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Motion Planning for a Class of Planar Closed-Chain Manipulators

Guanfeng Liu, J.C. Trinkle†, and N. Shvalb‡
Dept of CS, Stanford University, liugf@cs.stanford.edu
† Dept of CS, Rensselaer Polytechnic Institute, trink@cs.rpi.edu
‡ Dept of ME, Technion-Israel Institute of Technology, Israel, shvalbn@techunix.technion.ac.il

***Abstract*— We study the motion problem for planar *star-shaped* manipulators. These manipulators are formed by joining k "legs" to a common point (like the thorax of an insect) and then fixing the "feet" to the ground. The result is a planar parallel manipulator with $k-1$ independent closed loops. A topological analysis is used to understand the global structure of the configuration space so that planning problem can be solved exactly. The worst-case complexity of our algorithm is $O(k^3N^3)$, where N is the maximum number of links in a leg. A simple example illustrating our method is given.**

I. Introduction

Due to the computational complexity and difficulty of implementing general exact motion planning algorithms, such as Canny's [1], today sample-base algorithms, such as Kavraki's [5] dominate motion planning research. However, there are important classes of problems for which these algorithms do not perform well. These arise in systems whose configuration space (C-space) cannot effectively be represented as a set of parameters with simple bounds, but rather is most naturally represented as a variety of co-dimension one or greater embedded in a higher-dimensional ambient space [8]. Examples of the systems include manipulators with one or multiple closed loops, whose configuration space is defined by loop closure constraints. The RLG method [2], [3] improves the sampling techniques through estimating the regions of sampling parameters. However, its efficiency relies on the accuracy of the estimation, which often varies case by case. Moreover, it ignores the global structure of C-space, and may fail to sample globally important regions.

Recent advances in the understanding of the global structure of C-spaces of single-loop closed chains [6], [7] allows us to develop an effective exact algorithm for a class of manipulators with multiple loops, namely, the planar *star-shaped* manipulators. These manipulators are formed by joining k planar "legs" to a common point (like the thorax of an insect) and then fixing the "feet" to the ground. The result is a planar parallel manipulator with $k-1$ independent closed loops. Each independent loop imposes an algebraic kinematic constraint equation on the system, and so the C-space of star-shaped manipulators is an algebraic variety embedded in the joint space. Walking robots whose legs are SCARA robots with axes perpendicular to the ground can be modeled as a planar star-shaped manipulators.

Here, we extend the previous topological methods in [6], [7] for C-space connectivity analysis to the case of planar star-shaped manipulators without obstacle. We derive important global properties of C-space and use these results to derive a necessary and sufficient condition for path existence problem. A complete algorithm based on the properties is implemented and examples are presented.

II. Notation

Manipulator Notation	
M	- Manipulator
A	- Root junction or thorax of M
o_i	- Grounding point of foot i of M
n_j	- Number of links in M_j
$l_{j,i}$	- Length of link i of M_j; $i = 1, ..., n_j$
$\theta_{j,i}$	- Angle of link i relative to link $i-1$
M_j	- Leg j of M with foot fixed at o_j and other end free, $j = 1, ..., k$
$\tilde{M}_j(p)$	- Leg j of M with foot fixed at o_j and other end fixed at p
$\tilde{M}(p)$	- Manipulator with A fixed at p
L_j	- Sum of lengths of links of M_j
$\mathcal{L}_j(p)$	- A set of long links of $\tilde{M}_j(p)$
$\lvert\mathcal{L}_j^*(p)\rvert$	- Number of long links of $\tilde{M}_j(p)$
Workspace Notation	
W_A	- Workspace of A
${}^dU_i$	- Cell of dimension d of W_A
p	- Point in the plane of M
$\gamma = p(t)$	- Curve in the plane of M
f	- Kinematic map of A
f_j	- Kinematic map of endpoint of M_j
Σ_A	- Singular set of f in W_A
Σ_j	- Singular set of f_j
Configuration Space (C-space) Notation	
$\mathcal{C}$	- C-space of M
$\tilde{\mathcal{C}}(p)$	- C-space of $\tilde{M}(p)$
$\mathcal{C}_j$	- C-space of M_j
$\tilde{\mathcal{C}}_j(p)$	- C-space of $\tilde{M}_j(p)$
c	- Point in C-space

