On the Geometry of Contact Formation Cells for Systems of Polygons *

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Abstract

The efficient planning of contact tasks for intelligent robotic systems requires a thorough understanding of the kinematic constraints imposed on the system by the contacts. In this paper, we derive closed-form analytic solutions for the position and orientation of a passive polygon moving in sliding and rolling contact with two or three active polygons whose positions and orientations are independently controlled. This is accomplished by applying a simple elimination procedure to solve the appropriate system of contact constraint equations. The benefits of having analytic solutions are numerous. For example, they eliminate the need for iterative nonlinear equation solving algorithms to determine the position and orientation of the passive polygon given the positions and orientations of the active ones. Also, because they contain the configuration variables of the active polygons and the relevant geometric parameters, models of geometric and control uncertainty can be readily incorporated into the solutions. This will facilitate the analysis of the effects of these uncertainties on the kinematic constraints.

We also prove that the set of solutions to the contact constraint equations is a smooth submanifold of the system's configuration space and we study its projection

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onto the configuration space of the active polygons (i.e., the lower-dimensional configuration space of controllable parameters). By relating these results to the wrench matrices commonly used in grasp analysis, we discover a previously unknown and highly nonintuitive class of nongeneric contact situations. In these situations, for a specific fixed configuration of the active polygons, the passive polygon can maintain three contacts on three mutually nonparallel edges while retaining one degree of freedom of motion.

1 Introduction

Consider a planar system of rigid polygonal bodies in contact (see Figure 1) and assume that the positions and orientations of all but one of the polygons are actively controlled. The polygon that is not actively controlled, but rather is "grasped," is referred to as the workpiece or the passive polygon, and those that are actively controlled through joint actuation are collectively referred to as the manipulator or the active polygons. The workpiece and the manipulator taken together are referred to as the manipulation system, or just the system. The process of reorienting the workpiece by controlling the manipulator is known as dexterous manipulation. In contrast to the classical piano movers' problem [23] in which contact is avoided, every solution to a dexterous manipulation planning problem must include contact.

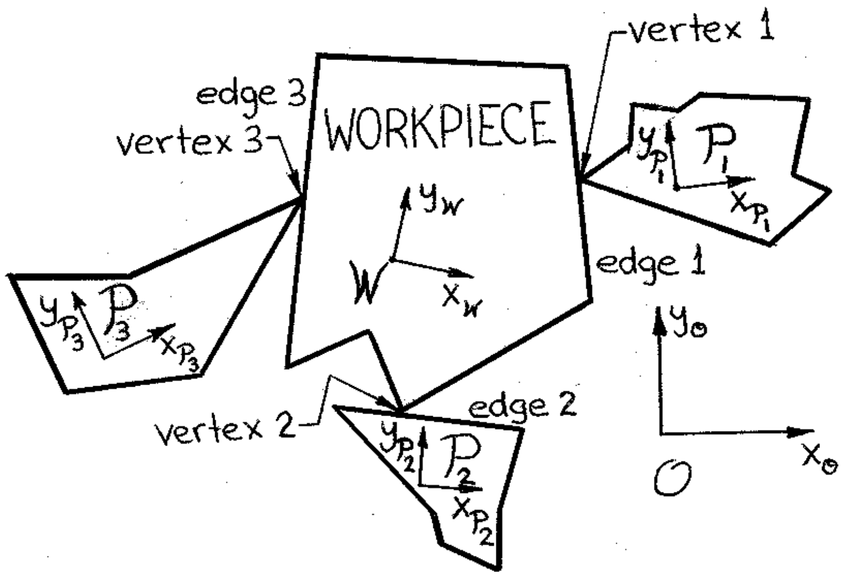

Figure 1: A workpiece and a three-polygon manipulator in contact.

When the level of uncertainty is a significant factor in the planning and execution of a dexterous manipulation task, then we refer to it as a fine motion or fine manipulation task. Automatic fine manipulation planning is one of the most important unsolved problems in the field of robotics. Tasks in this class include mechanical assembly/disassembly and grasping operations. The development of a practical, reliable planner for this class of tasks would facilitate the automation of large portions of various manufacturing and service industries and would expand our ability to work in Space and other hazardous environments.

The worst case running times of general motion planning algorithms increase exponentially with the dimension of the system's configuration space (C-space). Nonetheless, nonpathological planning problems have been solved in several minutes indicating that the exponential worst-case complexity should not deter the future development of planning algorithms [7]. Since fine manipulation requires contacts, algorithms for fine motion planning can be more efficient, because the kinematic constraint equation associated with each contact effectively reduces the dimension of C-space by one. However, fine manipulation planning algorithms can only fully benefit from this reduction if the relevant systems of kinematic constraints are thoroughly understood.

Besides the effective reduction in the dimension of C-space due to the contacts, there are two other good reasons to believe that practical fine manipulation planning algorithms can be developed: first, to find a plan, the entire C-space need not be decomposed (for example, see [27, 10, 7]); and second, cell decomposition techniques lend themselves to parallel and distributed computation.

In this paper, we present the geometric characterization and analytic representation of eight fundamental types of kinematic constraint "surfaces" in C-space (called "contact formation cells" [30] [29]). These constraint "surfaces" are the most important ones that arise during the planar manipulation of a passive polygonal workpiece by a manipulator composed of up to three active polygons whose positions and orientations are independently controlled. Using simple techniques from algebraic geometry, we derive analytic representations of these CF-cells which yield a number of benefits. For example, they provide us with an in-depth understanding of the geometry of the cells and their projections onto the space of controllable variables. Also, they obviate the need for an iterative solution procedure to determine the position and orientation of the workpiece given the configuration of manipulator. Finally, they reveal configurations (analogous to the singular configurations of typical serial-link manipulators) in which controlling the motion of the system requires more careful planning. Kinematic constraints among the polygons due to linkage connections are not considered (see "Future Work" below). If present, those constraints would further restrict the subset of C-space, potentially making planning even more efficient.

Each of the eight types of contact formation cells (CF-cells) that we will be studying, corresponds to a distinct combination of the contact constraint equations associated with sliding and/or rolling, type A and/or type B contacts [17] (see definitions below). Each sliding contact yields one constraint equation, while each rolling contact yields two. Under our assumption of pure position control of the manipulator, the number of contact constraints is generally three, because all degrees of freedom of the system except for the three associated with the workpiece are directly controlled. Contact combinations with four or more contact constraints could be maintained through compliant control, but those combinations are beyond the scope of this paper.

Even though we do not study three-dimensional systems, our analysis will be useful in planning the manipulation of solid objects with constant cross-sectional geometry, also it will be possible to apply our results to manipulating an extended solid which does not have constant cross-sectional geometry. For example, imagine a long slender object with a slightly varying polygonal cross-section and suppose that due to its length, one would prefer to pick it up at two locations distributed along its length (with two nominally identical dexterous manipulators). Next, suppose that a manipulation plan is generated for the nominal cross-section. Applying techniques from deformation theory and differential topology, it will be possible to identify an envelope of geometric uncertainty and control error within which the system must remain for successful task execution. If both dexterous manipulators execute the plan "simultaneously" and remain in the envelope, then the task will succeed.

1.1 Previous Work

Much previous research has been motivated by the desire to automatically generate robot programs that can reliably accomplish tasks specified at a high level. As a result many task-level robot programming languages have been developed, (*e.g.*, RAPT [20], LAMA [16], AUTOPASS [15], AML [25], and LM [13]). Our work is especially relevant to RAPT and a geometric constraint propagation system developed by Taylor (that influenced the development of AML) [24]. Both RAPT and Taylor's system dealt with geometric constraint relationships between the bodies in an assembly. Given the geometric models of the set of bodies and a set of spatial constraint relationships among them (*e.g.*, "Face A of Body 1 against Face B of Body 2," these systems were able to produce parameterized equations describing all the configurations of the bodies satisfying the constraint relationships. These relationships were simplified through the application of certain rules. For example, RAPT was able to solve systems of equations linear in a subset of the configuration variables to eliminate those variables.

There is, however, an important difference between Taylor's system and RAPT. RAPT focused on simplifying the system of geometric constraint *equations.* In Taylor's system, such simplification was just the first step. The goal was to produce systems of constraint *inequalities* representing geometric uncertainties, sensing errors, disturbance forces, and other variations [24]. This was accomplished by including error parameters in the constraint inequalities, expanding them, and retaining only the terms which were constant or linear in the error parameters. The result was a high-dimensional polytope in the space of error parameters which could be used to identify the most important uncertain parameters for the given task.

Because the "bags" of solution "tricks" used by RAPT and Taylor's system were not complete, they were not able to fully simplify all systems of equations. An alternative system was developed by Corner, Ambler, and Popplestone [4]. This system was able to recognize standard combinations of relationships among the bodies and apply the corresponding standard solutions. The result was a more efficient system. The closed-form solutions given in this paper for eight common feature relationships (which we call contact formations) represent new solution "tricks" that could be added to any symbolic constraint solving system to improve efficiency further. Note also, that because these solutions are closed-form, their incorporation into Taylor's system would not prevent the generation of the uncertainty polytope.

Another concern in fine manipulation planning is the smoothness of the C-surfaces corresponding to the contacts. In many problems, such as parts mating, a manipulation plan will begin with the part in freespace, move it along some path until a contact occurs, then move it along the corresponding C-surface until another contact occurs, and so on, until the part achieves a goal configuration. In order to plan paths of this sort, the C-surfaces and their intersections must be smooth. Otherwise the tangent space will be ill-defined making it difficult to employ any numerical technique which uses a move along a tangent as a first approximation to a move along the "surface." In her dissertation, Koutsou noted that C-surfaces corresponding to single type A, type B, and type C contacts are manifolds (and thus have a well-defined tangent space at each point), but that one cannot assume that the same is true for intersections of these manifolds [11]. In this paper, we show that the eight CF-cells studied are smooth manifolds. However, we note that the same cannot be said when one or more of the contacting surfaces are curved. We also note that RAPT, Koutsou's extension of RAPT, and Taylor's system dealt with spatial relationships and that our results are applicable only to planar motions, smoothness results for polyhedral bodies moving in three-space can be found in [14].

The results presented in this paper can also be viewed as an extension of Brost's work [2]. In his terminology, Brost developed techniques for computing and representing the configuration obstacle (C-obstacle) of two polygons in the plane. The C-obstacle is the three-dimensional subset of C-space (also three-dimensional) whose boundary is composed of curved surface patches, curved edges, and vertices corresponding to one, two, and three elemental contacts. [1] The C-obstacle is a complete description of all possible relative configurations of the two polygons. Points on the surface of the C-obstacle correspond to configurations in which the polygons are in contact (at one or more points). Integrating the equations of motion of the system on the surface allowed Brost to plan the controlled motion of one polygon, the "pusher," to achieve a desired final relative configuration of the two polygons.

