Some Remarks on the Geometry of Contact Formation Cells †

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Abstract

The contact formation cells of a polygonal planar system of rigid bodies in contact have been studied in [1]. There, it was shown that the CF-cells are smooth manifolds, but the methods used were too complicated to extend to three-dimensional polygonal rigid body systems. In this paper, we develop an alternative way to define contact formation cells. Under the new definition, we show that the contact formation cells are smooth manifolds, and further that all intersections of contact formation cells are smooth manifolds. The simplicity of the new definition makes it easy to prove the smoothness results for three-dimensional systems. Also, we investigate other extensions of the results in [1].

Section 1: Introduction

The concept of a contact formation cell (CF-cell) of a two-dimensional polygonal workpiece-manipulator system has been studied in [1], where they are shown to be smooth submanifolds of configuration space (C-space). The primary goal of this paper is to extend the study of CF-cells to three-dimensional systems, obtain the corresponding smoothness results, and derive other useful facts. The principal application of this work is to dexterous manipulation planning.

Section 2 concentrates on a new construction for CF-cells. We begin in section 2.1 with a review of the necessary background on the original construction of CF-cells. Next we give the new method of construction (section 2.2). We show that the CF-cells, under this new construction, are smooth manifolds. Also, the CF-cells as constructed originally and the new construction are shown to be equivalent geometrically, in the sense that there is a one-to-one correspondence between the CF-cells that preserves their geometric structure (i.e. a diffeomorphism). With this new definition, we are able to show that all the intersections of various CF-cells are also smooth manifolds (section 2.3). Finally, we extend the notion of CF-cell to three-dimensional systems (section 3) and show that they too are smooth manifolds.

Besides the smoothness results, we also study the extension of other results in [1] to three-dimensional systems. In section 4, we examine the number of possible workpiece configurations for a fixed manipulator configuration. The results for a planar system are known, but the general result for a three-dimensional system is not known. We investigate the case of a cube as the workpiece and find that the number is bounded by 16. Under some special conditions, we show in section 5 that there can be infinite number of workpiece configurations that maintain the contacts with a fixed manipulator configuration.

Section 2: Contact Formation Cells for Two-dimensional Systems

The original CF-cells are constructed by using C-functions [1]. However, this construction is very complicated to work with when dealing with three-dimensional systems. An alternative construction is developed in this section. We prove that the two constructions are equivalent by establishing an diffeomorphism between them. The new construction is easy to extend to three-dimensional systems. We also show that all intersections of CF-cells are smooth manifolds in the ambient C-space.

2.1 Basic definition and the original construction of CF-cells

Consider a two-dimensional system consisting of a workpiece and three manipulators (Figure 1). We assume that all are polygonal rigid bodies, and that only the manipulators are under active control.

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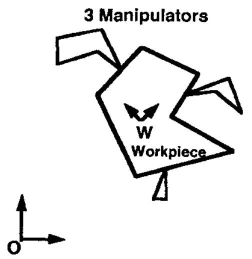


Figure 1. A two-dimensional system.

There are two types of elementary contacts[2]. A *type A* contact is formed when an edge of the workpiece contacts a vertex of a manipulator. A *type B* contact is formed when a vertex of the workpiece contacts an edge of a manipulator. Each collection of elementary contacts constitutes a contact formation and leads to a CF-cell. Note that we assume that all of the contacts are only *geometrically admissible*, that is, we consider a contact to occur when the specified vertex is in contact with any point on the straight line supporting the specified edge without regard to penetration.

A CF-cell composed of three type A contacts is called a 3A CF-cell. The other possible types of CF-cells studied in [1] are 2AB, 2BA and 3B. We will discuss only 3A CF-cells here. The other types of CF-cells can be treated similarly.

Consider now a vertex of the i^{th} manipulator polygon and an edge of the workpiece (Fig. 2). Let $\hat{x}_i, \hat{y}_i$ and $\hat{\theta}_i$ denote the position and orientation of a frame on the i^{th} manipulator polygon relative to a world frame O. Let $\hat{x}, \hat{y}$ and $\hat{\theta}$ denote the position and orientation of a frame W on the workpiece relative to O.

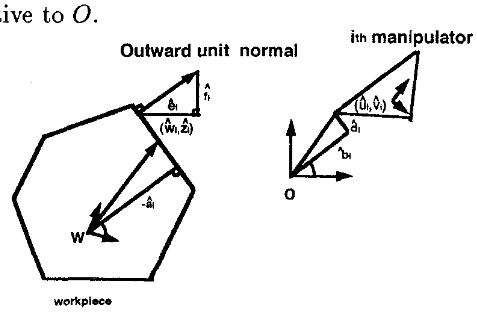


Figure 2. Illustration of parameters relevant to a type A contact.

