A Complementarity Approach to a Quasistatic Multi-Rigid-Body Contact Problem

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Abstract

In this paper, we study the problem of predicting the quasistatic planar motion of a passive rigid body in frictional contact with a set of active rigid bodies. The active bodies can be thought of as the links of a mechanism or robot manipulator whose positions can be actively controlled by actuators. The passive body can be viewed as a "grasped" object, which moves only in response to contact forces and other external forces such as those due to gravity. We formulate this problem as a certain uncoupled complementarity problem, and show that it belongs to the class of NP-complete problems. Finally, numerical results of our proposed linear programming-based solution algorithm for this class of problems are presented and compared to the only other currently available solution algorithm.

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1 Introduction

In the field of robotics, one of the most important unsolved problems is that of automatically planning the motions of a robot or system of robots to accomplish a given task involving the manipulation of one or more objects. Since practical planners for general tasks of this sort do not currently exist, robot motion plans are created primarily in two ways. First, programs in the robot's control language are written by hand. This is a laborious, error-prone undertaking. Second, programs can be developed by recording the sequences of joint positions and end effector[1] forces experienced by the robot as its operator manually "walks" it through its task. These sequences are translated automatically into the robot's control language and then modified by trial and error until the robot can reliably complete the task (if possible).

These approaches are best suited to tasks involving no contacts between the robot and its surroundings or just one contact with an immovable object (as would be the case if the robot performed grinding or deburring operations on a fixtured part). Neither approach is suitable when the intended task has any of the following characteristics:

- The task is to take place in an environment hazardous to human health (such as Space, waste disposal/clean-up sites, and sub-oceanic environments).
- The task is not repetitive (as would be the case in small-batch manufacturing enterprises).
- The task involves one or more robots cooperating to manipulate one or more objects (such tasks require manual dexterity and are referred to as dexterous manipulation tasks).

One method applicable to tasks with the first two characteristics is tele-operation. While this method has been used successfully for simple tasks since World War II, it is extremely difficult and costly to build tele-operation systems with sufficient fidelity in force-feedback to allow the operator to execute tasks involving multiple sliding and/or rolling contacts. The same comments apply to manipulation systems employing the technologies of virtual reality.

Tasks exhibiting the last characteristic cannot be planned and executed by any of the techniques described above (or other existing techniques), so new approaches must be developed. In several recent articles [9, 10, 11], a new approach based on the mechanics of quasistatic multi-rigid-body systems has been proposed. These articles demonstrated that by embedding a model of quasistatic contact mechanics in a dexterous manipulation planner, it is possible to automatically generate programs (without manual editing) that can be

[1] The "end effector" of a robot is the link furthest from the base, *i.e.*, the most distal link. It may be a robotic hand, grinding wheel, paint-spraying nozzle, or any other device required to perform the intended task.

successfully executed by a real multi-robot system. However, the planners developed could be improved if quasistatic rigid body mechanics were better understood.

The purpose of this paper is to advance our understanding of the quasistatic multi-rigid-body contact problem (or just, the quasistatic contact problem), so that future manipulation planners can be made more efficient, capable, and reliable than the prototype planner discussed above. Our improved understanding comes as a result of the three main contributions of this paper. First, a new complementarity formulation of the quasistatic contact problem is introduced, which we refer to as the *uncoupled complementarity problem* (*UCP*) in order to distinguish it from the standard linear complementarity problem [2]. Second, it is established that the UCP lies in the class of NP-complete problems. Finally, we develop a linear programming-based algorithm for solving the UCP. The application of this algorithm to the quasistatic contact problem is discussed and numerical results are reported. We show that our proposed algorithm is capable of finding a solution much more quickly than the only other existing algorithm [11]. In fact, the other algorithm is enumerative, requiring the evaluation of a number of potential solutions, which grows exponentially with the number of contacts. While this algorithm can find multiple solutions when more than one exists, it would take years of cpu time to execute on the largest problems for which our new algorithm found solutions.