The ultimate goal of our work (automatic fine manipulation planning) is the same as Brost's except for one crucial complicating factor – our "pusher" is to be a manipulator of any desired kinematic structure. Thus, in the context of Brost's work, we are interested in the extended problem in which the geometry of the "pusher" is a function of a number of control parameters, in particular, the positions and orientations of the polygons composing the manipulator (*i.e.*, the active polygons) in contact with the workpiece (*i.e.*, the passive or "pushed" polygon). In this paper, we limit our discussion to manipulator composed of three independent polygons, so the manipulator has nine control parameters.

[1] The term "elemental contact" [5] refers to either a type A or a type B contact.

Brost represented the patches and edges of the C-obstacle in parametric form and found the vertices by numerical solution. In the dexterous manipulation planning problem that we have been pursuing [27, 28, 30, 29], the vertices are extremely important, because during manipulation, the workpiece configuration commonly resides at a vertex or (when rolling is involved) at a fixed point on an edge of the deforming C-obstacle. Actual manipulation of the workpiece corresponds to the deliberate deformation of the C-obstacle (caused by varying the configuration of the manipulator) to cause desired motions of the vertex or edge point. Controlling two vertex or edge points of the C-obstacle so as to come together and then separate corresponds to a discrete change in the contact topology of the actual system. Such changes typically take place during dexterous manipulation to effect significant reorientations of the workpiece.

In this paper, we derive closed-form, analytic solutions for nine-dimensional "surfaces" of the C-space of the system. These submanifolds correspond to the vertices and the individual edge points of deformable C-obstacles in Brost's three-dimensional configuration space. Our solutions are written as functions of the nine configuration variables of the manipulator, thereby facilitating future studies of the possible motions of the workpiece over finite, as opposed to infinitesimal, manipulation trajectories. We also show that the set of all possible workpiece configurations, the CF-cell, forms a submanifold of the system's C-space (the space of all possible configurations of the manipulator and the workpiece) and that the CF-cell forms a generically finite branched covering[2] (with up to four sheets) of the nine-dimensional C-space of the manipulator. By relating the results obtained through our algebro-geometric analysis to the existing research results on grasp analysis, we also discover a new nonintuitive, nongeneric contact situation that otherwise might not have been revealed.

1.2 Layout of Paper

This paper is organized as follows. In Section 2, we define the class of systems that will be considered and state our assumptions. In Section 3, we derive the sets of contact constraints for the four relevant combinations of three sliding contacts and give a noniterative solution procedure for obtaining the workpiece's possible configurations given the configuration of the manipulator. This is done in detail mainly for the case of three type A contacts (in Section 3.1), because the other three combinations of contact constraints can be analyzed in the same way. We also derive expressions for the relevant wrench matrices (commonly used in the analysis of grasps) as functions of the manipulator configuration and relate them to the multiplicity of the solution obtained. In Section 4, we discuss the four combinations of

[2]A branched covering of a space can be thought of as several sheets "above" the space. For example, the graph of the curve $y^2 = x$ in the plane forms a finite branched cover of the x-axis (under orthogonal projection onto the x-axis), because every point on the x-axis is "covered" by two, one, or zero points on the curve $y^2 = x$. Locally the sheets are diffeomorphic to the x-axis except at the branch point $(0, 0)$ where the sheets come together smoothly, but with infinite slope. Note that in higher dimensions, removing a branch point does not necessarily disconnect the sheets (*e.g.*, $z \to z^2$ mapping the complex plane to itself, which branches as $z = 0$.).

contact constraints relevant to situations with one rolling and one sliding contact and show that they are special cases of the four contact combinations with three sliding contacts. Finally, in Section 5, we conclude and recommend avenues for further research.

2 System Model and Problem Statement

In studies of polygonal mobile robots operating in a plane among polygonal obstacles, two types of elemental contacts have been defined: type A, which is a contact between an edge of the robot and a vertex of an obstacle; and type B, which is a contact between a vertex of the robot and an edge of an obstacle [12]. In the work presented here, we view the workpiece as a mobile robot and the manipulator as a deformable obstacle. Thus we are led to define type A and B elemental contacts as follows:
Type A Contact: an edge of the workpiece is in contact with a vertex of the manipulator.
Type B Contact: a vertex of the workpiece is in contact with an edge of the manipulator.

Figure 1 shows a workpiece and a three-polygon manipulator with three elemental contacts: edge 1 of the workpiece contacts vertex 1 of the first manipulator polygon (a type A contact), vertex 2 of the workpiece contacts edge 2 of the second manipulator polygon (a type B contact), and edge 3 of the workpiece contacts vertex 3 of the third manipulator polygon (a type A contact).

A set of elemental contacts constitutes a contact formation, CF [5]. Based on the numbers of type A and B contacts, we classify CF's into various types. For example, the CF shown in Figure 1 is of type 2AB. When the configurations of the manipulator polygons are independently controlled, a CF of this type can be maintained *only if* all three contacts slide as the manipulator moves. The same is true for every CF of type 2BA, 3A, or 3B; all of which involve three elemental contacts. We will be studying all four of these types in this paper. The other CF types of interest will be denoted by: A_RA, A_RB, B_RA, and B_RB. Every CF of each of these types has only two elemental contacts, one of which rolls as indicated by the subscript "R." Note that these CF types have, in addition to the constraints specifying the two elemental contacts, a third constraint specifying the location of the rolling contact point.

In what follows, we will assume that as many as three rigid manipulator polygons are in contact with a rigid workpiece polygon; that the positions and orientations of the manipulator polygons can be controlled directly and independently; and that the position and orientation of the workpiece is controlled purely as a byproduct of maintaining (if possible) the specified CF. For each of the eight types of CF's identified above, we will, among other things, answer the following questions:

1. Does the CF-cell (defined below) form a submanifold of the system's C-space (the configuration space of the manipulator and workpiece taken together)?

2. For a given configuration of the manipulator, how many configurations of the workpiece achieve the specified CF and what are the conditions under which there is exactly one such configuration, or perhaps an infinite number of such configurations?

3. How does the CF-cell project to the C-space of the manipulator? In particular, how does the CF-cell branch over the C-space of the manipulator and when does it fail to be a finite covering? These are the places where controlling the manipulator requires special care.

3 CF-Cells with Three Contacts

In this section, we begin our study of CF's with three elemental contacts. Such CF's fall into four basic types: 3A, 3B, 2AB, and 2BA. It turns out that the 3A and 3B cases and the 2AB and 2BA cases are dual in a sense that will be clarified later. For that reason, we will give complete derivations only for the 3A and 2AB cases and simply state our results for the other two.

3.1 3A CF-Cells

Referring again to Figure 1, let x_l, y_l, and θ_l denote the position and orientation of a frame $\mathcal{P}_l$, attached to the l^{th} manipulator polygon, measured with respect to the world frame, $\mathcal{O}$. Similarly, let x, y, and θ, without any subscripts, denote the position and orientation of a workpiece-fixed frame $\mathcal{W}$ with respect to $\mathcal{O}$. Next define the workpiece configuration vector $\mathbf{q}$, the manipulator configuration vector $\mathbf{r}$, and the system configuration vector $\mathbf{p}$ as follows:

$$\mathbf{q} = [x, y, c, s] \quad (1)$$

$$\mathbf{r} = [x_1, x_2, x_3, y_1, y_2, y_3, \theta_1, \theta_2, \theta_3] \quad (2)$$

$$\mathbf{p} = [x, y, c, s, x_1, x_2, x_3, y_1, y_2, y_3, \theta_1, \theta_2, \theta_3] \quad (3)$$

where c and s "represent" $cos(\theta)$ and $sin(\theta)$, respectively. The variables c and s are to be thought of as independent variables and are used to represent the orientation of the workpiece in place of θ, so that the contact constraints may be written as algebraic, rather than trigonometric equations. This, however, will require the introduction of an additional algebraic constraint, namely, $c^2 + s^2 - 1 = 0$.

In this spirit, we define the system's *modified C-space*, $\mathcal{Z}$, to be the set of all possible vectors $\mathbf{p}$, and denote the three contact constraints by $C_l(\mathbf{p}) = 0$ for $l = 1, 2, 3$. We then define the resulting CF-cell, $\mathcal{CF}$, as follows:

$$\mathcal{CF} = \{\mathbf{p} \in \mathcal{Z} \mid c^2 + s^2 - 1 = 0 \text{ and } C_l(\mathbf{p}) = 0 \text{ for every } l = 1, 2, 3\}. \quad (4)$$

$\mathcal{Z}$ is topologically the product space $R^{10} \times T^3$, where R^{10} is ten-dimensional Euclidean space and T^3 is the three-torus (the product of three circles parametrized by the variables $\theta_1, \theta_2, \theta_3$).

Note that $\mathcal{Z}$ is a manifold [12]. The usual C-space $\mathcal{X}$ for our four-polygon system would be the submanifold of $\mathcal{Z}$ cut out by the equation $c^2 + s^2 - 1 = 0$. It would be 12-dimensional, while $\mathcal{Z}$ is 13-dimensional.

Consider Figure 2. Let the pair (u_l, v_l) denote the position of vertex l, on polygon l of

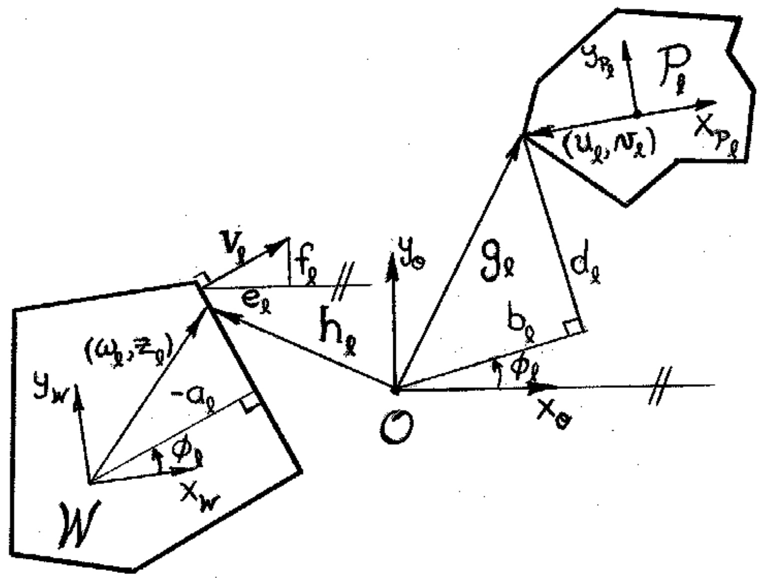

Figure 2: Illustration of parameters relevant to type A contacts.

the manipulator, which is intended to contact edge l of the workpiece. Here u_l and v_l are measured with respect to the frame $\mathcal{P}_l$. Let the pair (w_l, z_l) denote the coordinates of an arbitrarily selected point on edge l of the workpiece measured with respect to $\mathcal{W}$. Also let ϕ_l denote the angle between the outward normal to edge l and the positive x-axis of $\mathcal{W}$. The signed distance between vertex l and the line supporting edge l, the so called C-function [12], is given by:

$$C_l(\mathbf{p}) = \mathbf{v}_l \cdot (\mathbf{g}_l - \mathbf{h}_l) \tag{5}$$

where the vectors $\mathbf{g}_l$, $\mathbf{h}_l$, and $\mathbf{v}_l$ are defined as follows:

$$\mathbf{g}_l = (x_l + cos(\theta_l)u_l - sin(\theta_l)v_l \ , \ y_l + sin(\theta_l)u_l + cos(\theta_l)v_l) \tag{6}$$

$$\mathbf{h}_l = (x + cos(\theta)w_l - sin(\theta)z_l \ , \ y + sin(\theta)w_l + cos(\theta)z_l) \tag{7}$$

$$\mathbf{v}_l = (cos(\theta + \phi_l) \ , \ sin(\theta + \phi_l)). \tag{8}$$

Here $\mathbf{g}_l$ is the position of vertex l, $\mathbf{h}_l$ is the position of the selected point on edge l, and $\mathbf{v}_l$ is the outward unit normal of edge l; all of these quantities are expressed with respect to the frame $\mathcal{O}$.

