CF and the manipulator configuration $\mathbf{r}$, we determine the intersection of the line (12) and the unit circle (13). When $G \neq 0$, we solve equation (12) for c and substitute into equation (13) to get a quadratic equation for s:

$$(H^2 + G^2)s^2 + 2HIs + (I^2 - G^2) = 0 \quad \text{when} \quad G \neq 0. \quad (14)$$

When $H \neq 0$, we solve for s in equation (12) to find:

$$(H^2 + G^2)c^2 + 2GIc + (I^2 - H^2) = 0 \quad \text{when} \quad H \neq 0. \quad (15)$$

The discriminants of these equations are $-4G^2(-G^2 - H^2 + I^2)$ when $G \neq 0$ and $-4H^2(-G^2 - H^2 + I^2)$ when $H \neq 0$. For geometrically admissible workpiece configurations to exist (*i.e.*, for the system of equations, (12) and (13) to have real solutions) when $G^2 + H^2 \neq 0$, the discriminant must be nonnegative. Thus we require that the following inequality be satisfied:

$$G^2 + H^2 \geq I^2. \quad (16)$$

When $G = 0$ and $H = 0$, we get real solutions only if $I = 0$. In that case, there will be an infinite number of solutions, because any pair (c, s) satisfying equation (13) also satifies equation (12) which is identically zero, and x and y can be found by back substitution into equations (10) and (11). When $G = 0$ and $H = 0$, but $I \neq 0$, the equation will be inconsistent and there will be no solutions. Note that even though inequality (16) is satisfied when elimination fails (because $G = H = I = 0$), it is *not* a necessary and sufficient condition for solutions to exist when elimination fails (see Proposition 2 below).

Proposition 2: *If elimination fails, then* $G = H = I = 0$ *and for a fixed* $\mathbf{r}$, *if there is any workpiece configuration which attains the 3A CF, then there will be an infinite number.*

Proof: This result was proved by Brost in [1]. ... □

The functions $G(\mathbf{r})$, $H(\mathbf{r})$, and $I(\mathbf{r})$ turn out to be closely related to the "wrench matrix" which arises in the analysis of multi-fingered grasps. This matrix is particularly useful in determining the stability and mobility of the grasped workpiece [5]. If the system under consideration is planar with n_c contacts, then the wrench matrix $\mathbf{W}$, can be partitioned into normal and tangential components $\mathbf{W}_n$ and $\mathbf{W}_t$, which have size $(3 \times n_c)$.

The wrench matrices have the following form:

$$\mathbf{W}_n = \begin{bmatrix} \hat{\mathbf{n}}_1 & \hat{\mathbf{n}}_2 & \hat{\mathbf{n}}_3 \\ \mathbf{p}_1 \wedge \hat{\mathbf{n}}_1 & \mathbf{p}_2 \wedge \hat{\mathbf{n}}_2 & \mathbf{p}_3 \wedge \hat{\mathbf{n}}_3 \end{bmatrix} \quad (17)$$

$$\mathbf{W}_t = \begin{bmatrix} \hat{\mathbf{t}}_1 & \hat{\mathbf{t}}_2 & \hat{\mathbf{t}}_3 \\ \mathbf{p}_1 \wedge \hat{\mathbf{t}}_1 & \mathbf{p}_2 \wedge \hat{\mathbf{t}}_2 & \mathbf{p}_3 \wedge \hat{\mathbf{t}}_3 \end{bmatrix} \quad (18)$$

where $\hat{\mathbf{n}}_l$ is the inward pointing unit normal vector to edge l, $\hat{\mathbf{t}}_l$ is the tangential unit vector defined so that $\hat{\mathbf{n}}_l \times \hat{\mathbf{t}}_l$ points out of the page, $\mathbf{p}_l$ is the position of the contact point, and $\mathbf{p}_l \wedge \hat{\mathbf{n}}_l$ is given by $p_{l_x} n_{l_y} - p_{l_y} n_{l_x}$. Here p_{l_x} and p_{l_y} are the components of $\mathbf{p}_l$. The components of $\hat{\mathbf{n}}_l$ are defined analogously.