III. Preliminaries

The class of planar manipulators studied here are referred to as planar *star-shaped* manipulators (see Fig. 1). A star-shaped manipulator is composed of k serial chains with all revolute joints. Leg M_j is composed of n_j links of lengths $l_{j,i}, i =$

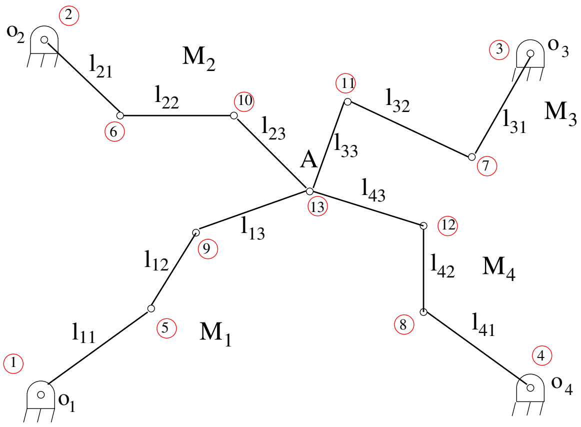

Fig. 1. Star-shaped manipulator with $k = 4$.

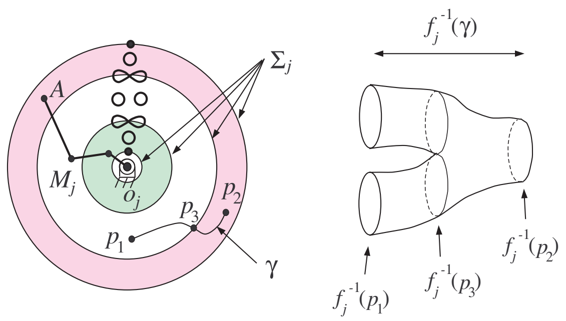

Fig. 2. **Left:** The workspace W_j of a three-link open chain M_j based at o_j. The singular set Σ_j of the kinematic map f_j is four concentric circles. **Right:** The inverse image of the curve γ - a "pair of pants."

$1, ..., n_j$ with angles $\theta_{j,i}, i = 1, ..., n_j$. At one end (the foot), M_j is connected to ground by a revolute joint fixed at the point o_j. At the other end, it is connected by another revolute joint to a junction point denoted by A. Note that when k is one, a star-shaped manipulator is an open serial chain. When k is two, it is a single-loop closed chain.

Assuming that the foot of M_j is fixed at o_j, let $f_j(\Theta_j) = p$ denote the kinematic map of M_j, where $\Theta_j = (\theta_{j,1}, \cdots, \theta_{j,n_j})$ is the tuple of joint angles, and p is the location of the endpoint of the leg (the thorax end). When M_j is detached from the junction A, the image of its joint space is the reachable set of positions of the free end of the leg, called the workspace W_j. In the absence of joint limits, the workspace W_j is an annulus if and only if there exists one link with length strictly greater than the sum of all the other link lengths. Otherwise it is a disk. Clearly, the workspace W_A of A when all the legs are connected to A is given by:

$$W_A = \bigcap_{j=1}^{k} W_j. \tag{1}$$

In our study of $\mathcal{C}$, it will be convenient to refer to several other C-spaces. The C-space of leg M_j when detached from the rest of the manipulator will be denoted by $\mathcal{C}_j$. When the endpoint is fixed at the point p, leg j will be denoted by $\tilde{M}_j(p)$. Note that $\tilde{M}_j(p)$ is a single-loop planar closed chain, about which much is known (see [6]), including global structural properties of its C-space, denoted by $\tilde{\mathcal{C}}_j(p) = f_j^{-1}(p)$.

When the junction A is fixed at point p, its C-space will be denoted by $\tilde{\mathcal{C}}(p)$. Since collisions are ignored, the motions of the legs are independent, and therefore the C-space of the manipulator (with fixed junction) is the product of the C-spaces of the legs with all endpoints fixed at p:

$$\left.\begin{aligned} \tilde{\mathcal{C}}(p) &= \tilde{\mathcal{C}}_1(p) \times \cdots \times \tilde{\mathcal{C}}_k(p) \\ &= f_1^{-1}(p) \times \cdots \times f_k^{-1}(p) \\ &= f^{-1}(p) \end{aligned}\right\} \tag{2}$$

where by analogy, f is a total kinematic map of the star-shaped manipulator. Loosely speaking, the union of the C-spaces $\tilde{\mathcal{C}}(p)$ at each point p in W_A gives the C-space of a star-shaped manipulator:

$$\mathcal{C} = \bigcup_{p \in W_A} \tilde{\mathcal{C}}(p). \tag{3}$$

Several properties of the C-spaces $\mathcal{C}_j$ and $\tilde{\mathcal{C}}_j(p)$ are highly relevant and so are reviewed here before analyzing the C-space of star-shaped manipulators. It is well known that the C-space of M_j is a product of circles (*i.e.*, $\mathcal{C}_j = (S^1)^{n_j}$). The workspace W_j contains a singular set Σ_j which is composed of all points p in W_j for which the Jacobian of the kinematic map $Df_j(\Theta_j)$ drops rank for some $\Theta_j \in f_j^{-1}(p)$. These points form concentric circles of radii $|l_{j,1} \pm l_{j,2} \pm \cdots \pm l_{j,n_j}|$, as shown in Fig 2. When A coincides with a point in Σ_j, the links can be arranged such that they are all colinear, in which case the number of instantaneous degrees of freedom of the endpoint of the leg is reduced from two to one.

Now consider the case where the endpoint of leg j is fixed to the point p. In other words, we are interested in the C-space $\tilde{\mathcal{C}}_j(p)$ of $\tilde{M}_j(p)$. In the 12 o'clock position in Fig. 2, points, circles, and figure eights are drawn to represent the global structures of $\tilde{\mathcal{C}}_j(p)$ in the seven regions of W_j. Specifically, when A is fixed to a point p on the outer-most singular circle, $\tilde{\mathcal{C}}_j(p)$ is a single point. For p fixed to any point in the largest open annular region, C-space is a single circle. Continuing inward, the possible C-space types are a figure eight (on the second largest singular circle), two disconnected circles, a figure eight again, a single circle, and a single point (on the inner-most singular circle).

A detailed analysis of $\tilde{\mathcal{C}}_j(p)$ with an arbitrary number of links in $\tilde{M}_j(p)$ can be found in [6]. The results that will be particularly useful in the analysis of star-shaped manipulators follow. First, the connectivity of $\tilde{\mathcal{C}}_j(p)$ is uniquely determined by the number of "long links." Consider the augmented link set composed of the links of M_j and $\overline{o_j p}$, which will be called the fixed base link with length denoted by $l_{j,0}$. Let L_j be the sum of all the link lengths including the fixed base link (*i.e.*, $L_j = \sum_{i=0}^{n_j} l_{j,i}$). Further, let $\mathcal{L}_j(p)$ be a subset of $\{0, 1, ..., n_j\}$ such that $l_{j,\alpha} + l_{j,\beta} > L_j/2$; $\alpha, \beta \in \mathcal{L}_j(p)$, $\alpha \neq \beta$. Over all such sets, let $\mathcal{L}_j^*(p)$ be a set of maximal cardinality. Then the number of long links of $\tilde{M}_j(p)$ is defined as $|\mathcal{L}_j^*(p)|$, where $|\cdot|$ denotes set cardinality.

Lemma *1:* **Kapovich and Milson [4], Trinkle and Milgram [6]**
The C-space $\tilde{\mathcal{C}}_j(p) = f_j^{-1}(p)$ has two components if and only if $|\mathcal{L}_j^*(p)| = 3$, and is connected if and only if $|\mathcal{L}_j^*(p)| = 2$ or 0. No other cardinality is possible.

Return to Fig. 2. Viewing W_j as a base manifold and the C-space corresponding to each end point location as a fibre, one can see that Σ_j partitions W_j into regions over which $\tilde{\mathcal{C}}_j(p)$ forms a trivial fibration. This is useful in determining the C-space of more complicated mechanisms. Consider a modification to $\tilde{M}_j(p)$ that constrains its endpoint to move along a curve γ within W_j. As long as γ is contained in one of the regions defined by the singular circles, $\tilde{\mathcal{C}}_j(\gamma) = \tilde{\mathcal{C}}_j(p) \times I$, where I is the interval. If γ crosses a singular circle transversally, then $\tilde{\mathcal{C}}_j(\gamma) = (\tilde{\mathcal{C}}_j(p_1) \times I) \bigcup \tilde{\mathcal{C}}_j(p_3) \bigcup (\tilde{\mathcal{C}}_j(p_2) \times I)$, where p_1 is a point in one of the two open angular regions containing γ, p_2 is a point in the other, and p_3 is a point on the singular circle crossed by γ, and $\bigcup$ denotes the standard "gluing" operation.