Notice that we have labeled the edges of the workpiece and the polygons so that contact l is between polygon l of the manipulator and the line supporting edge l of the workpiece.

Note also that we handle cases in which an edge of the workpiece is to be contacted by more than one vertex of the manipulator by labeling that edge more than once.

The C-function corresponding to each type A contact can be written as a function of the configuration of the workpiece to yield the following system of contact constraint equations:

$$C_l(\mathbf{p}) = a_l + b_l c + d_l s - e_l x c + f_l x s - f_l y c - e_l y s \;=\; 0\;;\quad l = 1,2,3 \tag{9}$$

$$c^2 + s^2 - 1 \;=\; 0 \tag{10}$$

where the coefficients a_l, b_l, d_l, e_l, and f_l, illustrated in Figure 2, are functions of the geometry of the bodies in the system (including the workpiece) and the configuration of the manipulator:

$$a_l = -cos(\phi_l)w_l - sin(\phi_l)z_l \tag{11}$$

$$b_l = cos(\phi_l - \theta_l)u_l + sin(\phi_l - \theta_l)v_l + sin(\phi_l)y_l + cos(\phi_l)x_l \tag{12}$$

$$d_l = -sin(\phi_l - \theta_l)u_l + cos(\phi_l - \theta_l)v_l + cos(\phi_l)y_l - sin(\phi_l)x_l \tag{13}$$

$$e_l = cos(\phi_l) \tag{14}$$

$$f_l = sin(\phi_l). \tag{15}$$

Note that the derivation of the equations in (9) uses equation (10) to eliminate terms involving c^2 and s^2 in $C_l(\mathbf{p})$.

Each C-function, $C_l(\mathbf{p})$, as well as the unit circle equation (equation (10)), defines a quadric hypersurface in C-space. The intersection of these four hypersurfaces defines the set of *geometrically admissible* 3A configurations of the workpiece as a function of the manipulator configuration vector, (*i.e.*, the intersection defines the CF-cell, $\mathcal{CF}$, in $\mathcal{Z}$). The reader should be cautioned that "geometrically admissible" means that contact is maintained along the line supporting the edge, not necessarily the actual physical edge of the workpiece. In addition, we allow the bodies to overlap. Extra constraints to prevent overlap would take the form of inequalities that would restrict us to a portion of our CF-cell.

To obtain an analytic solution which describes the 3A CF-cell, notice that each equation in (9) is linear in x and y so that the system of equations (9-10) can be rewritten as:

$$D_l(\mathbf{r}, c, s) + E_l(\mathbf{r}, c, s)x + F_l(\mathbf{r}, c, s)y \;=\; 0\;;\quad l = 1,2,3 \tag{16}$$

$$c^2 + s^2 - 1 \;=\; 0 \tag{17}$$

where

$$D_l = a_l + b_l c + d_l s \qquad E_l = -e_l c + f_l s \qquad F_l = -f_l c - e_l s. \tag{18}$$

The first two of the three equations in (16) can be solved for x and y yielding:

$$x = \frac{(F_1 D_2 - F_2 D_1)}{E_1 F_2 - F_1 E_2} \qquad y = \frac{(E_2 D_1 - E_1 D_2)}{E_1 F_2 - F_1 E_2} \tag{19}$$

where the denominator $E_1F_2 - F_1E_2$ simplifies to $e_1 f_2 - e_2 f_1 = sin(\phi_2 - \phi_1)$. Note that if $sin(\phi_2 - \phi_1)$ is zero, a different pair of equations in (16) must be used. Thus solving for x

and y will fail if and only if the rank of $\begin{bmatrix} E_1 & F_1 \\ E_2 & F_2 \\ E_3 & F_3 \end{bmatrix} = \begin{bmatrix} e_1 & f_1 \\ e_2 & f_2 \\ e_3 & f_3 \end{bmatrix} \begin{bmatrix} -cos(\theta) & -sin(\theta) \\ sin(\theta) & -cos(\theta) \end{bmatrix}$ is less than two. In this event, we shall say that *elimination fails.*

Proposition 1: *For a 3A CF, elimination fails, i.e., the rank of* $\begin{bmatrix} e_1 & f_1 \\ e_2 & f_2 \\ e_3 & f_3 \end{bmatrix}$ *is less than two, if and only if the lines supporting the edges designated for contact are parallel. (This includes the possibility that two or three of the lines are coincident.)*

Proof: If the three contacted edges are parallel, then the angle between any two normals to the contacted edges is 0 or π radians. Consequently, $sin(\phi_l - \phi_m) = 0$ for any choice of l and m in the set $\{1, 2, 3\}$. This in turn implies that the determinant of every (2×2) minor of $\begin{bmatrix} e_1 & f_1 \\ e_2 & f_2 \\ e_3 & f_3 \end{bmatrix}$ is zero and that the rank of this matrix is less than two.

Conversely, the rank of $\begin{bmatrix} e_1 & f_1 \\ e_2 & f_2 \\ e_3 & f_3 \end{bmatrix}$ can only be less than 2 if the determinants of all (2×2) minors are zero. Thus elimination fails only if $sin(\phi_2 - \phi_1) = 0$, $sin(\phi_3 - \phi_2) = 0$, and $sin(\phi_1 - \phi_3) = 0$, that is, only if the three contacted edges are parallel. .... □

Apart from the case when elimination fails, the contacts can always be relabeled so that $sin(\phi_2 - \phi_1)$ is not zero and equations (19) are valid. Substituting x and y into the third equation of the system (16) gives a polynomial in c and s of the form:

$$G(\mathbf{r})c + H(\mathbf{r})s + I(\mathbf{r}) = 0 \quad (20)$$

$$c^2 + s^2 - 1 = 0 \quad (21)$$

where G, H, and I are given by:

$$G = b_1 e_2 f_3 - b_1 e_3 f_2 - b_2 e_1 f_3 + b_2 e_3 f_1 + b_3 e_1 f_2 - b_3 e_2 f_1 \quad (22)$$

$$H = d_1 e_2 f_3 - d_1 e_3 f_2 - d_2 e_1 f_3 + d_2 e_3 f_1 + d_3 e_1 f_2 - d_3 e_2 f_1 \quad (23)$$

$$I = a_1 e_2 f_3 - a_1 e_3 f_2 - a_2 e_1 f_3 + a_2 e_3 f_1 + a_3 e_1 f_2 - a_3 e_2 f_1. \quad (24)$$

It is important to note that equation (20) can be rewritten as follows:

$$Det \begin{bmatrix} D_1 & E_1 & F_1 \\ D_2 & E_2 & F_2 \\ D_3 & E_3 & F_3 \end{bmatrix} = 0, \quad (25)$$

which is a necessary and sufficient condition that the three equations (16), when homogenized (by multiplying each of the coefficients, $D_l(\mathbf{r}, c, s)$, by a homogenizing variable z) have a nontrivial solution. The additional condition, $rank \begin{bmatrix} E_1 & F_1 \\ E_2 & F_2 \\ E_3 & F_3 \end{bmatrix} = 2$, guarantees that the homogenizing variable z will not be zero. Linearly scaling x, y, and z to make z equal to one gives an x and y satisfying the original system of equations (16).

Expanding the determinant in equation (25) and recalling that $E_iF_j - E_jF_i = e_if_j - e_jf_i$, we arrive at alternate forms for G, H, and I that are useful in the proofs which will follow:

$$G = Det \begin{bmatrix} b_1 & e_1 & f_1 \\ b_2 & e_2 & f_2 \\ b_3 & e_3 & f_3 \end{bmatrix} = Det \begin{bmatrix} b_1 & cos(\phi_1) & sin(\phi_1) \\ b_2 & cos(\phi_2) & sin(\phi_2) \\ b_3 & cos(\phi_3) & sin(\phi_3) \end{bmatrix} \quad (26)$$

$$H = Det \begin{bmatrix} d_1 & e_1 & f_1 \\ d_2 & e_2 & f_2 \\ d_3 & e_3 & f_3 \end{bmatrix} = Det \begin{bmatrix} d_1 & cos(\phi_1) & sin(\phi_1) \\ d_2 & cos(\phi_2) & sin(\phi_2) \\ d_3 & cos(\phi_3) & sin(\phi_3) \end{bmatrix} \quad (27)$$

$$I = Det \begin{bmatrix} a_1 & e_1 & f_1 \\ a_2 & e_2 & f_2 \\ a_3 & e_3 & f_3 \end{bmatrix} = Det \begin{bmatrix} a_1 & cos(\phi_1) & sin(\phi_1) \\ a_2 & cos(\phi_2) & sin(\phi_2) \\ a_3 & cos(\phi_3) & sin(\phi_3) \end{bmatrix}. \quad (28)$$

To determine the geometrically admissible workpiece configurations, $\mathbf{q}$, given a specific 3A CF and the manipulator configuration, $\mathbf{r}$, we must determine the intersection of the line (20) and the unit circle (21). When $G \neq 0$, we solve equation (20) for c and substitute into equation (21) to get a quadratic equation for s:

$$(H^2 + G^2)s^2 + 2HIs + (I^2 - G^2) = 0 \quad \text{when} \quad G \neq 0. \quad (29)$$

When $H \neq 0$, we solve for s in equation (20) to find:

$$(H^2 + G^2)c^2 + 2GIc + (I^2 - H^2) = 0 \quad \text{when} \quad H \neq 0. \quad (30)$$

The discriminants of these equations are $-4G^2(-G^2 - H^2 + I^2)$ when $G \neq 0$ and $-4H^2(-G^2 - H^2 + I^2)$ when $H \neq 0$. For geometrically admissible workpiece configurations to exist (*i.e.*, for the system of equations, (20) and (21) to have real, as opposed to complex, solutions) when $G^2 + H^2 \neq 0$, the discriminant must be nonnegative. Thus we require that the following inequality be satisfied:

$$G^2 + H^2 \geq I^2. \quad (31)$$

When $G = 0$ and $H = 0$, we get real solutions only if $I = 0$. In that case, there will be an infinite number of solutions, because any pair (c, s) satisfying equation (21) also satisfies equation (20) which is identically zero, and x and y can be found by back substitution into equations (19). When $G = 0$ and $H = 0$, but $I \neq 0$, equation (20) will be inconsistent and there will be no solutions. Note that even though inequality (31) is satisfied when elimination fails (because $G = H = I = 0$), it is *not* a necessary and sufficient condition for solutions to exist in that case (see Proposition 2 below).