Given the geometric definitions of the coefficients shown in Figure 2, the wrench matrices can be rewritten as explicit functions of the system configuration [4].

In [4] it is shown that G and H are related to the determinants of ${}^{\mathcal{O}}\mathbf{W}_n$ and ${}^{\mathcal{O}}\mathbf{W}_{t,verts}$ by the following simple formulas:

$$Det({}^{\mathcal{O}}\mathbf{W}_n) = cH - sG \qquad Det({}^{\mathcal{O}}\mathbf{W}_{t,verts}) = -(sH + cG). \quad (19)$$

where the superscript ${}^{\mathcal{O}}$ indicates that the matrices are expressed with respect to frame $\mathcal{O}$. Note that this definition of ${}^{\mathcal{O}}\mathbf{W}_{t,verts}$ uses for the $\mathbf{p}_i$, the positions of the vertices of the manipulator polygons designated to contact the workpiece even though contact may be impossible for the manipulator configuration under consideration.

Therefore, the important quantity, $G^2 + H^2$, that arose in the discriminant of equations (14) and (15), is equal to the sum of the squares of the determinants of the normal and tangential wrench matrices.

Similarly, $I = Det({}^{W}\mathbf{W}_{t,edges})$, where the subscript "*edges*" indicates that the contacts are assumed to be on the designated edges of the workpiece.

The values of G, H, and I give us information about the existence and the number of geometrically admissible workpiece configurations for a given 3A CF and manipulator configuration $\mathbf{r}$. Clearly, the existence of geometrically admissible workpiece configurations (and the number of configurations, if any exist) is independent of the specific choices of the coordinate frames. This motivates the following result.

Proposition 3: *The quantities I^2 and $G^2 + H^2$ and $Det(\mathbf{W}_n)$, $Det(\mathbf{W}_{t,edges})$, and $Det(\mathbf{W}_{t,verts})$ are independent of the choice of all coordinate frames.*

Proof: the proof is given in [4]. .... □

The tangential wrench matrices, $\mathbf{W}_{t,edges}$ and $\mathbf{W}_{t,verts}$, each represent three lines parallel to the edges of the workpiece specified by the 3A CF. The lines corresponding to $\mathbf{W}_{t,edges}$ are those supporting the actual edges of the workpiece, while the lines corresponding to $\mathbf{W}_{t,verts}$ contain the designated vertices, which do not necessarily lie on the edges specified by the 3A CF. Again the condition that $Det({}^{O}\mathbf{W}_{t,verts}) = 0$ or $Det({}^{O}\mathbf{W}_{t,edges}) = 0$ is that the three lines involved intersect at a point (possibly infinity).

Figure 3 shows a workpiece in several geometrically admissible configurations for a single configuration of the manipulator. Note that every possible orientation of the workpiece corresponds to a point on the unit circle, equation (12), so that for every orientation, there is a workpiece position, (x, y), which achieves the specified CF. This position can be found by back substituting into equations (10) and (11).

Generic situations are those for which $G^2 + H^2$ is positive and not equal to I^2. In these cases, regardless of the particular values of G, H, and I, there will always be two distinct solutions to the system of equations, (12) and (13). If the solutions are real, then there will be two geometrically admissible configurations of the workpiece; if they are complex there will be none.

Nongeneric situations are those which satisfy $G = 0$ and $H = 0$, or which satisfy inequality (16) by strict equality. Thus we can view the equation $G^2 + H^2 - I^2 = 0$ as a hypersurface in the manipulator configuration space where nongenericity occurs. Two instances of nongenericty should be noted. The first occurs when two manipulator vertices are to contact a common edge of the workpiece and those vertices coincide. In this situation, since the two edges corresponding to the coinciding vertices are identical, all three edges intersect the other edge at a point (possibly at infinity) regardless of the orientation of the third edge. Essentially, one of the coincident vertices may be discarded as redundant, and the 3A CF degenerates to a 2A CF. The second occurs when three manipulator vertices are to contact three mutually nonparallel edges which intersect at a point and the vertices coincide at that point of intersection. In this case, the workpiece may rotate freely about the intersection point while maintaining the 3A CF. However, as in the last case, one of the vertices may be discarded without changing the set of geometrically admissible workpiece configurations.