IV. Analysis of Star-Shaped Manipulators

For star-shaped manipulators with one or two legs, the global topological properties of the C-space $\mathcal{C}$ are fully understood (see [6]). The goals of this section are to study the global properties of $\mathcal{C}$ when M has more than two legs and to derive necessary and sufficient conditions for solution existence to the motion planning problem.

1) Local Analysis: As a direct generalization of the singular set of a single leg, we define the singular set of a star-shaped manipulator as a subset Σ of W_A such that for every $p \in \Sigma$, there exists a configuration c such that at least one of the Jacobians $\{Df_1(c), \cdots, Df_k(c)\}$ drops rank. By definition we have:

$$\Sigma = \left(\bigcup_{i=1}^{k} \Sigma_i \right) \bigcap W_A. \tag{4}$$

An advantage of this definition is that Σ can be used to stratify W_A such that each stratum is trivially fibred. Figure 3 shows a star-shaped manipulator with two legs. The singular set Σ is the boundary of the lune formed by the intersection of the outer singular circles of their individual workspaces. For every point interior to the lune, the fibre is two circles (the direct product of two points with one circle). The fibres associated to the vertices of the lune are single points, which correspond to simultaneous full extension of the two legs.

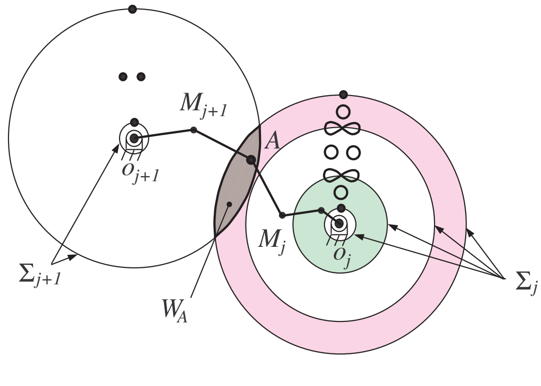

Fig. 3. The workspace W_A of A for a star-shaped manipulator with $k = 2$ is the intersection of the workspaces of A for each leg considered separately. The singular set Σ is composed of the black circular arcs where they bound or intersect the gray area.

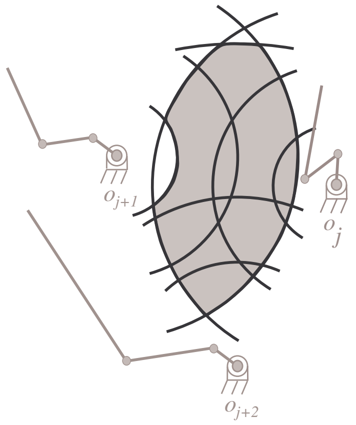

Fig. 4. Workspace (shaded gray) of a star-shaped manipulator with three legs. The singular set partitions W_A into 12 two-dimensional, 32 one-dimensional, and 21 zero-dimensional chambers.

Fig. 4 shows a possible workspace for a star-shaped manipulator with three legs. The singular set defines 65 distinct sets ${}^dU_i$ of varying dimension d, where i is an arbitrarily assigned index that simply counts components. We will refer to these sets as *chambers*. There are 12 two-dimensional, 32 one-dimensional, and 21 zero-dimensional chambers, each of which is trivially fibred. More generally, the intersections among the arcs composing Σ are zero-dimensional chambers, denoted ${}^0U_i,\ i = 1, \cdots, {}^0m$. Removing the ${}^0U_i$ from Σ partitions it into open one-dimensional chambers ${}^1U_i,\ i = 1, \cdots, {}^1m$. Removing ${}^0U_i$ and ${}^1U_i$ from W_A yields open two-dimensional sets ${}^2U_i,\ i = 1, \cdots, {}^2m$, for which the following relationships hold:

$$\Sigma = \left(\bigcup_{i=1}^{{}^0m} {}^0U_i \right) \bigcup \left(\bigcup_{i=1}^{{}^1m} {}^1U_i \right) \tag{5}$$

$$W_A - \Sigma = \bigcup_{i=1}^{{}^2m} {}^2U_i. \tag{6}$$

Proposition. *1:* For all $d = 0, 1, 2$ and i, $f^{-1}({}^dU_i) = {}^dU_i \times f^{-1}(p)$, where p is any point in ${}^dU_i$ and the operator $\times$ denotes the direct product. Gluing the $f^{-1}({}^dU_i)$ for all i and d gives the total C-space $\mathcal{C}$.