Proposition 2: *If elimination fails, then $G = H = I = 0$ and for a fixed* $\mathbf{r}$*, if there is any workpiece configuration which attains the specified 3A CF, then there will be an infinite number.*

Proof: This proposition refers to the case of three parallel edges and its truth is obvious from the discussion above. .... □

When elimination fails, we will say that the particular manipulator configuration vector $\mathbf{r}$, is *nongeneric*. Brost identified this nongeneric situation and noted that either there are no geometrically admissible workpiece configurations or an infinite number [2]. In the latter case, the workpiece can be translated along the contacted edges while maintaining the specified CF.

By relating the results of the above analysis (especially G, H, and I) to the wrench matrices used in grasp analysis, we have found an overlooked nongeneric contact situation with three contacts on nonparallel edges for which there is an infinite number of admissible workpiece configurations. This new contact situation is discussed below.

Our use of the term "nongeneric" is similar to Brost's [2]: the system of contact constraint equations does not satisfy "general position." Loosely speaking, a loss of general position implies that either a system of n equations in m unknowns ($m > n$) has more than the expected $m - n$ degrees of freedom, or that solutions occur with multiplicities, or that the equations are inconsistent. In the context of this work, a system of three contact constraint equations in general position (*i.e.*, the generic case) for a fixed $\mathbf{r}$ constrains the workpiece to a finite number of configurations.

In practice, nongeneric situations would appear to be rare, because it is impossible to manufacture the perfect arrangement of geometric features required (*e.g.*, a perfectly straight edge, or three perfectly parallel and straight edges) or to precisely control the position of the manipulator. Mathematically, this fact is related to standard results in algebraic geometry which say that every nongeneric system of polynomial equations can be made generic by a suitably small perturbation of the coefficients [3]. However, from a computational and modeling standpoint, nongeneric situations are common, and if they are not understood completely, they will compromise the robustness of any dexterous manipulation planning algorithm based on geometric models. This is especially true in situations where the initial and goal configurations happen to be separated by the locus of nongeneric configurations, which can form co-dimension 1 "barriers" in our CF-cells. One encounters an analogous situation when the planned trajectory of an industrial robot passes through a singular configuration in its workspace.

In the 3A case, we will find that the generic situations are those in which there are two distinct real solutions to the system of contact constraints (9) and (10) or two distinct complex solutions, implying that there are two or zero real geometrically admissible workpiece configurations, respectively. The nongeneric situations are those for which there is an infinite number of solutions or just one solution (of multiplicity two), or inconsistent equations leading to no solutions. The one exception to our definition of genericity (over the eight CF-cells studied) occurs in the 3A case when elimination fails and there are no solutions. Strictly speaking, such a case is generic, *i.e.*, small perturbations of $\mathbf{r}$ will not produce solutions. However, the geometry itself is nongeneric because the three edges are parallel, and for that reason we do not consider this case as generic. (In future work when uncertainty is involved, we will be varying the geometry and the case of three parallel edges will certainly be nongeneric in that context.)

In contrast to the nongeneric situations, generic situations are those for which $G^2 + H^2$ is positive and not equal to I^2. In these cases, regardless of the particular values of G, H, and I, there will always be two distinct solutions to the system of equations, (20) and (21). If the solutions are real, then there will be two geometrically admissible configurations of the workpiece; if they are complex there will be none. Moreover, small changes in $\mathbf{r}$ will not change the number of solutions.

Theorem 1: *For a generic positioning,* $\mathbf{r}$, *of the manipulator, we will have in the 3A case, either zero or two geometrically admissible workpiece configurations.*

3.2 The Wrench Matrices

The functions $G(\mathbf{r})$, $H(\mathbf{r})$, and $I(\mathbf{r})$ turn out to be closely related to the "wrench matrix" which arises in the analysis of multi-fingered grasps. This matrix is particularly useful in determining the stability and mobility of the grasped workpiece [9]. If the system under consideration is planar with n_c contacts, then the wrench matrix $\mathbf{W}$, can be partitioned into normal and tangential components $\mathbf{W}_n$ and $\mathbf{W}_t$, which have size $(3 \times n_c)$. The partitions $\mathbf{W}_n$ and $\mathbf{W}_t$ appear in the kinematic relationships constraining the normal and tangential components of the relative velocities of the contact points and in the summations of the normal and tangential components of the contact forces [28].

The wrench matrices have the following form:

$$\mathbf{W}_n = \begin{bmatrix} \hat{\mathbf{n}}_1 & \hat{\mathbf{n}}_2 & \hat{\mathbf{n}}_3 \\ \mathbf{p}_1 \wedge \hat{\mathbf{n}}_1 & \mathbf{p}_2 \wedge \hat{\mathbf{n}}_2 & \mathbf{p}_3 \wedge \hat{\mathbf{n}}_3 \end{bmatrix} \qquad \mathbf{W}_t = \begin{bmatrix} \hat{\mathbf{t}}_1 & \hat{\mathbf{t}}_2 & \hat{\mathbf{t}}_3 \\ \mathbf{p}_1 \wedge \hat{\mathbf{t}}_1 & \mathbf{p}_2 \wedge \hat{\mathbf{t}}_2 & \mathbf{p}_3 \wedge \hat{\mathbf{t}}_3 \end{bmatrix} \tag{32}$$

where $\hat{\mathbf{n}}_l$ is the inward pointing unit normal vector to edge l, $\hat{\mathbf{t}}_l$ is the tangential unit vector defined so that $\hat{\mathbf{n}}_l \times \hat{\mathbf{t}}_l$ points out of the page, $\mathbf{p}_l$ is the position of the l^{th} contact point, and $\mathbf{p}_l \wedge \hat{\mathbf{n}}_l$ is given by $p_{l_x} n_{l_y} - p_{l_y} n_{l_x}$. Here p_{l_x} and p_{l_y} are the components of $\mathbf{p}_l$ and n_{l_x} and n_{l_y} are the components of $\hat{\mathbf{n}}_l$ relative to some specified frame (usually $\mathcal{O}$).

Given the geometric definitions of the coefficients shown in Figure 2, the wrench matrices can be rewritten as explicit functions of the system configuration as follows:

$$ {}^{\mathcal{O}}\mathbf{W}_n = \begin{bmatrix} -cos(\theta+\phi_1) & -cos(\theta+\phi_2) & -cos(\theta+\phi_3) \\ -sin(\theta+\phi_1) & -sin(\theta+\phi_2) & -sin(\theta+\phi_3) \\ d_1c - b_1s & d_2c - b_2s & d_3c - b_3s \end{bmatrix} \tag{33} $$

$$ {}^{\mathcal{O}}\mathbf{W}_{t,verts} = \begin{bmatrix} sin(\theta+\phi_1) & sin(\theta+\phi_2) & sin(\theta+\phi_3) \\ -cos(\theta+\phi_1) & -cos(\theta+\phi_2) & -cos(\theta+\phi_3) \\ -(b_1c + d_1s) & -(b_2c + d_2s) & -(b_3c + d_3s) \end{bmatrix} \tag{34} $$

where the superscript ${}^{\mathcal{O}}$ indicates that the matrices are expressed with respect to the frame $\mathcal{O}$. Note that in this definition of ${}^{\mathcal{O}}\mathbf{W}_{t,verts}$, $\mathbf{p}_i$ is the position of the vertex l of the manipulator polygon designated to contact the workpiece *even though achieving all three contacts may be impossible* for the manipulator configuration under consideration.

Using standard trigonometric identities, one can rewrite the wrench matrices as follows:

$$ {}^{\mathcal{O}}\mathbf{W}_n = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -cos(\phi_1) & -cos(\phi_2) & -cos(\phi_3) \\ -sin(\phi_1) & -sin(\phi_2) & -sin(\phi_3) \\ d_1c - b_1s & d_2c - b_2s & d_3c - b_3s \end{bmatrix} \tag{35} $$

$$ {}^{\mathcal{O}}\mathbf{W}_{t,verts} = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} sin(\phi_1) & sin(\phi_2) & sin(\phi_3) \\ -cos(\phi_1) & -cos(\phi_2) & -cos(\phi_3) \\ -(b_1c + d_1s) & -(b_2c + d_2s) & -(b_3c + d_3s) \end{bmatrix}. \tag{36} $$

Finally, comparing these expressions with equations (26) and (27), we find that G and H are related to the determinants of ${}^{\mathcal{O}}\mathbf{W}_n$ and ${}^{\mathcal{O}}\mathbf{W}_{t,verts}$ by the following simple formulas:

$$ Det({}^{\mathcal{O}}\mathbf{W}_n) = cH - sG \qquad Det({}^{\mathcal{O}}\mathbf{W}_{t,verts}) = -(sH + cG). \tag{37} $$

Therefore, the important quantity, $G^2 + H^2$, that arose in the discriminant of equations (29) and (30), is equal to the sum of the squares of the determinants of the normal and tangential wrench matrices:

$$ G^2 + H^2 = Det^2({}^{\mathcal{O}}\mathbf{W}_n) + Det^2({}^{\mathcal{O}}\mathbf{W}_{t,verts}). \tag{38} $$

Similarly, one can deduce that:

$$ {}^{\mathcal{W}}\mathbf{W}_{t,edges} = \begin{bmatrix} sin(\phi_1) & sin(\phi_2) & sin(\phi_3) \\ -cos(\phi_1) & -cos(\phi_2) & -cos(\phi_3) \\ a_1 & a_2 & a_3 \end{bmatrix} \tag{39} $$

where the subscript "*edges*" indicates that the contacts are assumed to be on the designated edges of the workpiece. (The matrix does not depend on the specific edge point chosen.) Comparing equations (28) and (39), we see that:

$$ I = Det({}^{\mathcal{W}}\mathbf{W}_{t,edges}). \tag{40} $$

The values of G, H, and I give us information about the existence and the number of geometrically admissible workpiece configurations for a given 3A CF and manipulator configuration $\mathbf{r}$. Clearly, the existence of geometrically admissible workpiece configurations (and the number of configurations, if any exist) is independent of the specific choices of coordinate frames. This motivates the following result.