2 Problem Formulation

In what follows, we shall first develop the governing equations of the quasistatic multi-rigid-body contact problem and then show how they lead to an uncoupled complementarity system. For in-depth discussions of the advantages of the quasistatic multi-rigid-body contact model over its dynamic counterpart and of previous related work, we refer the reader to [11].

A body of arbitrary shape (the object or workpiece) moves quasistatically due to frictional contact with one or more actively-controlled bodies (see Figure 1). The actively-controlled bodies are viewed as the links of a *manipulator* composed of any number of serial and branching kinematic chains. The joints are either revolute (allowing only rotational relative motion between the bodies) or prismatic (allowing only translational relative motion). The manipulator and the object are collectively referred to as the *system*. We assume that:

1. The positions, orientations, and geometries of all bodies are known.
2. The bodies are rigid and restricted to move in a plane.
3. The external forces and torques applied to all the bodies in the system (other than those arising at the contacts) are known.
4. The kinetic energy of the system and all dynamic effects are negligible (this assumption makes the system quasistatic).

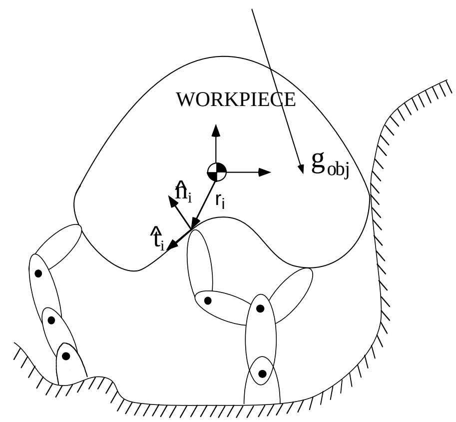

Figure 1: Workpiece in Contact with Manipulator

5. Each joint may be position- or effort-controlled; effort control implies force control of a prismatic joint and torque control of a revolute joint.

6. The friction forces at the contacts satisfy Coulomb's Law.

Then, given the instantaneous velocities of a subset of joints and the efforts applied at the remaining joints, our goal is to determine the instantaneous velocity of the object. We would also like to determine the contact forces and the unspecified joint efforts and velocities.

To formulate the governing equations, a "world" frame may be chosen arbitrarily. For convenience, we choose it so that its origin coincides with the center of gravity of the workpiece (see Figure 1). Let n_c be the number of contact points on the object and let n_q be the number of degrees of freedom of the object's motion. Further, let contact point i be the origin of a coordinate frame i, whose axes, $\hat{n}_i$ and $\hat{t}_i$, are aligned with the contact normal (pointing inward with respect to the object) and the contact tangent such that the cross product of $\hat{n}_i$ and $\hat{t}_i$, points out of the plane of motion.

Let the (unknown) vector $c_i = [c_{in} \;\; c_{it}]^T$, $i \in \{1, \ldots, n_c\}$, represent the force applied to the object at the i^{th} contact such that c_{in} and c_{it} are the normal and tangential components, where ^{T} indicates matrix transposition. The vector c_i is known as the wrench intensity vector [8] of the i^{th} contact. The wrench matrix, W_i, transforms the i^{th} contact force into the world frame. In the planar case, W_i has dimension (3×2) and is defined as follows:

$$W_i = [w_{in} \;\; w_{it}] = \left[\begin{array}{cc} \hat{n}_i & \hat{t}_i \\ r_i \otimes \hat{n}_i & r_i \otimes \hat{t}_i \end{array} \right] \qquad (1)$$

where $\hat{n}_i$, $\hat{t}_i$, r_i are all expressed in the world coordinate frame, r_i is the position of the i^{th} contact point, and the $\otimes$ operator applied to two vectors, $[a_1, a_2] \otimes [b_1, b_2]$, is defined as $a_1 b_2 - a_2 b_1$.