Proposition 1 and the fact that ${}^dU_i$ is a simply connected set, reveal that each component of $f^{-1}({}^dU_i)$ is a direct product of one component of $\tilde{\mathcal{C}}_j(p),\ j = 1, \cdots, k$, with a d-dimensional disk. Using $|\mathcal{L}_j^*(p)|,\ j = 1, \cdots, k$ and Lemma 1, one can show that the number of components of $f^{-1}({}^dU_i)$ is 2^{k_0}, where $k_0 \leq k$ is the number of legs for which $|\mathcal{L}_j^*(p)| = 3$.

2) Local Path Existence: Before considering the global path existence problem, consider motion planning between two valid configurations c_{init} and c_{goal} for which the junction A

lies in the same chamber. Since the fibre over every point in ${}^dU_i$ is equivalent, path existence amounts to checking the component memberships of the configurations c_{init} and c_{goal}.

For a single leg $\tilde{M}_j(p)$, if the number of long links $|\mathcal{L}_j^*(p)|$ is not three, then any two configurations of $\tilde{M}_j(p)$ are in the same component. When $|\mathcal{L}_j^*(p)| = 3$, choose any two long links and test the sign of the angle between them (with full extension taken as zero). There are two possible signs, one corresponding to *elbow-up* and the other to *elbow-down*. If for two distinct configurations of $\tilde{M}_j$, A lies in the same chamber, there is a continuous motion between them while keeping A in this chamber, if and only if the elbow sign is the same at both configurations (naturally, one must perform the sign test with the same two links and in the same order for both configurations). Considering all the legs together, a continuous motion of A in ${}^dU_i$ exists if and only if a motion exists for each leg individually. The previous discussion serves to prove the following result.

Proposition. *2:* Restricted to $f^{-1}({}^dU_i)$, two configurations $c_1, c_2 \in f^{-1}({}^dU_i)$ are path connected if and only if for each leg $\tilde{M}_j$ with $|\mathcal{L}_j^*| = 3$ in ${}^dU_i$, the elbow angle of $\tilde{M}_j$ has the same sign at c_1 and c_2.

Proposition 2 completely solves the path existence problem if W_A consists of a single chamber. However, things become complex when W_A has more than one chamber.

3) Singular Set and Global C-space Analysis: Recall that the C-space $\mathcal{C}$ is a union of $f^{-1}({}^dU_i)$, $d \in \{0, 1, 2\}$, $i = 1, \cdots, {}^dm$ and that $f^{-1}(p)$, $p \in {}^dU_i$ for $d \neq 2$ and all i is a set containing at least a singularity of f. Combining the local C-space and singular set analysis yields the global structure of C-space.

Proposition. *3:* For all $p \in \Sigma_j$, $f_j^{-1}(p)$ is a singular set containing isolated singularities. If a singularity separates its neighborhood V in $f_j^{-1}(p)$, then it is these singularities which glue the two separated components in $f_j^{-1}(q)$ where $q \in W_A - \Sigma_j$ is a point sufficiently close to p.

Next, we establish necessary and sufficient conditions for the connectivity of $\mathcal{C}$. Let $\mathcal{J}$ be the index set such that for all $j \in \mathcal{J}$, $|\mathcal{L}_j^*| = 3$ for at least one chamber ${}^dU_i$. We prove the following theorem.

Theorem *1:* Suppose $W_A = \bigcup_{d=0}^{2} \left(\bigcup_{i=1}^{{}^dm} {}^dU_i \right)$. Then $\mathcal{C} = f^{-1}(W_A)$ is connected if and only if:

1) W_A is connected;
2) $\Sigma_j \bigcap W_A \neq \emptyset$ for all $j \in \mathcal{J}$.