Proposition 3: *The quantities I^2 and G^2+H^2, and the quantities $Det({}^{\mathcal{W}}\mathbf{W}_{t,edges})$, $Det({}^{\mathcal{O}}\mathbf{W}_n)$, and $Det({}^{\mathcal{O}}\mathbf{W}_{t,verts})$ are independent of the choice of all coordinate frames.*

Proof: None of the vectors, $\hat{\mathbf{n}}_l$, $\hat{\mathbf{t}}_l$, and $\mathbf{p}_l$, appearing in the wrench matrix definitions given in equation (32) depend on the choice of frames $\mathcal{P}_l$. Also, it is known that a wrench matrix, expressed in an arbitrary frame $\mathcal{A}$, can be expressed in an arbitrary frame $\mathcal{B}$ by premultiplying it by a force transformation matrix ${}^{\mathcal{B}}_{\mathcal{A}}\mathbf{T}_f$ [26]:

$$ {}^{\mathcal{B}}\mathbf{W} = {}^{\mathcal{B}}_{\mathcal{A}}\mathbf{T}_f \, {}^{\mathcal{A}}\mathbf{W} \tag{41} $$

where in the planar case, ${}^{\mathcal{B}}_{\mathcal{A}}\mathbf{T}_f$ is $\begin{bmatrix} cos(\psi) & sin(\psi) & 0 \\ -sin(\psi) & cos(\psi) & 0 \\ \Delta y & -\Delta x & 1 \end{bmatrix}$. Here ψ is the angle of rotation of frame $\mathcal{B}$ with respect to frame $\mathcal{A}$ and Δx and Δy are the components of the displacement of the origin of frame $\mathcal{B}$ relative to the origin of frame $\mathcal{A}$ expressed in terms of frame $\mathcal{A}$. Since the determinant of ${}^{\mathcal{B}}_{\mathcal{A}}\mathbf{T}_f$ is one, the determinants of the (3×3) matrices ${}^{\mathcal{O}}\mathbf{W}_n$, ${}^{\mathcal{W}}\mathbf{W}_{t,edges}$, and ${}^{\mathcal{O}}\mathbf{W}_{t,verts}$ are all invariant under a change of $\mathcal{O}$ or $\mathcal{W}$ as appropriate. The frame invariance of I^2 and G^2+H^2 now follow from formulas (39) and (40). We emphasize, however, that G, H, and the wrench matrices are not frame invariant. .... □

The invariance of the wrench matrix determinants can also be seen geometrically. Each column of ${}^{\mathcal{O}}\mathbf{W}_n$ represents a line perpendicular to an edge of the workpiece and containing the vertex of the manipulator intended to contact that edge. The condition that $Det({}^{\mathcal{O}}\mathbf{W}_n) = 0$ is exactly that these three lines meet at a point (possibly at infinity). Since these lines depend purely on the geometry of the bodies and their relative arrangement, it is clear that the determinant of ${}^{\mathcal{O}}\mathbf{W}_n$ is independent of any choice of frames.

The tangential wrench matrices, ${}^{\mathcal{W}}\mathbf{W}_{t,edges}$ and ${}^{\mathcal{O}}\mathbf{W}_{t,verts}$, each represent three lines parallel to the edges of the workpiece specified by the 3A CF. The lines corresponding to ${}^{\mathcal{W}}\mathbf{W}_{t,edges}$ are those supporting the actual edges of the workpiece, while the lines corresponding to ${}^{\mathcal{O}}\mathbf{W}_{t,verts}$ are those that contain the designated vertices and are parallel to the actual edges. Note that the vertices may or may not lie on the actual edges of the workpiece. Again, the condition that $Det({}^{\mathcal{O}}\mathbf{W}_{t,verts}) = 0$ or $Det({}^{\mathcal{W}}\mathbf{W}_{t,edges}) = 0$ is that the three lines involved intersect at a point (possibly at infinity).

3.3 A New Nongeneric Contact Situation

Recall that nongeneric situations are those which satisfy $G = 0$ and $H = 0$, or which satisfy inequality (31) by strict equality. Thus we can view the equation $G^2 + H^2 - I^2 = 0$ as defining the hypersurface in the manipulator configuration space where nongenericity occurs.

Proposition 4: *If elimination succeeds, then for a fixed* $\mathbf{r}$*, we will have an infinite number of solutions on the 3A CF-cell if and only if* $Det(\mathbf{W}_n) = Det({}^{\mathcal{O}}\mathbf{W}_{t,verts}) = Det({}^{\mathcal{W}}\mathbf{W}_{t,edges}) = 0$*, or equivalently,* $G = H = I = 0$*.*

Proof: Our elimination procedure reduced the search for geometrically admissible configurations of the workpiece for a given $\mathbf{r}$ to finding the intersections between a line and a circle. When elimination succeeds and $G = H = I = 0$, we are left with only the circle. Given the relationships between G, H, and I and the wrench matrices, it is clear that the determinants of all three wrench matrices must be zero and conversely. .... □

One can easily construct examples meeting the conditions of Proposition 4. First, construct a workpiece polygon that has three edges which intersect at a point. Second, place two manipulator vertices anywhere on two of those three edges. Third, find the intersection point of the two contact normals and project it perpendicular to the third edge to locate the third manipulator vertex. Figure 3 shows a workpiece (light gray) in several geometrically admissible configurations for a fixed configuration of the manipulator (dark gray). Note that every orientation of the workpiece corresponds to a point on the unit circle, equation (20), so that for any orientation, there is a workpiece position, (x, y), which achieves the specified CF. This position can be found by back substituting into equations (19).

The points, $\mathbf{r}$, in the manipulator configuration space, where Proposition 4 is satisfied will be referred to as *exceptional points*. These points "blow up" [22] to circles on the CF-cell. Each direction in manipulator configuration space in a neighborhood of an exceptional point potentially gives rise to a different limit point (in the CF-cell). Thus special care must be taken when planning a manipulation trajectory which passes near or through exceptional points to ensure that the workpiece follows its desired trajectory. Physically, the configuration of the workpiece when $\mathbf{r}$ is exceptional depends not only on the current $\mathbf{r}$, but on the direction from which $\mathbf{r}$ was reached. For a detailed analysis of the geometry of this particular "blow-up" see [6].

We now turn to the CF-cell as a whole.

3.4 Global Properties of 3A CF-cells

Theorem 2: *Every 3A CF-cell is a nine-dimensional manifold.*

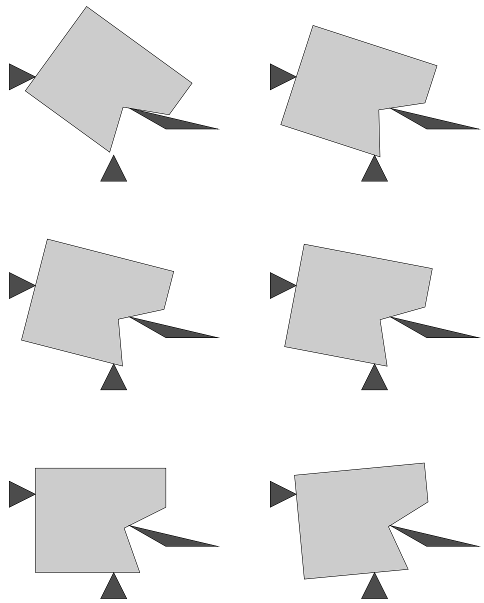

Figure 3: A nongeneric situation for which elimination succeeds.

Proof: Let $\mathbf{f} = 0$ be the vector of constraint equations given by:

$$\mathbf{f}(\mathbf{p}) = [C_1(\mathbf{p}), C_2(\mathbf{p}), C_3(\mathbf{p}), c^2 + s^2 - 1]^T. \tag{42}$$

For $\mathcal{CF}$ to be a manifold, it is sufficient that at each point in $\mathcal{CF}$, the rank of the Jacobian matrix $(\frac{\partial \mathbf{f}}{\partial \mathbf{p}})$ be four [19]. The Jacobian matrix has the following form:

$$\begin{bmatrix} m_{1,1} & m_{1,2} & m_{1,3} & m_{1,4} & m_{1,5} & 0 & 0 & m_{1,8} & 0 & 0 & m_{1,11} & 0 & 0 \\ m_{2,1} & m_{2,2} & m_{2,3} & m_{2,4} & 0 & m_{2,6} & 0 & 0 & m_{2,9} & 0 & 0 & m_{2,12} & 0 \\ m_{3,1} & m_{3,2} & m_{3,3} & m_{3,4} & 0 & 0 & m_{3,7} & 0 & 0 & m_{3,10} & 0 & 0 & m_{3,13} \\ 0 & 0 & 2c & 2s & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{43}$$

The entries $m_{1,5}$ and $m_{1,8}$ are $cos(\theta + \phi_1)$ and $sin(\theta + \phi_1)$ respectively. Thus it is clear that both $m_{1,5}$ and $m_{1,8}$ cannot simultaneously be zero. An identical argument shows that at least one of $m_{2,6}$ and $m_{2,9}$ must be nonzero, as well as one of $m_{3,7}$ and $m_{3,10}$. Therefore, we can always form a nonsingular (4×4) minor as follows:

$$\begin{bmatrix} m_{1,\beta} & m_{1,i} & 0 & 0 \\ m_{2,\beta} & 0 & m_{2,j} & 0 \\ m_{3,\beta} & 0 & 0 & m_{3,k} \\ 2\alpha & 0 & 0 & 0 \end{bmatrix} \tag{44}$$

where $(\alpha, \beta) \in \{(c, 3), (s, 4)\}$, $i \in \{5, 8\}$, $j \in \{6, 9\}$, and $k \in \{7, 10\}$. .... □

Asada and By [1] showed that the matrix of partial derivatives of the functions describing the surface of a part (taken with respect to the position and orientation variables of the workpiece) is a normal wrench matrix. In this paper, the orientation of the workpiece is represented by $cos(\theta)$ and $sin(\theta)$, which are treated as independent variables. Thus with our definitions, the four-by-thirteen Jacobian matrix contains a modified normal wrench matrix, $\mathbf{W}_n^*$, which has dimension (4×4). Its rows are the first four columns of that Jacobian matrix:

$$\mathbf{W}_n^* = \begin{bmatrix} m_{1,1} & m_{2,1} & m_{3,1} & 0 \\ m_{1,2} & m_{2,2} & m_{3,2} & 0 \\ m_{1,3} & m_{2,3} & m_{3,3} & 2c \\ m_{1,4} & m_{2,4} & m_{3,4} & 2s \end{bmatrix}. \tag{45}$$

The first two rows of the first three columns of $\mathbf{W}_n^*$ are the same as those in the traditional normal wrench matrix (expressed in frame $\mathcal{O}$). They represent the x- and y-components of the three contact normals. The last two rows of the first three columns are the moments (about the origin of frame $\mathcal{W}$) of the x- and y-components of the normal vectors at the contact points, respectively.

Using the chain rule relationship, $\frac{\partial f_i(c,s)}{\partial \theta} = -\frac{\partial f_i}{\partial c}s + \frac{\partial f_i}{\partial s}c$, we can derive the following simple relationship between $\mathbf{W}_n^*$ and the usual normal wrench matrix $\mathbf{W}_n$:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -s & c \\ 0 & 0 & -c & -s \end{bmatrix} \mathbf{W}_n^* = \begin{bmatrix} & \mathbf{W}_n & & 0 \\ & & & 0 \\ & & & 0 \\ (\cdot) & (\cdot) & (\cdot) & -2 \end{bmatrix}. \tag{46}$$

From this relationship, it is clear that the deficiency in the rank of $\mathbf{W}_n$ and $\mathbf{W}_n^*$ is the same.