We define the workpiece configuration $\hat{\mathbf{q}}$, the manipulator configuration $\hat{\mathbf{r}}$, and the system configuration $\hat{\mathbf{p}}$ as follows:

$$\hat{\mathbf{q}} = [\hat{x}, \hat{y}, c, s]^T, \ \hat{\mathbf{r}} = [\hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3]^T$$

$$\hat{\mathbf{p}} = [\hat{x}, \hat{y}, c, s, \hat{x}_1, \hat{x}_2, \hat{x}_3, \hat{y}_1, \hat{y}_2, \hat{y}_3, \hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3]^T$$

where $c = cos(\hat{\theta})$ and $s = sin(\hat{\theta})$. The distance between the vertex and the edge is given by

$$C_i(\hat{\mathbf{p}}) = \hat{a}_i + \hat{b}_i c + \hat{d}_i s - \hat{e}_i \hat{x} c + \hat{f}_i \hat{x} s - \hat{f}_i \hat{y} c - \hat{e}_i \hat{y} s$$

where

$$\hat{a}_i = -cos(\hat{\phi}_i)\hat{w}_i - sin(\hat{\phi}_i)\hat{z}_i$$

$$\hat{b}_i = cos(\hat{\phi}_i - \hat{\theta}_i)\hat{u}_i + sin(\hat{\phi}_i - \hat{\theta}_i)\hat{v}_i + sin(\hat{\phi}_i)\hat{y}_i + cos(\hat{\phi}_i)\hat{x}_i$$

$$\hat{d}_i = -sin(\hat{\phi}_i - \hat{\theta}_i)\hat{u}_i + cos(\hat{\phi}_i - \hat{\theta}_i)\hat{v}_i + cos(\hat{\phi}_i)\hat{y}_i - sin(\hat{\phi}_i)\hat{x}_i$$

$$\hat{e}_i = cos(\hat{\phi}_i), \ \hat{f}_i = sin(\hat{\phi}_i).$$

The C_i's are called contact functions (C-functions). The condition for a contact is $C_i(\hat{\mathbf{p}}) = 0$, and the CF-cell formed by a 3A contact formation is defined by:

$$\mathcal{CF}' = \{\hat{\mathbf{p}} \mid C_i(\hat{\mathbf{p}}) = 0 \ and \ c^2+s^2-1 = 0, \ i = 1,2,3\}.$$

2.2 A new construction for the CF-cells

We want to develop a method of constructing the CF-cells by studying the possible rigid body motions of the whole workpiece-manipulator system. First let us consider three edges E_i with supporting lines given by $L_i(x,y) = a_i x + b_i y + c_i = 0$ relative to the frame W, for i=1,2,3 (Figure 3). Let $P_i = (x_i, y_i)$ be the point on L_i that the vertex of the i^{th} manipulator contacts relative to the frame W. We define

$$X = \{(P_1, P_2, P_3) \in \mathbf{R}^6 \mid L_i(P_i) = 0, i = 1,2,3\}.$$

X contains the contact information of the CF-cell.

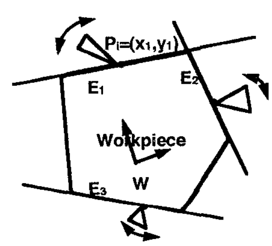


Figure 3. Rigid body motion of the whole workpiece-manipulator system.

Two pieces of information remain to be determined. First is the position and orientation of the workpiece. This information can be represented by the *rigid motion group* in two dimensions which is diffeomorphic to $\mathbf{R}^2 \times S^1$, where S^1 is the unit circle. Second is that relative to a fixed workpiece and a choice of $(P_1, P_2, P_3) \in X$, each manipulator has one degree of freedom, namely rotation around the contact point relative to the frame W. This information can be represented by S^1 for each manipulator. Hence, we define a 3A CF-cell as

$$\mathcal{CF} = X \times \mathbf{R}^2 \times (S^1)^4$$

Theorem 1. $\mathcal{CF}$ is a smooth manifold.

Proof. R^2 and S^1 are smooth manifolds. It remains to show that X is a smooth manifold, since the product of smooth manifolds is a smooth manifold. Let

$$\mathbf{p} = [x_1, y_1, x_2, y_2, x_3, y_3]^T$$
$$f_i(\mathbf{p}) = a_i x_i + b_i y_i + c_i, \ i = 1, 2, 3.$$

Then X is defined by $\mathbf{f} = \mathbf{0}$, where

$$\mathbf{f}(\mathbf{p}) = [f_1(\mathbf{p}), f_2(\mathbf{p}), f_3(\mathbf{p})]^T.$$

To show that X is a manifold, it suffices to show that the Jacobian matrix $\left(\frac{\partial \mathbf{f}}{\partial \mathbf{p}}\right)$ has a rank 3. It is easy to see that

$$\left(\frac{\partial \mathbf{f}}{\partial \mathbf{p}}\right) = \begin{pmatrix} a_1 & b_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & b_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_3 & b_3 \end{pmatrix}$$

Since a_i and b_i are not both zero for each i, there is a 3×3 non-singular minor of the form

$$\begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & \alpha_3 \end{pmatrix}$$

where $\alpha_i \in \{a_i, b_i\}$ $i = 1, 2, 3$. ∎

Note that $dim(X)$ = 6-3 = 3. And therefore $dim(\mathcal{CF})$ = 3+2+4 = 9. So a 3A CF-cell is a 9-dimensional smooth manifold.