Proof:(sketch) Notice that $\mathcal{C}$ is connected if and only if any two possible configurations of leg j are connected for all j. These are exactly what Item 1 and 2 imply. ■

Fig. 5 illustrates the global connectivity for an example W_A corresponding to a star-shaped manipulator with two legs and a workspace for which there are two chambers ${}^2U_1$ and ${}^2U_3$ where leg 1 has three long links and another chamber ${}^2U_4$ where both legs have three long links. Among these chambers, ${}^1U_1$ and ${}^1U_2$ belong to Σ_1, and ${}^1U_3$ belongs to Σ_2. According

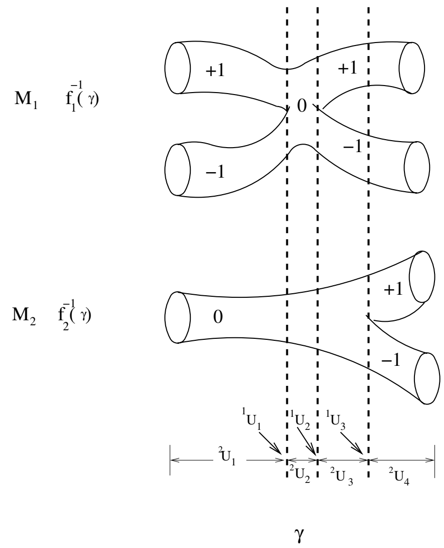

Fig. 5. C-space of a star-shape manipulator with two legs. For purposes of simplicity, only the portion of $f^{-1}(\gamma)$ is shown, where γ is a continuous curve in W_A that visits all chambers.

to Theorem 1, the C-space is path connected. In this example, the $\mathcal{C}$ is the product of the two structures shown.

Corollary *1:* Two configurations c_1 and c_2 of a star-shaped manipulator are in the same component if and only if

1) $f(c_1)$ and $f(c_2)$ are in the same component of W_A;
2) For each leg j with $|\mathcal{L}_j^*| = 3$ for all chambers ${}^dU_i$ in the component of W_A which contains $f(c_1)$ and $f(c_2)$, the elbow sign is same at both c_1 and c_2.

V. A Polynomial-Time, Exact, Complete Algorithm

Our algorithm uses two main routines, `PathExists` (illustrated in Fig 6) and `ConstructPath`. Its input is the topology and link lengths of a star-shaped manipulator and two valid configurations, c_{init} and c_{goal}. Its output is the answer to the path existence question. We will show that the complexity of `PathExists` is $O(k^3 + kN)$, where N is the maximum number of links in a leg and k is the number of legs.

The approach taken is to compute W_A and then, for each leg with its end point constrained to lie in W_A, to determine if its initial and goal configurations are path connected. Since the C-space of a leg is guaranteed to be connected if one of its singular circles Σ_j intersects W_A, the most straight forward way to test connectivity is to explicitly perform the intersections. However, since there are as many as $2^{n_j - 1}$ singular circles, any algorithm based on this approach will have worst-case complexity that is at least exponential in N. The key idea of `PathExists` is a polynomial-time algorithm for checking the existence of an intersection between W_A and a singular circle.

1. Construct W_A Recall that W_A is the intersection of the workspaces of the legs when they are disconnected from A. Each workspace is a disk or annulus, which can be determined by finding the length of the longest link and comparing it with the sum of all the other link lengths of that leg. The boundary

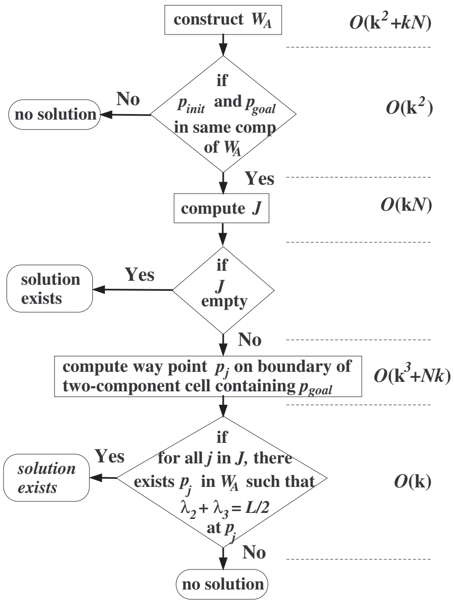

Fig. 6. Logical flow and complexity of the major steps of `PathExists`.

circles of these annuli decompose the plane into 2-D open cells, among which, those can be reached by all legs constitute W_A. We adopt a cell-decomposition algorithm (e.g., the line sweeping algorithm) to compute a graph representation for W_A. The complexity of constructing the graph is $O(k^2+kN)$.

2. Are p_{init} and p_{goal} in same component of W_A? A consequence of the cell decomposition is that this can be answered directly by searching the cell graph. This requires $O(k^2)$ since in worst case the number of nodes in the graph is $O(k^2)$.