Let g_{obj} be the external force and torque applied to the object; it includes all forces and torques not applied at the contact points. Summing all the forces and torques and setting them equal to zero (as required by Newton's Laws) yields the equations of equilibrium of the object:

$$Wc + g_{\text{obj}} = W_n c_n + W_t c_t + g_{\text{obj}} = 0 \tag{2}$$

where W and (the unknown) c are referred to as the global wrench matrix and the global wrench intensity vector [5] and have dimensions $(3 \times 2n_c)$ and $(2n_c \times 1)$, respectively. The normal and tangential wrench matrices, W_n and W_t, both of dimension $(3 \times n_c)$, are formed by the horizontal concatenation of all the individual normal and tangential contact wrenches w_{in} and w_{it} (defined in equation (1)). Correspondingly, the normal and tangential wrench intensity vectors, c_n and c_t, both have length n_c and are formed by the vertical concatenation of all the normal and tangential wrench intensity components, c_{in} and c_{it}, respectively.

The manipulator must satisfy similar equilibrium equations as described in [5]:

$$J^T c = J_n^T c_n + J_t^T c_t = \tau - g_{\text{man}} \tag{3}$$

where J is the global Jacobian of the manipulator, τ is the vector of joint efforts, and g_{man} is the vector of joint loads induced by the external forces. The dimension of the latter two vectors is equal to the number of joints which we denote n_θ. The partitions J_n^T and J_t^T are exactly analogous to the partitions, W_n and W_t, of the wrench matrix, W. In particular, J_n^T and J_t^T are of order n_θ by n_c.

The motion of the object is subject to kinematic velocity constraints, the satisfaction of which implies that the bodies in the manipulator do not penetrate the object's surface. For $i = 1, \ldots, n_c$, let $v_i = [v_{in}\ v_{it}]^T$ denote the relative linear velocity at the i^{th} contact point, expressed in terms of the i^{th} contact frame; similar to the definition of c_n and c_t, let v_n and v_t denote the n_c-vectors with components v_{in} and v_{it}, respectively. Let $\dot{q}$ be the (unknown) n_q-vector of (linear and angular) velocity of the point on the object coincident with the origin of the world frame. Further, let $\dot{\theta}$ be the n_θ-vector of angular and linear velocities of the joints of the manipulator: angular velocities for revolute joints and linear velocities for prismatic joints. We have the following defining equations:

$$v_n = W_n^T \dot{q} - J_n \dot{\theta} \tag{4}$$

$$v_t = W_t^T \dot{q} - J_t \dot{\theta}. \tag{5}$$

The nonpenetration constraint mentioned before is given by the nonnegativity of the vector v_n; *i.e.*,

$$v_n \geq 0. \tag{6}$$

Note that there is no such requirement on the vector v_t, because while sliding at a contact may be prevented by friction, it is not prevented by a geometric constraint as penetration is.

Among the components of the two vectors $\dot{\theta}$ and τ, some are given as inputs while others are to be computed. The motivation for this partitioning of $\dot{\theta}$ and τ, is that it allows us to model compliant control modes. These modes are implemented by position-controlling some joints (those corresponding to the given elements of $\dot{\theta}$) and effort-controlling the others. Compliant control modes increase the range of tasks that the manipulator can perform by providing the capability to maintain desired contact modes (*i.e.*, sliding or rolling) at specific contact points.

Continuing with our development, given is a subset α of $\{1, \ldots, n_\theta\}$ such that $\dot{\theta}_i$ and τ_j are given constants for all $i \in \alpha$ and $j \notin \alpha$; $\dot{\theta}_j$ and τ_i are unknowns. Note that as far as the individual constraints in the equation (3) are concerned, only those that correspond to known values of τ_i are effective restrictions of the problem.

The remaining constraints enforce the Coulomb friction law, which for the planar case, stipulates that each contact force lies within or on the boundary of its respective friction cone:

$$|c_{it}| \le \mu_i c_{in}, \quad i = 1, \ldots, n_c, \tag{7}$$

where μ_i is the effective coefficient of friction at the i^{th} contact point; each μ_i is a positive constant. More specifically, Coulomb's Law states that if contact i is sliding, then the contact force must be on the boundary of the friction cone such that the friction force component, c_{it}, opposes the relative motion. If contact i is rolling, there is no additional restriction. Note that the inequality (7) implies that the contact forces are nontensile (*i.e.*, $c_{in} \ge 0$ for all i). Also, the reader should be aware that it is physically meaningful for one or more of the constants μ_i to be zero; nevertheless, a zero μ_i clearly renders the corresponding c_{it} equal to zero and the constraint (7) trivially satisfied.