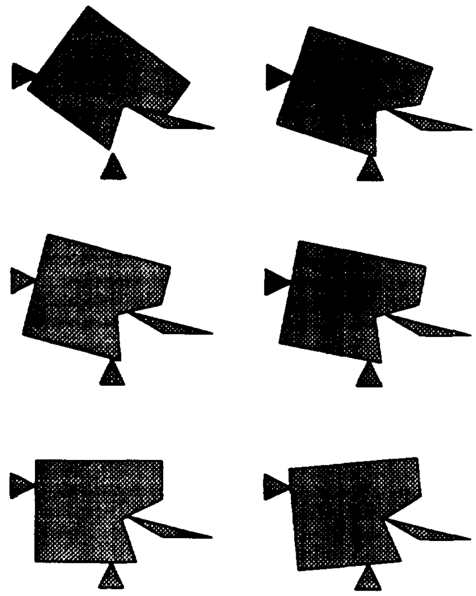

Figure 3: A Nongeneric Situation for which Elimination Succeeds

Proposition 4: *If elimination succeeds, then for a fixed $\mathbf{r}$, we will have an infinite number of solutions on the 3A CF-cell if and only if $Det(\mathbf{W}_n) = Det(\mathbf{W}_{t,verts}) = Det(\mathbf{W}_{t,edges}) = 0$, or equivalently, $G = H = I = 0$.*

Proof: the proof is given in [4]. .... □

Theorem 1: *For a generic positioning, $\mathbf{r}$, of the manipulator, we have either zero or two geometrically admissible workpiece configurations.*

We now turn to the CF-cell as a whole.

Theorem 2: *The 3A CF-cell is a nine-dimensional manifold.*

Proof: the proof is given in [4]. .... □

Theorem 3: *For the 3A CF, a configuration of the workpiece associated to a point* $\mathbf{r}$ *in manipulator configuration space, will be a branch point of multiplicity two if and only if* $G^2 + H^2 \neq 0$ *and* $\mathbf{W}_n$ *is singular. In that case the configuration will be the only geometrically admissible configuration. (This is equivalent to* $G^2 + H^2 \neq 0$ *and* $G^2 + H^2 - I^2 = 0$*.)*

Proof: the proof is given in [4]. .... □

Situations that satisfy Theorem 3 are those for which the contact normals intersect at a point, but the edges designated for contact do not.

Corollary 1 *The branch locus of a 3A CF-cell under projection to the space of manipulator configurations is precisely the locus where* $G^2 + H^2 \neq 0$ *and* $\mathbf{W}_n$ *is singular. By contrast, when* $\mathbf{W}_n$ *is nonsingular at* $\mathbf{p} \in \mathcal{CF}$*, we are forced to have* $G^2 + H^2 \neq 0$*, and the projection will be a local diffeomorphism.*

Proof: the proof is given in [4]. .... □

4 Conclusion

In summary, the 3A CF-cell, $\mathcal{CF}$, is a smooth "surface" sitting over the space of manipulator configurations. For most values of $\mathbf{r}$, there are zero or two points on the $\mathcal{CF}$-cell (These correspond to the geometrically admissible workpiece configurations.). However, there are a "few" manipulator configurations for which there is 1 or an infinite number of points. Nonetheless, the CF-cell is smooth everywhere. In the regions of manipulator configuration space where there is a finite number of workpiece configurations, the CF-cell can be viewed as two sheets which occasionally come together smoothly (without discontinuity between their normals).

Analogous results apply to the other 7 CF types.

5 Acknowledgements

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