Theorem 3: *For a 3A CF, a point* $\mathbf{p} \in \mathcal{CF}$, *mapping to a point* $\mathbf{r}$ *in manipulator configuration space, will be a branch point of the mapping of multiplicity two if and only if* $G^2+H^2 \neq 0$ *and* $\mathbf{W}_n$ *is singular. In that case, the configuration will be the only geometrically admissible configuration. (This is equivalent to* $G^2 + H^2 \neq 0$ *and* $G^2 + H^2 - I^2 = 0$.*)*

Proof: We have shown that $Det(\mathbf{W}_n) = cH - sG$. But notice that $cH - cG = 0$ is the equation of a line through the origin in the c-s plane which is perpendicular to the line defined by $Gc + Hs + I = 0$. Since the set of C-functions reduces to the following system of equations:

$$Gc + Hs + I = 0 \tag{47}$$

$$c^2 + s^2 - 1 = 0, \tag{48}$$

the above system will have a unique solution if and only if the line represented by equation (47) is tangent to the circle represented by the equation (48). In that case, the line defined by $-Gs + Hc = 0$ contains the point of tangency. This implies that the determinant of the normal wrench matrix is zero.

Conversely, if the determinant of the normal wrench matrix is zero, then the line (47) is forced to be tangent to the unit circle. .... □

Situations that satisfy Theorem 3 are those for which the contact normals intersect at a point, but the edges designated for contact do not. For example, imagine placing three contacts at the midpoints of the three sides of an equilateral triangle. The contact normals intersect at the center of the triangle. This situation is nongeneric, because moving any one of the vertices straight through its edge and into the triangle even slightly results in no geometrically admissible workpiece configurations. Similarly, moving one vertex out from its edge results in two admissible configurations.

Corollary 1: *The branch locus of a 3A CF-cell under projection to the manipulator configuration space is precisely the locus where $G^2 + H^2 \neq 0$ and $\mathbf{W}_n$ is singular. By contrast, when $\mathbf{W}_n$ is nonsingular at $\mathbf{p} \in \mathcal{CF}$, we are forced to have $G^2 + H^2 \neq 0$, and the projection will be a local diffeomorphism.*

Proof: The singularity of the normal wrench matrix at points where $G^2 + H^2 \neq 0$ indicates that the projection from the CF-cell to the manipulator configuration space is not a local diffeomorphism. Since the CF-cell itself is a smooth manifold, the singularity of the wrench matrix indicates that the two-sheeted cover branches. .... □

In summary, every 3A CF-cell, $\mathcal{CF}$, is a smooth nine-dimensional "surface" sitting over the space of manipulator configurations, which is also nine-dimensional. For most values of $\mathbf{r}$, there are zero or two points on the $\mathcal{CF}$-cell corresponding to the geometrically admissible workpiece configurations. However, there are a "few" special manipulator configurations for which there is 1 or an infinite number of workpiece configurations. Despite these special configurations, every 3A CF-cell is smooth everywhere. Over the regions of manipulator configuration space where there is a finite number of workpiece configurations, the CF-cell can be viewed as two sheets which occasionally come together smoothly (without discontinuity between their normals). When generating manipulation plans (*i.e.*, trajectories in $\mathbf{r}$) that pass near or through these branch points or the points corresponding to an infinite number of workpiece configurations, extra care must be taken to ensure that the workpiece moves as desired.

3.5 3B CF-Cells

Suppose we are given a contact formation with three type B contacts. Consider Figure 4. Let (w_l, z_l) denote the position, with respect to the frame $\mathcal{W}$, of the vertex on the workpiece

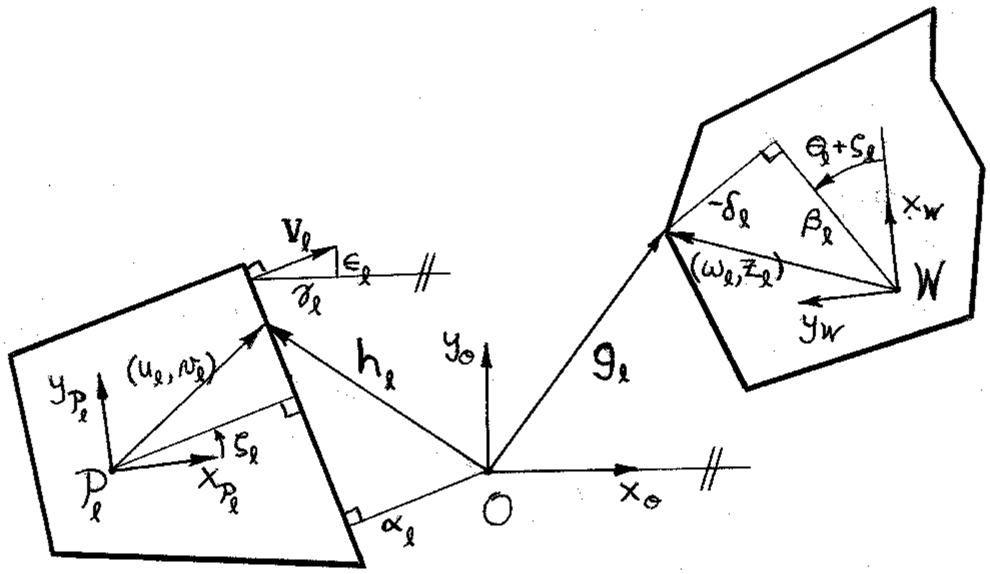

Figure 4: Illustration of parameters relevant to type B contacts.

which is to be contacted by the designated edge of manipulator polygon l. Select a point on edge l, and let (u_l, v_l) denote its position with respect to the frame $\mathcal{P}_l$. Let $\mathbf{g}_l$, $\mathbf{h}_l$, and $\mathbf{v}_l$ be respectively, the position of vertex l, the position of the selected point (u_l, v_l) on edge l, and the unit outward normal of edge l, all expressed with respect to the world frame $\mathcal{O}$. The C-function for a single B type contact is then given as in equation (5):

$$C_l(\mathbf{p}) = \mathbf{v}_l \cdot (\mathbf{g}_l - \mathbf{h}_l). \tag{49}$$

To make this explicit, let ζ_l denote the angle between $\mathbf{v}_l$ and the x-axis of frame $\mathcal{P}_l$. Then the C-functions for the three B-type contacts can be expanded to yield:

$$\begin{aligned} C_l(\mathbf{p}) = \alpha_l + \beta_l c + \delta_l s + \gamma_l x + \epsilon_l y &= 0\,; \quad l = 1,2,3 & (50) \\ c^2 + s^2 - 1 &= 0 & (51) \end{aligned}$$

where the coefficients are given by:

$$\begin{aligned} \alpha_l &= -cos(\zeta_l + \theta_l)x_l - sin(\zeta_l + \theta_l)y_l - cos(\zeta_l)u_l - sin(\zeta_l)v_l & (52) \\ \beta_l &= cos(\zeta_l + \theta_l)w_l + sin(\zeta_l + \theta_l)z_l & (53) \\ \delta_l &= sin(\zeta_l + \theta_l)w_l - cos(\zeta_l + \theta_l)z_l & (54) \\ \gamma_l &= cos(\theta_l + \zeta_l) & (55) \\ \epsilon_l &= sin(\theta_l + \zeta_l). & (56) \end{aligned}$$

Proceeding as we did in our study of 3A CF-cells, we again find (assuming elimination succeeds) that the feasible workpiece configurations correspond to the intersection of a line and the unit circle:

$$G(\mathbf{r})c + H(\mathbf{r})s + I(\mathbf{r}) = 0 \quad (57)$$

$$c^2 + s^2 - 1 = 0 \quad (58)$$

where $G = det\begin{bmatrix} \beta_1 & \gamma_1 & \epsilon_1 \\ \beta_2 & \gamma_2 & \epsilon_2 \\ \beta_3 & \gamma_3 & \epsilon_3 \end{bmatrix}$, $H = det\begin{bmatrix} \delta_1 & \gamma_1 & \epsilon_1 \\ \delta_2 & \gamma_2 & \epsilon_2 \\ \delta_3 & \gamma_3 & \epsilon_3 \end{bmatrix}$, and $I = det\begin{bmatrix} \alpha_1 & \gamma_1 & \epsilon_1 \\ \alpha_2 & \gamma_2 & \epsilon_2 \\ \alpha_3 & \gamma_3 & \epsilon_3 \end{bmatrix}$.

Not surprisingly, results similar to those obtained for 3A CF-cells can be proved for 3B CF-cells. It will become clear in the propositions and theorems below that the vertices and edges in the 3B case are simply the duals of the edges and vertices, respectively, in the 3A case. This duality can be explained in terms of the natural duality between lines and points in the projective plane [18]. Under this duality, the workpiece (viewed as a set of lines) gets mapped to a "dual" workpiece (given as a set of vertices) and a manipulator polygon (viewed as a set of vertices) gets mapped to a "dual" manipulator polygon (given as a set of lines).

Because the proofs for the propositions and theorems below exactly follow the logic of the proofs offered for the 3A CF-cell, they will not be given explicitly.

Proposition 5: *For a 3B CF, elimination fails, i.e., the rank of* $\begin{bmatrix} \gamma_1 & \epsilon_1 \\ \gamma_2 & \epsilon_2 \\ \gamma_3 & \epsilon_3 \end{bmatrix}$ *is less than two, if and only if the lines supporting the edges designated for contact are parallel. (This includes the possibility that two or three of the lines are coincident.)*

Proposition 6: *If elimination fails, then $G = H = I = 0$ and for a fixed $\mathbf{r}$, if there is any workpiece configuration which attains the 3B CF, then there will be an infinite number.*

Once again, the functions G, H, and I are related to the determinants of certain wrench matrices and $G^2 + H^2$ and I are frame invariant. In effect, Proposition 3 carries over to the case of a 3B CF-cell:

Proposition 7: *The quantities I^2 and G^2+H^2, and the quantities $Det({}^{\mathcal{W}}\mathbf{W}_{t,edges})$, $Det({}^{\mathcal{O}}\mathbf{W}_n)$, and $Det({}^{\mathcal{O}}\mathbf{W}_{t,verts})$ are independent of the choice of all coordinate frames.*

The relationships among G, H, I, and the wrench matrices are:

$$Det({}^{\mathcal{O}}\mathbf{W}_n) = cH - sG \qquad Det({}^{\mathcal{O}}\mathbf{W}_{t,verts}) = sH + cG \qquad Det({}^{\mathcal{W}}\mathbf{W}_{t,edges}) = -I \quad (59)$$

where

$${}^{\mathcal{O}}\mathbf{W}_n = \begin{bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \epsilon_1 & \epsilon_2 & \epsilon_3 \\ \delta_1 c - \beta_1 s + x\epsilon_1 - y\gamma_1 & \delta_2 c - \beta_2 s + x\epsilon_2 - y\gamma_2 & \delta_3 c - \beta_3 s + x\epsilon_3 - y\gamma_3 \end{bmatrix} \quad (60)$$

$$
{}^{\mathcal{O}}\mathbf{W}_{t,verts} = \begin{bmatrix} -\epsilon_1 & -\epsilon_2 & -\epsilon_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ \delta_1 s + \beta_1 c + y\epsilon_1 + x\gamma_1 & \delta_2 s + \beta_2 c + y\epsilon_2 + x\gamma_2 & \delta_3 s + \beta_3 c + y\epsilon_3 + x\gamma_3 \end{bmatrix} \quad (61)
$$

$$
{}^{\mathcal{W}}\mathbf{W}_{t,edges} = \begin{bmatrix} -\epsilon_1 & -\epsilon_2 & -\epsilon_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \\ -\alpha_1 & -\alpha_2 & -\alpha_3 \end{bmatrix}. \quad (62)
$$

Proposition 8: *If elimination succeeds, then for a fixed* $\mathbf{r}$*, we will have an infinite number of solutions on the 3B CF-cell if and only if* $Det({}^{\mathcal{O}}\mathbf{W}_n) = Det({}^{\mathcal{O}}\mathbf{W}_{t,verts}) = Det({}^{\mathcal{W}}\mathbf{W}_{t,edges}) = 0$*, or equivalently,* $G = H = I = 0$*.*

In cases when elimination succeeds, the nongeneric situations are dual to those for the 3A case. To highlight this fact, note that the nongeneric case shown in Figure 3 is nongeneric for the 3B type CF if one views the (dark) manipulator polygons as rigidly connected to become the workpiece and one cuts up the (light) workpiece polygon into disconnected pieces, each containing one of the contact edges, to become three manipulator polygons.