Next we show that the two constructions of CF-cells are essentially the same, by establishing a diffeomorphism between $\mathcal{CF}$ and $\mathcal{CF}'$.

Theorem 2. $\mathcal{CF}$ is diffeomorphic to $\mathcal{CF}'$.

Proof. Consider that the workpiece is put in a position where the frame W coincides with the world frame O. And suppose that each frame attached to a manipulator is placed at the contact point with the x-axis lined up with one of the edges of the manipulator adjacent to the contact vertex (Figure 4). Note that $(\hat{u}_i, \hat{v}_i) = 0$, i=1,2,3. Let $\mathcal{CF}'$ and $\mathcal{CF}$ be defined by the old and new methods respectively for a 3A contact formation.

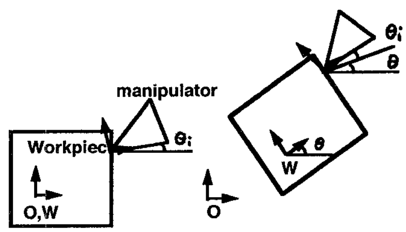


Figure 4. The workpiece in a position where W coincides with O.

Note also that since the frame W coincides with the world frame O, the position and orientation of the workpiece is (0,0) and 0, respectively. So for fixed contact points $P_i = (x_i, y_i)$ and manipulator orientations θ_i (i=1,2,3), we have a point

$$\mathbf{q_o} = (P_1, P_2, P_3, 0, 0, 0, \theta_1, \theta_2, \theta_3) \in \mathcal{CF}.$$

It corresponds to a point

$$\hat{\mathbf{q}}_\mathbf{o} = [0, 0, 1, 0, x_1, x_2, x_3, y_1, y_2, y_3, \theta_1, \theta_2, \theta_3]^T \in \mathcal{CF}'.$$

We also have the relations $C_i(\hat{\mathbf{q}}_\mathbf{o}) = 0$, for i=1,2,3. Or explicitly,

$$-cos(\hat{\phi}_i)\hat{w}_i - sin(\hat{\phi}_i)\hat{z}_i + sin(\hat{\phi}_i)y_i + cos(\hat{\phi}_i)x_i = 0. \quad (1)$$

Now, translate the whole workpiece-manipulator system by (u, v) and rotate it about W by an amount θ. The new configuration gives us a point

$$\mathbf{q} = (P_1, P_2, P_3, u, v, \theta, \theta_1, \theta_2, \theta_3) \in \mathcal{CF}.$$

We define a map from $\mathcal{CF}$ to $\mathcal{CF}'$ by sending $\mathbf{q}$ to $\hat{\mathbf{q}}$, where

$$\hat{\mathbf{q}} = [u, v, cos(\theta), sin(\theta), T_x(P_1), T_x(P_2), T_x(P_3), T_y(P_1), T_y(P_2), T_y(P_3), \theta_1 + \theta, \theta_2 + \theta, \theta_3 + \theta]^T$$

and

$$T_x(P_i) = x_i cos(\theta) - y_i sin(\theta) + u$$
$$T_y(P_i) = x_i sin(\theta) + y_i cos(\theta) + v, \ i = 1, 2, 3.$$

It is easy to see that this map is one-to-one, onto and C^∞-differentiable. So if the map is actually well-defined, then we get the required diffeomorphism. That is, we need to show that

$$C_i'(\hat{\mathbf{q}}) = \hat{a}_i' + \hat{b}_i' cos(\theta) + \hat{d}_i' sin(\theta) - \hat{e}_i' ucos(\theta) + \hat{f}_i' usin(\theta) - \hat{f}_i' vcos(\theta) - \hat{e}_i' vsin(\theta) = 0$$

where

$$\begin{aligned}
\hat{a}_i' &= -cos(\hat{\phi}_i)\hat{w}_i - sin(\hat{\phi}_i)\hat{z}_i \\
\hat{b}_i' &= sin(\hat{\phi}_i)T_y(P_i) + cos(\hat{\phi}_i)T_x(P_i) \\
\hat{d}_i' &= -cos(\hat{\phi}_i)T_y(P_i) - sin(\hat{\phi}_i)T_x(P_i) \\
\hat{e}_i' &= cos(\hat{\phi}_i) \quad \hat{f}_i' = sin(\hat{\phi}_i).
\end{aligned}$$

This is easy to check by using (1). ∎

2.3 The intersection of CF-cells

The intersection of two 3A CF-cells represents the intermediate state of moving one or more manipulators from one edge to another edge of the workpiece (Figure 5). There are different types of intersections that arise from different contacts and different numbers of manipulators moving across edges. Our new construction allows us to show that all of these intersections are in fact smooth manifolds of the expected dimensions. Again, we do this here only for the simple case of one manipulator moving across an edge for a 3A-contact. In other words, we consider the intersection of two 3A CF-cells.