3. Compute J This step is used to filter out easy solution existence checks, based on the cardinality and members of the sets $\mathcal{L}_j^*(p_{\text{init}})$ and $\mathcal{L}_j^*(p_{\text{goal}})$. For each leg $\tilde{M}_j(p_{\text{init}})$, compute L_j (see Section III) and find the three longest links of the set $\{l_{j,0},...,l_{jn_j}\}$. Denote these links by $(p_{\text{init}};\lambda_{j,1},\lambda_{j,2},\lambda_{j,3})$. Do the same for (p_{goal}) and define $(p_{\text{goal}};\lambda_{j,1},\lambda_{j,2},\lambda_{j,3})$. This requires $O(N)$ work. Finally, $|\mathcal{L}_j^*(p_{(\cdot)})| = 3$ if and only if $\lambda_{j,2}+\lambda_{j,3} > L_j/2$. If $\mathcal{L}_j^*(p_{\text{init}}) = \mathcal{L}_j^*(p_{\text{goal}})$ and $|\mathcal{L}_j^*(p_{\text{init}})| = 3$, and if the signs of the long links are different at c_{init} and c_{goal}, then add j into J. Computing J is $O(kN)$.

4. Does the set of long links vary for all $j \in J$? If and only if a way point $p_j \in W_A$ exists such that $\mathcal{L}_j^*(p_j) \neq \mathcal{L}_j^*(p_{\text{init}})$, then it is possible to make the long links colinear and thus change the signs of their relative angles. This can be done by computing a point $p_j \in W_A$ on the boundary of the cell which contains p_{goal} and keeps the same constant $\mathcal{L}_j^*(p)$ for all p in this cell. This boundary is characterized by $\lambda_{j,2}+\lambda_{j,3} = L_j/2$. Using the fact that the base link is the only link with variable link length, $\{p_j \mid j \in J\}$ can be computed in $O(k^3+kN)$.

The basic idea of `ConstructPath` is to use two kinds of motion generation algorithms: *accordion move* and *sign-adjust move*. The former moves the thorax endpoint (at A) along a specified path segment with all legs moving compliantly so that all loop closures are maintained. The latter keeps the endpoint fixed at a way point $q_j \in \Sigma_j$ while moving leg j into a colinear configuration and then to a nearby configuration with the sign of the relative angle between a pair of long links in this leg chosen to match those of c_{goal}.

The input of `ConstructPath` is W_A and its cell graph, c_{init}, c_{goal}, and the set of way points $p_j \in W_A$, $j \in J$ computed during the execution of `PathExist`.

1. Construct an initial path `ConstructPath` explores the cell graph of W_A, and constructs a path in W_A connecting p_{init} to p_{goal} and visiting all of the way points. Since there are at most k way points, this can be done in $O(k^3)$ time (the path has k segments each with $O(k^2)$ arcs).

2. Construct *guards* and insert the guards into the path Notice that accordion moves keep the sign of the relative angle between a pair of long links of leg j only when the thorax endpoint moves in a cell where $\tilde{C}_j$ has two components. For this reason, we set *guards* $\{q_j\}$ for legs which have three long links at p_{goal}. The set of such legs is denoted I. $\{q_j\}$ is computed as the last intersection point between the above constructed path in W_A and the boundary of the two-component cell of leg j containing p_{goal}, for all leg $j \in I$. Inserting these *guards* into the path. Assuming each arc in the path is approximated by fixed number of line segments, finding guards is $O(k^3)$.

3. Accordion moves and sign-adjust moves The path in $\mathcal{C}$ then is produced by using accordion moves along the path and sign-adjust moves at the *guards*. At each *guard* q_j, $j \in I$, one checks the sign between a pair of long links of leg j. If it does not match the goal one, do sign-adjust move, otherwise, accordion moves keep going on. Once A is coincident with p_{goal}, one is assured by the previous steps, that with A fixed at p_{goal}, the configuration of each leg is in the same component of its current C-space $\tilde{C}_j(p_{\text{goal}})$ as c_{goal}. The final move can be accomplished using a special accordion move algorithm found in [6].

The complexity of the accordion move algorithms reported in [6] are $O(N^3)$. Since the path has $O(k^3)$ line segments the complexity of `ConstructPath` is $O(k^3N^3)$.

Overall, our path planning algorithm is $O(k^3N^3)$.