Theorem 4: *For a generic positioning,* $\mathbf{r}$*, of the manipulator, we will have either zero or two geometrically admissible workpiece configurations.*

Theorem 5: *Every 3B CF-cell is a nine-dimensional manifold.*

Theorem 6: *For a 3B CF, a point* $\mathbf{p} \in \mathcal{CF}$*, mapping to a point* $\mathbf{r}$ *in manipulator configuration space, will be a branch point (necessarily of multiplicity two) of the mapping if and only if* $G^2+H^2 \neq 0$ *and* $\mathbf{W}_n$ *is singular. In that case, the configuration will be the only geometrically admissible configuration. (This is equivalent to* $G^2 + H^2 \neq 0$ *and* $G^2 + H^2 - I^2 = 0$*.)*

A nongeneric situation satisfying Theorem 6 is as follows. Let the workpiece be an equilateral triangle and let the edges of three manipulator polygons touch the three vertices such that the contact normals bisect the angles of the triangle.

Corollary 2: *The branch locus of the 3B CF-cell under projection to the manipulator configuration space is precisely the locus where* $G^2 + H^2 \neq 0$ *and* $\mathbf{W}_n$ *is singular.*

3.6 2AB CF-Cells

2AB CF-cells are characteristically different from 3A and 3B CF-cells, because finding the geometrically admissible configurations of the workpiece boils down to determining the intersections between a circle and a general quadric in c and s. Nonetheless, most of the steps followed in analyzing the 3A case apply directly to the 2AB case. Thus in this section, we will only include the steps that are different.

Building on the results of our analysis of the 3A and 3B CF-cells given above, the C-functions in the 2AB case are given by:

$$\begin{aligned} a_l + b_l c + d_l s - e_l x c + f_l x s - f_l y c - e_l y s &= 0\,; \quad l = 1, 2 && (63) \\ \alpha_3 + \beta_3 c + \delta_3 s + \gamma_3 x + \epsilon_3 y &= 0 && (64) \\ c^2 + s^2 - 1 &= 0. && (65) \end{aligned}$$

Eliminating x and y yields:

$$\begin{aligned} Jc + Ks + Lc^2 + Ms^2 + Ncs &= 0 && (66) \\ c^2 + s^2 - 1 &= 0 && (67) \end{aligned}$$

where (using equations (67)):

$$\begin{aligned} J &= +\epsilon_3 e_1 a_2 - \gamma_3 a_2 f_1 - \epsilon_3 a_1 e_2 + \gamma_3 f_2 a_1 - \beta_3 e_2 f_1 + \beta_3 f_2 e_1 && (68) \\ K &= +\gamma_3 e_2 a_1 - \epsilon_3 f_1 a_2 - \gamma_3 a_2 e_1 + \epsilon_3 a_1 f_2 - \delta_3 e_2 f_1 + \delta_3 f_2 e_1 && (69) \\ L &= +\gamma_3 f_2 b_1 - \epsilon_3 b_1 e_2 + \epsilon_3 e_1 b_2 - \gamma_3 b_2 f_1 - \alpha_3 e_2 f_1 + \alpha_3 e_1 f_2 && (70) \\ M &= -\gamma_3 d_2 e_1 + \epsilon_3 d_1 f_2 + \gamma_3 e_2 d_1 - \epsilon_3 f_1 d_2 - \alpha_3 e_2 f_1 + \alpha_3 e_1 f_2 && (71) \\ N &= -\gamma_3 b_2 e_1 + \gamma_3 f_2 d_1 + \epsilon_3 b_1 f_2 - \epsilon_3 d_1 e_2 - \epsilon_3 f_1 b_2 - \gamma_3 d_2 f_1 + \epsilon_3 e_1 d_2 + \gamma_3 e_2 b_1. && (72) \end{aligned}$$

Equation (66) can be rewritten as:

$$Det \begin{bmatrix} a_1 + b_1 c + d_1 s & -e_1 c + f_1 s & -f_1 c - e_1 s \\ a_2 + b_2 c + d_2 s & -e_2 c + f_2 s & -f_2 c - e_2 s \\ \alpha_3 + \beta_3 c + \delta_3 s & \gamma_3 & \epsilon_3 \end{bmatrix} = 0 \qquad (73)$$

and elimination will fail if and only if the rank of $\begin{bmatrix} -e_1 c + f_1 s & -f_1 c - e_1 s \\ -e_2 c + f_2 s & -f_2 c - e_2 s \\ \gamma_3 & \epsilon_3 \end{bmatrix}$ is less than two. Note that the rows of this (2×3) matrix are unit normals to the edges written with respect to the frame $\mathcal{O}$.

Proposition 9: *For a 2AB CF, elimination fails if and only if the three designated edges are mutually parallel.*

Proposition 10: *If elimination fails for a fixed* $\mathbf{r}$*, then if there is any workpiece configuration which attains the 2AB CF, there will be an infinite number of workpiece configurations which attain the 2AB CF.*

Proposition 11: *If elimination succeeds, then we will have an infinite number of solutions for the 2AB CF if the two designated vertices of the manipulator contact a single edge of the workpiece at the same point. The designated edge of the manipulator can contact any vertex of the workpiece.*

Proof: If the two vertices of the manipulator are designated to coincide and contact the same edge, then one is redundant and may be removed. Thus we are left with two contacts. Using the two C-functions, we can solve for x and y as functions of c and s. Therefore, any c and s on the unit circle correspond to a valid (x, y) pair. .... □

The above proposition represents a sufficient condition for the existence of an infinite number of geometrically admissible configurations, but it *may* not be necessary. One might think that when considering all three contacts, the necessary and sufficient condition would be as before; the determinants of the normal and tangential wrench matrices are zero. This is not true. The 2AB configuration on the left side of Figure 5 has zero normal and tangential wrench matrix determinants, but this example has only two geometrically admissible workpiece configurations. One of the admissible configurations is the one shown on the left. The other can be obtained from the first by rotating the square π radians about its lower left corner. Clearly, when the square is in the orientation shown on the right, it cannot be translated to achieve the desired 2AB CF.

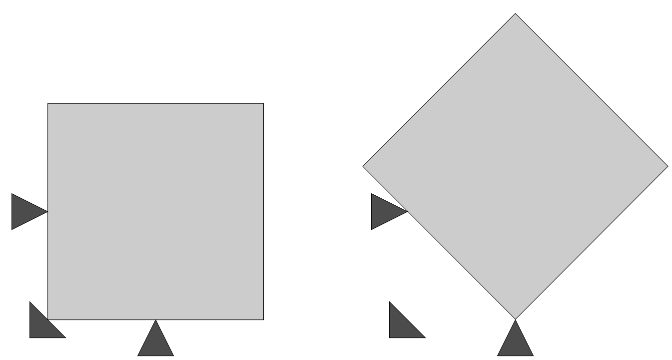

Figure 5: Counter-Example that $Det(\mathbf{W}_n) = Det(\mathbf{W}_{t,edges}) = Det(\mathbf{W}_{t,edges}) = 0$ is a sufficient condition for an infinite number of geometrically admissible workpiece configurations.

Theorem 7: *For a generic positioning* $\mathbf{r}$ *of the manipulator, we will have zero, two, or four geometrically admissible workpiece configurations.*

Proof: A valid configuration must satisfy equations (66) and (67) each of which has degree 2. By Bezout's Theorem [8], the number of intersections is 4 counting points at infinity, complex intersections, and multiplicities. For the case where all intersections are real, finite and of multiplicity one, the number of configurations is precisely 4. This is the maximum possible number of geometrically admissible configurations. The other cases occur when pairs of distinct solutions are complex. .... □

Theorem 8: *Every 2AB CF-cell is a nine-dimensional manifold.*

Proof: The proof is analogous to the proof of Theorem 2. .... □

Theorem 9: *For a 2AB CF, a point* $\mathbf{p} \in \mathcal{CF}$*, mapping to a point* $\mathbf{r}$ *in manipulator configuration space, will be a branch point of multiplicity greater than one if and only if the normal wrench matrix,* $\mathbf{W}_n$*, is singular.*

Proof: The condition that the normal wrench matrix is singular is given by:

$$-Js + Kc - Ns^2 + Nc^2 + 2(M-L)cs = 0. \tag{74}$$

We assume that elimination does not fail and that equation (66) is non-trivial.

Recall that the valid solutions are given by the intersection of the conic (66) and the unit circle (67). A point of intersection of these two conics will have multiplicity greater than one if and only if they are tangent at that point. Let U denote the equation for the unit circle and V denote the equation of the general conic, then the condition for tangency is given by:

$$-\frac{\frac{\partial U}{\partial c}}{\frac{\partial U}{\partial s}} = -\frac{\frac{\partial V}{\partial c}}{\frac{\partial V}{\partial s}} \tag{75}$$

which expands to yield:

$$-\frac{J + 2Lc + Ns}{K + 2Ms + Nc} = -\frac{2c}{2s}. \tag{76}$$

Equation (76) which can be manipulated to give equation (74), is precisely the condition that the wrench matrix is singular. .... □

Singularity of the normal wrench matrix does not indicate that we have a unique solution. However, it does indicate that we *don't* have four distinct solutions. Singularity is indicative of the fact that two or more of the sheets of the 2AB CF-cell over the manipulator configuration space come together (branch) and hence, that the projection to the manipulator configuration space is not a local diffeomorphism.

3.7 2BA CF-Cells

The 2BA CF-cell is dual to the 2AB CF-cell with results carrying over in the obvious manner.

4 CF-Cells with Two Contacts

So far we have only discussed the cases involving three sliding contacts. In this section, we will consider the cases in which one contact rolls and one slides. Since our analysis is limited to polygonal bodies, rolling reduces to pivoting about a vertex and the associated kinematic constraints are holonomic.