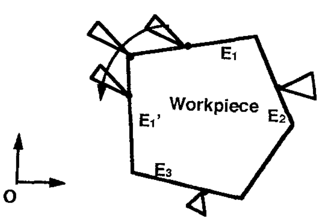


Figure 5. The intersection of two 3A CF-cells.

Theorem 3. The intersection of two 3A CF-cells is a smooth manifold.

Proof. Refer to figure 5. Suppose we move the first manipulator from edge E_1 of the workpiece to edge E_1'. Let the supporting line for E_1' be given by $L_1'(x, y) = a_1'x + b_1'y + c_1'$. Let

$$X_{int} = \{(P_1, P_2, P_3) \in R^6 \mid L_i(P_i) = 0, i = 1, 2, 3, \; and \; L_1'(P_1) = 0\}$$

and

$$\mathrm{p} = [x_1, y_1, x_2, y_2, x_3, y_3]^T.$$

Here P_1 is forced to be the common vertex. Then X_{int} is defined by $\mathbf{f}' = \mathbf{0}$ where

$$\begin{aligned}
\mathbf{f}'(\mathbf{p}) &= [f_1(\mathbf{p}), f_1'(\mathbf{p}), f_2(\mathbf{p}), f_3(\mathbf{p})]^T \\
f_i(\mathbf{p}) &= a_i x_i + b_i y_i + c_i, \; i = 1, 2, 3 \\
f_1'(\mathbf{p}) &= a_1' x_1 + b_1' y_1 + c_1'
\end{aligned}$$

To show that X_{int} is a manifold, it suffices to show that $\left(\frac{\partial \mathbf{f}'}{\partial \mathbf{p}}\right)$ has a rank 4. It is easy to see that

$$\left(\frac{\partial \mathbf{f}'}{\partial \mathbf{p}}\right) = \begin{pmatrix} a_1 & b_1 & 0 & 0 & 0 & 0 \\ a_1' & b_1' & 0 & 0 & 0 & 0 \\ 0 & 0 & a_2 & b_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_3 & b_3 \end{pmatrix}$$

a_i and b_i are not both zero for each i. Also a_1' and b_1' are not both zero. Since edge E_1 and E_1' are not parallel, the upper left 2×2 minor (whose rows are the normals of these lines) is not zero. Therefore, there exists a 4×4 non-singular minor of the form

$$\begin{pmatrix} a_1 & b_1 & 0 & 0 \\ a_1' & b_1' & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & \alpha_3 \end{pmatrix}$$

where $\alpha_i \in \{a_i, b_i\}$ $i = 2, 3$. ∎

Note that in general, when n ($n = 1, 2, 3$) manipulators are moved to the edge intersection positions, the dimension of the intersection of the CF-cells is given by $6 - (n + 3) + 2 + 4 = 9 - n$.

Section 3: Contact Formation Cells for Three-Dimensional Systems

Now we are ready to extend the above construction of CF-cells to a three-dimensional polyhedral workpiece-manipulator system. Note that for such a system, the workpiece has six degrees of freedom, so we will need six manipulator contacts to fully determine its position and orientation. There are three types of elementary contacts possible. A *type A* contact is formed when a face of the workpiece contacts with a vertex of a manipulator. A *type B* contact is formed when a vertex of the workpiece contacts a face of a manipulator. Finally, a *type C* contact is formed when an edge of the workpiece contacts an edge of a manipulator. Here, we will demonstrate the construction of a 6A CF-cell and show that it is a smooth manifold. All the others types of CF-cells can be treated similarly.

Let us consider six faces F_i with the supporting planes given by $Pl_i(x,y,z) = a_i x + b_i y + c_i z + d_i = 0$, for i=1,2,...,6. Let $P_i = (x_i, y_i, z_i)$ be the point on Pl_i at which the specified vertex of the i^{th} manipulator is in contact. We define

$$Y = \{(P_1, P_2, ..., P_6) \in \mathbf{R}^{18} \mid Pl_i(P_i) = 0, \\ i = 1, 2, ...6.\}$$

Again, Y contains the contact information.