VI. Example

In this example, the manipulator has three three-link legs, one of which has three long links when A is fixed at p_{goal}. Figure 7 shows the manipulator in its starting and goal configurations. Since when A is fixed at p_{goal}, the C-space of two of the legs have one component, our algorithm requires that A move from its initial location to one guard, and then to the goal. At the guard, the two lower legs can remain fixed while the other leg is moved to adjust the signs of its relative angles. This leg was then moved to make all its links colinear. Before leaving the guard via the next accordion move, leg's angles were adjusted to match the relative signs in the goal configuration. Figures 8, 9, 10, and 11 show the progress of the manipulation plan as the steps of the complete planning algorithm are carried out. Animation of the motion in this example can be found in `http://www.cs.rpi.edu/~liugf/multiloop`.

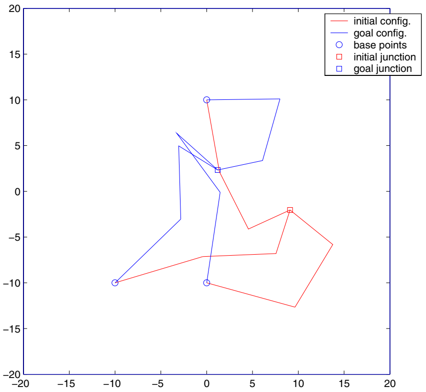

Fig. 7. Manipulator's initial configuration (junction on the right, drawn red) and goal configuration (junction just below the top foot, drawn blue.

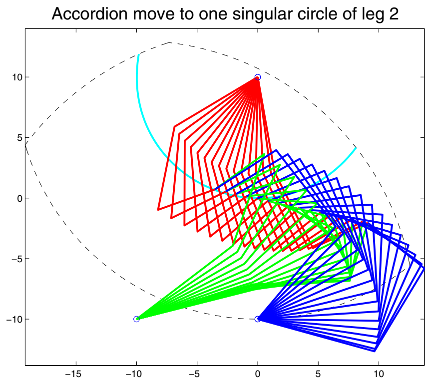

Fig. 8. All legs use an accordion move to move the junction A to the guard on a singular circle of leg 2.

VII. Conclusion

We have studied the global structural properties of planar star-shaped manipulators. Via the analysis of the singular set Σ, we derived the global connectivity of the C-space, and necessary and sufficient conditions for path existence. Based on these results, we devised a complete algorithm for motion planning. Simulation examples illustrate the key steps in motion planning for planar star-shaped manipulators.

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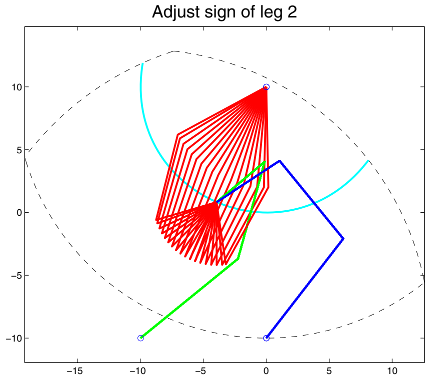

Fig. 9. With the junction A at the guard of leg 2, its join angles can be adjusted to achieve the signs required at the goal configuration.

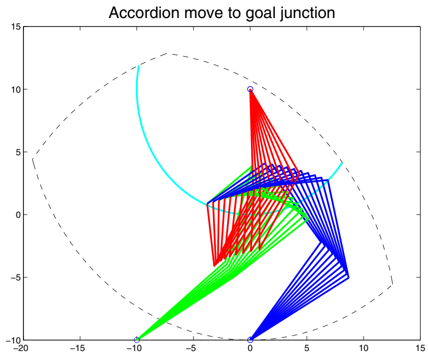

Fig. 10. All legs use an accordion move to move the junction A to its goal location. The sign of leg 2 are preserved guaranteeing that leg 2 will be in the correct C-space component once A is fixed at the goal position.

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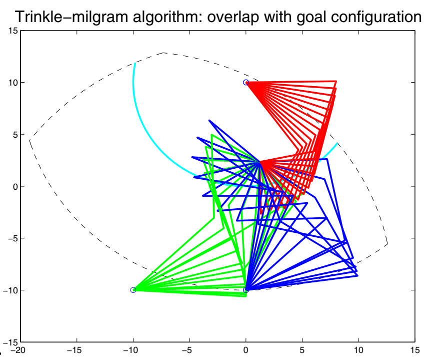

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Fig. 11. All legs use the Trinkle-Milgram algorithm with A fixed.