To complete the formulation of the quasistatic contact problem, we must specify the three types of relationships possible between the forces and the relative velocities at the contacts: for each $i = 1, \ldots, n_c$:

(i) rolling contact:

$$c_{in} > 0 \;\Rightarrow\; v_{in} = 0; \tag{8}$$

(ii) breaking contact:

$$v_{in} > 0 \;\Rightarrow\; c_{in} = 0 \; (\Rightarrow c_{it} = 0); \tag{9}$$

(iii) sliding contact:

$$\begin{aligned} v_{it} > 0 \;&\Rightarrow\; c_{it} = -\mu_i c_{in} \\ v_{it} < 0 \;&\Rightarrow\; c_{it} = \mu_i c_{in}. \end{aligned} \tag{10}$$

The quasistatic multi-rigid-body contact problem is to find vectors $c_n, c_t, v_n, v_t, \dot{q}$, and the unknown components of $\dot{\theta}$ and τ satisfying conditions (2) to (10). Incidentally, the contact conditions (8) and (9) are mathematically equivalent, but they have different physical meanings.

The contact conditions (8), (9), and (10) naturally lead one to suspect that this contact problem can be formulated as a complementarity problem. As we shall see below, this is indeed the case; nevertheless, the resulting complementarity problem is *not* of the standard type as discussed in [2].

Adding slack variables s_{it}^+ and s_{it}^- to inequality (7), we may rewrite it as the following pair of equations:

$$\mu_i c_{in} = s_{it}^- + c_{it}, \quad \text{and} \quad \mu_i c_{in} = s_{it}^+ - c_{it}, \quad i = 1, \ldots, n_c.$$

From these equations, we obtain $c_t = s_t^+ - Uc_n$ and $2Uc_n = s_t^+ + s_t^-$ where U is $n_c \times n_c$ diagonal matrix with the i^{th} diagonal entry given by μ_i, and s_t^+ and s_t^- are, respectively, the n_c-vectors with components s_{it}^+ and s_{it}^-. Physically, when the i^{th} contact is rolling and c_{it} is positive, s_{it}^- represents how much c_{it} can be increased before the onset of sliding in the negative tangential direction. Similarly, when the i^{th} contact is rolling and c_{it} is negative, s_{it}^+ represents how much c_{it} can be decreased before sliding begins in the positive tangential direction.

Eliminating c_t from the equations (2) and (3), we deduce the following equivalent formulation for the quasistatic multi-rigid-body contact problem.

Proposition 1 *Given the vector $\dot{q}$ and complementary partitions of $\dot{\theta}$ and τ (as described above), the vectors c_n, c_t, v_n, and v_t satisfy conditions (2) to (10) if and only if c_n and v_n, along with s_t^+, s_t^-, v_t^+, and v_t^-, satisfy the following conditions:*

$$\begin{aligned}
v_n &= W_n^T \dot{q} - J_n \dot{\theta} \\
v_t^+ - v_t^- &= W_t^T \dot{q} - J_t \dot{\theta} \\
0 &= 2Uc_n - s_t^+ - s_t^- \\
0 &= (W_n - W_t U) c_n + W_t s_t^+ + g_{\text{obj}} \\
\tau &= (J_n^T - J_t^T U) c_n + J_t^T s_t^+ + g_{\text{man}} \\
v_n, v_t^+, v_t^-, c_n, s_t^+, s_t^- &\geq 0 \\
(v_n)^T c_n = (v_t^+)^T s_t^+ &= (v_t^-)^T s_t^- = 0.
\end{aligned} \tag{11}$$

Proof. If c_n, c_t, v_n, and v_t satisfy (2) to (10), it suffices to define s_t^+ and s_t^- as the slack variables for (7) and v_t^+ and v_t^- be, respectively, the nonnegative and nonpositive part of v_t. It is easy to verify that the contact conditions (8), (9), and (10) imply the desired complementarity conditions $(v_n)^T c_n = (v_t^+)^T s_t^+ = (v_t^-)^T s_t^- = 0$; the other conditions in the system (11) are trivial.