Each of the four rolling cases can be thought of as a special case of one with three sliding contacts. For example, an $\mathrm{A_R A}$ case can be viewed as a 3A CF by imposing two additional fictitious constraints. First, add a fictitious edge to the workpiece which transversally intersects the edge on which rolling is to occur at the designated rolling point. Now assign one manipulator vertex to each of the three edges in question: the rolling edge, the sliding edge, and the fictitious edge. The second fictitious constraint is to fuse the two polygons whose vertices are to contact the rolling and fictitious edges such that the designated vertices coincide. The fused polygons become one rigid body. Requiring that the fused polygons contact their respective edges generates the desired rolling constraint. In a similar way, the $\mathrm{A_R B}$, $\mathrm{B_R A}$, and $\mathrm{B_R B}$ cases can be generated as special cases of the 2AB, 2BA, and 3B cases.

Despite the fact that the $\mathrm{A_R B}$ and $\mathrm{B_R A}$ cases are special cases of the 2AB and 2BA cases, for a fixed $\mathbf{r}$, they never admit more than two geometrically admissible workpiece configurations. This is because determining the admissible configurations in the rolling cases always reduces to determining the intersections between either a rotating edge with a vertex or a rotating vertex (a circle) with an edge. The results relating to the wrench matrices carry over to the rolling cases, however, one must be sure to include the contact on the fictitious edge when defining the wrench matrices.

Even though the rolling cases are special cases of the sliding cases, in the following subsection, we derive the results for the $\mathrm{B_R B}$ case to highlight one special feature common to all rolling cases; *elimination never fails.*

4.1 $\mathrm{B_R B}$ CF-Cells

Since there are only two active polygons, the system configuration vector $\mathbf{p}$ is:

$$\mathbf{p} = [x, y, c, s, x_1, x_2, y_1, y_2, \theta_1, \theta_2]\,. \tag{77}$$

Let (u_l, v_l) be the coordinates of the vertex of the workpiece that is maintaining the "$\mathrm{B_R}$" contact with respect to a frame $\mathcal{W}$ fixed to the workpiece. Let (w_l, z_l) be the coordinates (with respect to $\mathcal{P}_l$) of the point on the edge of the manipulator that is maintaining rolling contact with the workpiece. Let C_{R_1}, C_{R_2} denote the two C-functions which describe the constraints imposed by the rolling contact, which we label as contact l.

For the designated vertex of the workpiece to maintain a rolling contact with the prescribed point on edge l of manipulator polygon l, (u_l, v_l) must coincide with the point (w_l, z_l).

The constraints C_{R_1} and C_{R_2} can be written as:

$$C_{R_1} = j - u_l c + v_l s - x = 0 \quad (78)$$
$$C_{R_2} = k - v_l c - u_l s - y = 0 \quad (79)$$

where

$$j = x_l + cos(\theta_l)w_l - sin(\theta_l)z_l \quad (80)$$
$$k = y_l + cos(\theta_l)z_l + sin(\theta_l)w_l. \quad (81)$$

We label the sliding contact by m, and let C_m denote the constraint imposed by this contact. This C-function is given by equation (50):

$$\alpha_m + \beta_m c + \delta_m s + \gamma_m x + \epsilon_m y = 0 \quad (82)$$

where α_m, β_m, γ_m, δ_m, and ϵ_m are given by equations (52-56). The vector $\mathbf{f}$ of C-functions defining the $B_R B$ CF is given by:

$$\mathbf{f} = [C_{R_1}, C_{R_2}, C_m, c^2 + s^2 - 1]. \quad (83)$$

Proceeding in the now familiar way we find that the geometrically admissible workpiece configurations correspond to the intersections between a line and the unit circle in c and s:

$$(\beta_m - \gamma_m u_l - \epsilon_m v_l)c + (\delta_m + \gamma_m v_l - \epsilon_m u_l)s + (\alpha_m + \gamma_m j + \epsilon_m k) = 0 \quad (84)$$
$$c^2 + s^2 - 1 = 0. \quad (85)$$

Proposition 12: *For a* $\mathrm{B_R B}$ *CF, elimination always succeeds.*

Proof: Equation (84) is precisely the condition that the system (78,79,82) has a nontrivial solution. This condition is equivalent to

$$Det \begin{bmatrix} j - u_l c + v_l s & -1 & 0 \\ k - u_l s - v_l c & 0 & -1 \\ \alpha_m + \beta_m c + \delta_m s & \gamma_m & \epsilon_m \end{bmatrix} = 0. \quad (86)$$

For the elimination to succeed, the rank of $\begin{bmatrix} -1 & 0 \\ 0 & -1 \\ \gamma_m & \epsilon_m \end{bmatrix}$ must be two, which is clearly the case. .... □

This result that elimination always succeeds can be deduced from the fact that elimination only fails when the three contacted edges are parallel. Since the rolling constraint can be viewed as two sliding contacts on two distinct intersecting lines, elimination will always succeed.

Proposition 13: *For a fixed* $\mathbf{r}$*, we will have an infinite number of geometrically admissible workpiece configurations if and only if the sliding and rolling contact points coincide. (In this case Equation (84), as a linear equation is c and s, will be identically zero.)*

Proof: For a fixed $\mathbf{r}$, when one contact point is rolling, the other contact point moves on a circle whose radius is the distance between the contact points. There will be an infinite number of geometrically admissible workpiece configurations, if and only if every point of the circle is contained in the line supporting the edge on which the second contact is to take place. Clearly this is the case, if and only if the circle degenerates to a point on the line, indicating that contact points coincide. .... □

Note that using the fictitious edge in defining the relevant wrench matrices, there will be an infinite number of workpiece configurations if and only if $Det(\mathbf{W}_n) = Det(\mathbf{W}_{t,verts}) = Det(\mathbf{W}_{t,edges}) = 0$. Given that two polygons have been fused so their two contact points will always coincide, the determinants of all of the wrench matrices will be zero if and only if all three contact points coincide.

Proposition 14: *For a fixed* $\mathbf{r}$*, we will have a unique solution on the* $\mathrm{B_R B}$ *CF-cell if and only if the normal at the sliding contact passes through the rolling contact point and the contact points do not coincide.*

Proof: For a fixed $\mathbf{r}$, when one contact point is rolling, the other contact point moves on a circle whose radius is the distance between the contact points. There will be only one geometrically admissible workpiece configuration if and only if the circle is tangent to the line supporting the edge on which the second contact is to take place. In this case, the contact normal on that edge must pass through the rolling contact point (at the center of the circle) and the radius of the circle must be nonzero. .... □

Again, using the fictitious edge and the rolling and sliding edges to define the relevant wrench matrices, there will be a unique workpiece configuration if and only if $Det(\mathbf{W}_n) = 0$ and $Det(\mathbf{W}_{t,verts}) = Det(\mathbf{W}_{t,edges}) \neq 0$. Given that two polygons have been fused so their two contact points coincide, it is clear that $Det(\mathbf{W}_n)$ will be zero if and only if the normal at the sliding contact passes through the rolling contact. The other two wrench matrices are equal if and only if the designated contacts are achieved, and they will be nonzero if and only if the sliding edge does not contain the rolling point.

Theorem 10: *For a generic positioning,* $\mathbf{r}$*, of the manipulator, we will have zero or two geometrically admissible workpiece configurations.*

Theorem 11: *Every* $\mathrm{B_R B}$ *CF-cell is a six-dimensional manifold.*

Finally, we note that the analysis of the rolling constraints was carried out in a system C-space of dimension ten for which the corresponding manipulator configuration space has dimension six. However, one can show that in the nine-dimensional manipulator configuration space used to analyze the CF's for three sliding contacts, the equations effecting the fusing of the two polygons, as described at the beginning of this section, define a submanifold of dimension six. This manifold is diffeomorphic to the six-dimensional manipulator configuration space for a rolling contact. Moreover, in $\mathcal{Z}$, the thirteen-dimensional system configuration space for the 3B case, the polygon-fusing equations will define a ten-dimensional submanifold which intersects the 3B CF-cell transversally in a six-dimensional submanifold. This intersection is diffeomorphic to the $B_R B$ CF-cell as described above.

The proofs of all these facts come down to showing that the seven-by-thirteen Jacobian matrix of our seven constraints (the usual 3B constraints plus the three polygon-fusing constraints) always has rank 7. This can be done in a way similar to the four-by-thirteen case considered in Theorem 2.

4.2 $A_R B$ $B_R A$ and $A_R A$ CF-Cells

The remaining rolling cases can be analyzed in the same way that the $B_R B$ case was and the results obtained will be analogous.

5 Conclusion

The results of this paper provide an in-depth understanding of the kinematic constraints imposed on a dexterous manipulation system by eight fundamental systems of contact constraint equations. The CF-cells described by these constraint systems are relevant to the dexterous manipulation of a single passive workpiece by two or three position-controlled manipulator polygons. The CF-cells have been found to be manifolds in the system's C-space. This result implies that one can predict the motion of the workpiece using well-established techniques for the integration of differential algebraic systems [21]. Either a dynamic or quasistatic rigid body model can be used.

The planning of dexterous manipulation tasks utilizing pure position control entails the generation of trajectories in the space of controllable variables of the system (*i.e.*, the positions and orientations of the manipulator polygons in contact with the workpiece) that cause the workpiece to follow a desired trajectory. Thus the relationship between the manipulator configuration space and the space of workpiece configurations must be well understood. The results of this paper include analytic formulas which will permit the efficient computation of the workpiece configuration(s) given the manipulator configuration. They also identify the points where a given manipulator configuration "blows-up" to an infinite number of workpiece configurations. While further study of these "blow-up" points is needed, it is clear that special care must be taken when generating manipulation trajectories that pass near or through them. Another important benefit of our analytical formulas is that they contain the relevant geometric model parameters and control settings of the manipulator polygons. Thus, models of geometric uncertainty and control error can be incorporated into the solutions to study the effects of uncertainty on the kinematic constraints.

Through our algebro-geometric analysis of the eight CF-cells, we have discovered a previously unknown nonintuitive, nongeneric class of contact situations. In these situations, for a fixed configuration of the manipulator, (the lines of support of) three nonparallel edges of the workpiece can maintain contact with three distinct vertices of the manipulator while retaining one degree of freedom of motion. In fact, there is a unique position of the workpiece that achieves the three specified contacts for every orientation of the workpiece.

5.1 Future Work

Our results position us to study the connectivity of collections of CF-cells and the properties of their intersections, so that we can plan dexterous manipulation actions that utilize several different CF's. We plan to apply techniques from deformation theory and differential topology to determine regions in the space of uncertain parameters for which transverse intersections and connectivity are maintained. Such results will allow us to modify nominal manipulation plans generated from a deterministic nominal model of the system to make them robust to variations in the uncertain parameters.

Finally, notice that if our manipulator is a linkage of prismatic and revolute joints, the forward kinematic map will go from the usual joint space to our manipulator configuration space. The usual workspace, which projects onto joint space, will be the fiber product (in the sense of algebraic geometry) of the forward kinematic map and the projection of the CF-cell to our manipulator configuration space. We plan to study this construction making use of the known behavior of the forward kinematic map. The results of this study will be expected to further enhance the efficiency of dexterous manipulation planning.

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