The six degrees of freedom of the workpiece are represented by the rigid motion group which is diffeomorphic to $R^3 \times SO(3)$, where $SO(3)$ is the rotational group in three dimensions. Each manipulator has three degrees of freedom, namely it can rotate around the contact point. It is easy to see that these three degrees of freedom can be represented by $SO(3)$ for each manipulator. Hence, we can define a 6A CF-cell as

$$\mathcal{CF}_{6A} = Y \times R^3 \times SO(3) \times (SO(3))^6.$$

Theorem 4. $\mathcal{CF}_{6A}$ is a smooth manifold.

Proof. We need only show that Y is a smooth manifold. Let

$$\mathbf{p} = [x_1, y_1, x_2, \ldots, x_6, y_6]^T$$

Then Y is defined by $\mathbf{g} = \mathbf{0}$ where

$$\mathbf{g(p)} = [g_1(\mathbf{p}), g_2(\mathbf{p}), \ldots, g_6(\mathbf{p})]^T$$

and

$$g_i(\mathbf{p}) = a_i x_i + b_i y_i + c_i z_i + d_i, \; i = 1, 2, ..., 6.$$

We see that

$$\left(\frac{\partial \mathbf{g}}{\partial \mathbf{p}}\right) = \begin{pmatrix} a_1 & b_1 & c_1 & \ldots & 0 & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a_6 & b_6 & c_6 \end{pmatrix}$$

We know that not all of a_i, b_i and c_i are zero for each i. So there is a 6×6 non-singular minor

$$\begin{pmatrix} \alpha_1 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \ldots & 0 & \alpha_6 \end{pmatrix}$$

where $\alpha_i \in \{a_i, b_i, c_i\}$ i=1,2,...,6. Hence Y is a smooth manifold. ∎

Note that in general, $dim(Y) = 18 - 6 = 12$. And therefore $dim(\mathcal{CF}_{6A}) = 12 + 3 + 3 + 6(3) = 36$. So a 6A CF-cell is in general a 36-dimensional manifold. Also, all the intersections of CF-cells in a three-dimensional system are smooth manifolds. Proofs are similar to Theorem 3.

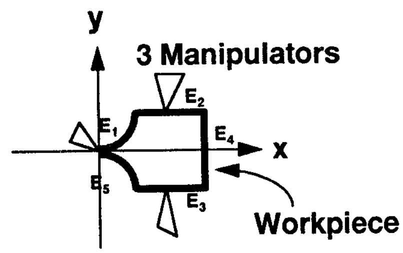


Figure 6. An example of non-smooth contact formation cell.

The fact that CF-cells are smooth manifolds depends on the geometry of the workpiece and the manipulators. The CF-cells that we consider in this paper are formed by polygonal (polyhedral) workpieces and manipulators. Workpiece-manipulator systems with different geometries may lead to non-smooth contact formation cells. For example, consider a workpiece with edges supported by $l_1(x,y) = y^2 - x^3 = 0$ with $y \geq 0$, $l_2(x,y) = y - 1 = 0$, $l_3(x,y) = y + 1 = 0$, $l_4(x,y) = x - 3 = 0$, and $l_5(x,y) = y^2 - x^3 = 0$ (Figure 6) with $y \leq 0$. Three polygonal manipulators contact (type A contacts) the first three edges at $(0,0)$, $(2,1)$ and $(2,-1)$ respectively. Let

$$\mathbf{X'} = \{(P_1, P_2, P_3) \in R^6 \mid l_i(P_i) = 0, \, i = 1, 2, 3\}$$

and

$$\mathbf{p} = [x_1, y_1, x_2, y_2, x_3, y_3]^T.$$

Then $\mathbf{X'}$ is defined by $\mathbf{h} = \mathbf{0}$ where

$$\mathbf{h(p)} = [h_1(\mathbf{p}), h_2(\mathbf{p}), h_3(\mathbf{p})]^T \text{ and}$$
$$h_1(\mathbf{p}) = y_1^3 - x_1^2, h_2(\mathbf{p}) = y_2 - 1, h_3(\mathbf{p}) = y_3 + 1.$$

We see that

$$\left(\frac{\partial \mathbf{h}}{\partial \mathbf{p}}\right) = \begin{pmatrix} -3x_1^2 & 2y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

And $\left(\frac{\partial \mathbf{h}}{\partial \mathbf{p}}\right)$ has rank $2 < 3$ at $\mathbf{p} = [0, 0, 2, 1, 2, -1]^T$. So the above configuration does not lead to a smooth contact formation cell. What is important here is that $\left(\frac{\partial \mathbf{h}}{\partial \mathbf{p}}\right)$ is ill-conditioned when the vertex of the manipulator slides to $(0,0)$.