Conversely, if $v_n, v_t^+, v_t^-, c_n, s_t^+$, and s_t^- satisfy (11), then it suffices to define $v_t = v_t^+ - v_t^-$ and $c_t = s_t^+ - Uc_n$. The verification that v_n, v_t, c_n, and c_t satisfy (2) to (10) is easy. Q.E.D.

With the above proposition, we can give an equivalent formulation of the quasistatic multi-rigid-body motion problem as a certain complementarity system. We introduce some notation. Let $\bar{\alpha}$ denote the complement of the index set α, a subset of $\{1, \ldots, n_\theta\}$; let J_α and $J_{\bar{\alpha}}$ denote, respectively, the columns of the matrix $J = \begin{bmatrix} J_n \\ J_t \end{bmatrix}$ indexed by α and $\bar{\alpha}$; write

$$P = \begin{bmatrix} W^T & J_{\bar{\alpha}} \end{bmatrix}$$

where $W = [W_n \; W_t]$. Note that P is of order $2n_c \times m$ where $m = n_q + |\bar{\alpha}|$. Let Z be any matrix with $2n_c$ columns such that its null space is equal to the column space of P; *i.e.*, a vector $a \in R^{2n_c}$ satisfies $Za = 0$ if and only if $a = Pb$ for some vector $b \in R^m$. One such matrix Z can be obtained as follows: if $\tilde{P}$ denotes the submatrix of P consisting of a maximal set of linearly independent columns, then

$$Z = I - \tilde{P}(\tilde{P}^T \tilde{P})^{-1} \tilde{P}^T,$$

where I is the identity matrix of order $2n_c$. The matrix Z can also be obtained by various rank-retaining factorizations (such as the QR or singular valued decompositions); see [3, section 6.8] for more details. In several numerical examples to be discussed in Section 5, both W_n and W_t are nonsingular (square) matrices and $\bar{\alpha} = \emptyset$; in this case, we have

$$Z = \begin{bmatrix} (W_n)^{-T} & -(W_t)^{-T} \end{bmatrix}.$$

In general, we may write $Z = [Z_n \; Z_t]$ where both Z_n and Z_t have n_c columns. Define the polyhedron:

$$X = \left\{ \begin{bmatrix} v_n \\ v_t^+ \\ v_t^- \end{bmatrix} \in R_+^{3n_c} \; : \; Z_n v_n + Z_t(v_t^+ - v_t^-) + Z J_\alpha \dot{\theta}_\alpha = 0 \right\}$$

where $R_+^{3n_c}$ is the positive orthant of Euclidean $3n_c$-space and $\dot{\theta}_\alpha$ is the vector formed from $\dot{\theta}$ by removing all unknown elements.

The polyhedron X represents all possible normal and tangential velocities at the contact points obtainable by varying the unknowns, $\dot{q}$ and $\dot{\theta}_{\bar{\alpha}}$; these variations are made without regard for their consistency with the equilibrium equations and our model of contact interactions which relate contact velocities and forces. Extreme points of X on the boundary of the nonnegative orthant represent system motions for which various combinations of sliding and rolling are taking place at the contacts. Note that all interior points of X correspond to system motions for which every contact is separating and therefore all the contact forces are zero. Problems that admit solutions with all contact forces equal to zero are degenerate in the sense that the external force applied to the object is zero, so no contact is required to maintain its equilibrium.