Section 4: A Bound For The Number of Workpiece Configuration Under A Fixed Manipulator Configurations While Maintaining Contacts

Refer to the two-dimensional system in section 2.1. Given a fixed manipulator configuration $\hat{\mathbf{r}}$, let

$$G = Det \begin{pmatrix} \hat{b}_1 & cos(\hat{\phi}_1) & sin(\hat{\phi}_1) \\ \hat{b}_2 & cos(\hat{\phi}_2) & sin(\hat{\phi}_2) \\ \hat{b}_3 & cos(\hat{\phi}_1) & sin(\hat{\phi}_3) \end{pmatrix}$$

$$H = Det \begin{pmatrix} \hat{d}_1 & cos(\hat{\phi}_1) & sin(\hat{\phi}_1) \\ \hat{d}_2 & cos(\hat{\phi}_2) & sin(\hat{\phi}_2) \\ \hat{d}_3 & cos(\hat{\phi}_1) & sin(\hat{\phi}_3) \end{pmatrix}$$

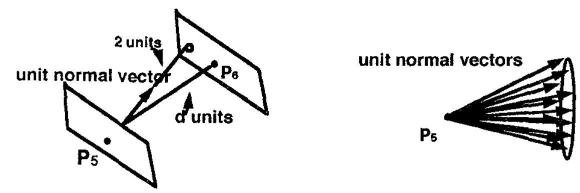
$$I = Det \begin{pmatrix} \hat{a}_1 & cos(\hat{\phi}_1) & sin(\hat{\phi}_1) \\ \hat{a}_2 & cos(\hat{\phi}_2) & sin(\hat{\phi}_2) \\ \hat{a}_3 & cos(\hat{\phi}_1) & sin(\hat{\phi}_3) \end{pmatrix}.$$

The vector $\hat{\mathbf{r}}$ is said to be *generic* if $G^2 + H^2 > 0$ and $G^2 + H^2 \neq I^2$. In this case, either there are two distinct workpiece configurations that maintain the specified contacts with the manipulators, or there are none.

The corresponding general result for a three-dimensional system is not known. We study this property for a simple case where the workpiece is a cube with a 6A contact formation. Let the number of possible workpiece configurations that maintain contact with a fixed manipulator configuration be N. In section 4.1, we show that N is bounded by 16. An example of N=8 is displayed in section 4.2, and we believe that the maximum is 8.

4.1 A bound for N

We start with a cube of two units length. It is placed such that the faces are supported by the following planes : face 1 : $y = 1$, face 2 : $z = 1$, face 3 : $y = -1$, face 4 : $z = -1$, face 5 : $x = 1$ and face 6 : $x = -1$. We consider a fixed 6A manipulator configuration where the i^{th} manipulator contacts face i, for i=1,2,...,6. The question we need to answer is : what is the maximum number of geometrically admissible workpiece configurations that maintain the contacts assuming this fixed position of the manipulators. In other words, we need to find the number of ways that one can translate and orientate the cube to achieve the contact formation with the fixed manipulator configuration.



Figures 7 and 8. Parallel planes that contact with P_5 and P_6 and the unit vectors from P_5 forming a cone.

Suppose $P_i = (x_i, y_i, z_i)$ is the position of the contact vertex of the i^{th} manipulator. Then the contact points on opposite faces, say P_5 and P_6, have a distance greater than or equal to 2. Using a rigid body motion, we can move P_5 to the origin and P_6 to $(-d, 0, 0)$ where d is the distance between P_5 and P_6. After doing this, we still have one degree of freedom, namely rotation about the x-axis. This set of choices is parameterized by a circle. Each choice amounts to finding a pair of parallel planes 2 units apart such that P_5 is on one of them and P_6 is on the other one (Figure 7). One can visualize all this by considering the collection of the unit normal vectors for each choice of planes as pointing out from P_5 to form a cone with circular base (Figure 8). Each angle $\theta \in [0, 2\pi)$ determine a point on a circle which in turn determines one unit normal vector and thus a choice of parallel planes.

Let Pl_θ be the plane passing through P_5 that has unit normal vector $\mathbf{n}_\theta$ pointed out from P_5 for the parameter θ. Let us project P_i, i=1,2,3,4, onto Pl_θ. Denote the resulting points as P_i' (Figure 9).

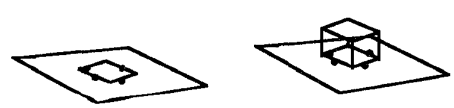


Figure 9. The projection of P_i, i=1,..,4 to Pl_θ.

If we can put a square of two units length in contact with P_i', i=1,..,4, then we will have a cube configuration that contacts the fixed manipulators. Thus we have to find out which θ will give us valid cube configurations. In order to make calculations easy, we further rotate (also a rigid body motion) $\mathbf{n}_\theta$ so that Pl_θ coincides with the yz-plane. Denote by $P_i'' = (u_i'', v_i'')$ the resulting points on the yz- plane that correspond to the P_i', i=1,..,4 (Figure 8). Then P_i'' is given by

$$\begin{cases} u_i'' = \frac{\sqrt{d^2-4}}{d}x_i + \frac{2y_i}{d}cos(\theta) + \frac{2z_i}{d}sin(\theta) \\ v_i'' = -y_i sin(\theta) + z_i cos(\theta). \end{cases}$$

Referring to the notation from section 2.1, let $(\hat{u}_i, \hat{v}_i) = (0,0)$, $(\hat{x}_i, \hat{y}_i) = (u_i'', v_i'')$, for i=1,..,4. Also let $(\hat{w}_1, \hat{z}_1) = (1,0)$, $(\hat{w}_2, \hat{z}_2) = (0,1)$, $(\hat{w}_3, \hat{z}_3) = (-1,0)$, $(\hat{w}_4, \hat{z}_4) = (0,-1)$, $\hat{\phi}_1 = 0$, $\hat{\phi}_2 = \frac{\pi}{2}$, $\hat{\phi}_3 = \pi$ and $\hat{\phi}_4 = \frac{3\pi}{2}$. Consider G, H and I (as defined above) for the points P_1'', P_2'' and P_3''. Denote by G', H' and I' the corresponding parameters for the points P_1'', P_2'' and P_4''. By a result from [1], P_i'', i=1,..4 are on the corresponding edges of the square iff

$$Gcos(\hat{\theta}) + Hsin(\hat{\theta}) + I = 0$$

and

$$G'cos(\hat{\theta}) + H'sin(\hat{\theta}) + I' = 0.$$

Or, explicitly

$$(u_1'' - u_3'')cos(\hat{\theta}) + (v_1'' - v_3'')sin(\hat{\theta}) - 2 = 0$$

and

$$(v_2'' - v_4'')cos(\hat{\theta}) + (u_4'' - u_2'')sin(\hat{\theta}) - 2 = 0.$$

If we put $t = tan(\frac{\theta}{2})$ and $\hat{t} = tan(\frac{\hat{\theta}}{2})$, the above equations can be rewritten as

$$(2) \quad A + Bt + Ct^2 = 0 \quad P + Qt + Rt^2 = 0$$

where

$$\begin{aligned} A &= \sqrt{d^2-4}(x_1 - x_3) + 2(y_1 - y_3) - 2d + 4(z_1 - z_3)\hat{t} + (\sqrt{d^2-4}(x_1 - x_3) - 2(y_1 - y_3) - 2d)\hat{t}^2, \\ B &= 2(d(z_1 - z_3) + 2d(y_3 - y_1)\hat{t} + d(z_3 - z_1)\hat{t}^2), \\ C &= -\sqrt{d^2-4}(x_1 - x_3) - 2(y_1 - y_3) - 2d - 4(z_1 - z_3)\hat{t} - (\sqrt{d^2-4}(x_1 - x_3) - 2(y_1 - y_3) + 2d)\hat{t}^2, \\ P &= d(z_2 - z_4) - 2d + 2d(y_4 - y_2)\hat{t} + (d(z_4 - z_3 - 2))\hat{t}^2, \\ Q &= 2((\sqrt{d^2-4}(x_4 - x_2) + 2(y_4 - y_2)) + 4(z_4 - z_2)\hat{t} + (\sqrt{d^2-4}(x_4 - x_2) + 2(y_4 - y_2))\hat{t}^2), \\ R &= -d(z_2 - z_4) - 2d + 2d(y_2 - y_4)\hat{t} - d(z_4 - z_2 + 2)\hat{t}^2. \end{aligned}$$

After eliminating t from (2), the equation in $\hat{t}$ is

$$det\begin{pmatrix} C & B & A & 0 \\ 0 & C & B & A \\ R & Q & P & 0 \\ 0 & R & Q & P \end{pmatrix} = 0.$$

This is known as the *resultant* of (2). It is a polynomial in $\hat{t}$ of degree 8. Thus in general, the maximum number of solutions $(t, \hat{t})$ of (2) is 16. And each pair $(t, \hat{t})$ determines a pair $(\theta, \hat{\theta})$. Note that only the real solutions contribute to workpiece configurations, so N is bounded by 16.

Working with examples leads us to believe that N should be bounded by 8. However, an analytical proof is so far not available. We give an example of N=8 in the next subsection.

4.2 Example of a 6A manipulator configuration with N=8

Let $P_1 = (0,1,0)$, $P_2 = (0,0,1)$, $P_3 = (-\frac{1}{3}, -1, -\frac{1}{3})$, $P_4 = (\frac{1}{3}, \frac{1}{3}, -1)$, $P_5 = (1,0,0)$ and $P_6 = (-1, \frac{1}{3}, -\frac{1}{3})$. The following is one of seven matrices (the other one is the identity matrix) representing the rigid body motions that take the cube from its original configuration to a new configuration that achieves the specified contact formation.

$$\begin{pmatrix} .946 & .3243 & -.43 \times 10^{-7} & -.06867 \\ -.3243 & .946 & -.21 \times 10^{-7} & -.03360 \\ .34 \times 10^{-7} & .34 \times 10^{-7} & 1.00 & -.4 \times 10^{-8} \\ 0 & 0 & 0 & 1.00 \end{pmatrix}$$

One can verify the results by actually acting on the cube by the rigid body motion given by each matrice and see that the points P_i, i=1,...,6 are contacting the correct faces. Equivalently, we can move the points P_i by the inverse of these matrices and see that they are contacting the corresponding faces of the cube in the original position. For example, after applying the inverse of the above matrix, call it M_1^{-1}, we have the following points:

$$\begin{aligned} M_1^{-1}P_1 &= (-.270270, 1.000000, -.000000) \\ M_1^{-1}P_2 &= (.054055, .054054, 1.000000) \\ M_1^{-1}P_3 &= (.0630631, -1.000000, -.333333) \\ M_1^{-1}P_4 &= (.261261, .477477. - 1.000000) \\ M_1^{-1}P_5 &= (1.000000, .378378, -.000000) \\ M_1^{-1}P_6 &= (-1.000000, .0450450, -.333333). \end{aligned}$$

One can easily see that they are on the correct faces, modulo round-off error.

Section 5: An Example of Infinite Number of Workpiece Configurations that Maintain Contact with a Fixed Manipulator Configuration

In a two dimensional-system, [1] gives an example of a non-generic manipulator configuration $\hat{\mathbf{r}}$ such that there is an infinite number of workpiece configurations that maintain the specified contacts with the manipulators (Figure 10). The characteristic of this example is that the lines supporting the contacting edges intersect at a point and the contact normals also intersect at a (possibly different) point.

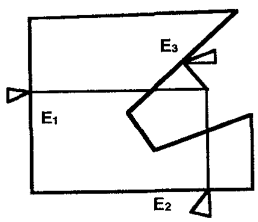


Figure 10. An example of infinite number of workpiece configurations that maintain contact with a fixed manipulator.

Refer to the notation in section 2.1. Each orientation of the workpiece corresponds to a point on the circle $cos^2(\hat{\theta})+sin^2(\hat{\theta})-1=0$, and for each $\hat{\theta}$, the workpiece position is calculated by (relabel the indices if necessary):

$$\hat{x}=\frac{\hat{F}_1\hat{D}_2-\hat{F}_2\hat{D}_1}{\hat{E}_1\hat{F}_2-\hat{F}_1\hat{E}_2}, \quad \hat{y}=\frac{\hat{E}_2\hat{D}_1-\hat{E}_1\hat{D}_2}{\hat{E}_1\hat{F}_2-\hat{F}_1\hat{E}_2}.$$

where $\hat{D}_i=\hat{a}_i+\hat{b}_ic+\hat{d}_is$, $\hat{E}_i=-\hat{e}_ic+\hat{f}_is$ and $\hat{F}_i=-\hat{f}_ic-\hat{e}_is$ for i=1,2. Note that the above expressions do not involve information about the third manipulator. Therefore, suppose there is a fourth manipulator contacting an edge E_4 such that the lines supporting all four contact edges intersect at a point and all four contact normals intersect at a point (Figure 11). Again, the configuration of the first, second and fourth manipulator allows an infinite number of workpiece configurations which maintain the specified contact formation. Also, given any orientation $\hat{\theta}$, the workpiece position is calculated by essentially the same formula as above. Hence, for this fixed configuration of four manipulators, there are an infinite number of workpiece configurations that maintain the specified 4A contact formation.

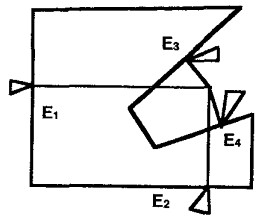


Figure 11. A fourth manipulator is added to the configuration.

Consider the workpiece in figure 12. We add two manipulators, one to contact each of the top and the bottom face, and we add four manipulators that contact the side faces in such a way that their projections to the bottom plane coincide with the configuration in figure 11. Now it is easy to see that this workpiece has one degree of freedom and so there are an infinite number of configurations that maintain the specified contact formation.

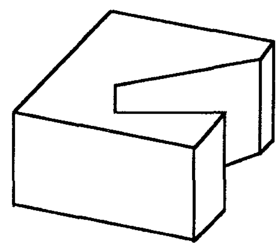


Figure 12. An example of a 6A configuration that has an infinite number of workpiece configurations maintaining contact with the fixed manipulators.

References

[1] A.O. Farahat, P.F. Stiller, and J.C. Trinkle, "On the Geometry of Contact Formation Cells for Systems of Polygons," *IEEE Transactions on Robots and Automation*, to appear.

[2] J.-C. Latombe, *Robot Motion Planning.* Kluwer Academic Publishers, 1991.

[3] B. O'Neil, *Semi-Riemannian Geometry.* Academic Press, 1983.