We need to define another polyhedron Y that plays the role of the dual of X. For this purpose, define

$$C = \begin{bmatrix} 2U & -I & -I \\ W_n - W_t U & W_t & 0 \\ (J_n)^T_{\bar{\alpha}} - (J_t)^T_{\bar{\alpha}} U & (J_t)^T_{\bar{\alpha}} & 0 \end{bmatrix}, \quad d = \begin{bmatrix} 0 \\ g_{\text{obj}} \\ (g_{\text{man}} - \tau)_{\bar{\alpha}} \end{bmatrix},$$

where as before, I is the identity matrix of order n_c. The polyhedron Y is

$$Y = \left\{ y = \begin{bmatrix} c_n \\ s_t^+ \\ s_t^- \end{bmatrix} \in R_+^{3n_c} \ : \ 0 = Cy + d \right\}.$$

This polyhedron represents the set of all possible contact forces without regard for the unknown velocities, $\dot{q}$ and $\dot{\theta}_{\bar{\alpha}}$, that might give rise to them. However, note that if the i^{th} contact point is sliding, one of s_{it}^+ or s_{it}^- is zero, and if it is breaking, all of c_{in}, s_{it}^+, and s_{it}^- are zero. Therefore, every system motion with at least one sliding or rolling contact is represented by an extreme point of Y. Only motions with all rolling contacts can correspond to interior points of Y, and in this case, the corresponding point in X is the origin.

We have the following result which does not require further justification.

Theorem 1 *The quasistatic multi-rigid-body motion problem is equivalent to the problem of finding a pair of vectors* $(x, y) \in X \times Y$ *such that* $x^T y = 0$.

This theorem indicates that only vectors in the velocity polyhedron, X, that are orthogonal to a vector in the force polyhedron, Y, represent quasistatic system motions consistent with the equilibrium of the object and manipulator, the kinematic constraints arising from the contacts, the Coulomb model of friction, and the contact interaction model (of sliding, rolling, and breaking contact).

3 The Uncoupled Complementarity Problem

Motivated by Theorem 1, we formally define the *uncoupled complementarity problem* (UCP) as follows. Let X and Y be two polyhedra in R^n_+; this is the problem of finding a pair of vectors $(x, y) \in X \times Y$ such that $x^T y = 0$. The term "uncoupled" refers to the fact that the variables x and y are independent of one another (except through the complementarity constraint). The use of the word "complementarity" is justified because both X and Y lie in the nonnegative orthant of R^n; hence a pair of vectors $(x, y) \in X \times Y$ satisfies the orthogonality condition $x^T y = 0$ if and only if the standard complementarity relation holds:

$$x_i y_i = 0, \quad i = 1, \ldots, n.$$

The well-known linear complementarity problem (LCP) is to find a vector $z \in R^n$ satisfying the following constraints [2]:

$$w = q + Mz \geq 0, \quad z \geq 0, \quad w^T z = 0,$$

where $q \in R^n$ and $M \in R^{n\times n}$ are given. The LCP is *coupled*. Indeed, let $X = R^n_+$ and $Y = q + MR^n_+$; then (z, w) solves the LCP if and only if $(z, w) \in X \times Y, z^T w = 0$, and $0 = w - Mz - q$; the last equation is a coupling between the variables z and w.

Besides the quasistatic contact problem, the problem of determining whether two disjoint point sets in R^n can be separated by two planes can also be formulated as an uncoupled complementarity problem [1]. In addition, the zero-one integer feasibility problem is also a special instance of the UCP; consequently, the uncoupled complementarity problem belongs to the class of NP-complete problems.

Proposition 2 *The uncoupled complementarity problem with integral data is NP-complete.*

Proof. The UCP clearly belongs to the class NP. We now show that the zero-one integer feasibility problem is a special instance of the UCP. More specifically, let A be an $m \times n$ matrix and b an m-vector. Consider the problem of finding a vector z satisfying

$$Az = b, \quad z \in \{0, 1\}^n. \tag{12}$$

Let e be the n-vector of all ones. Define

$$X = \{(z, w) \in R^{2n}_+ \ : \ Az = b, \ z + w = e\},$$
$$Y = \{(u, v) \in R^{2n}_+ \ : \ Av = b, \ u + v = e\}.$$

With the pair X and Y defined in this way, it is trivial to see that if z satisfies (12), then $(z, e - z) \in X$ and $(e - z, z) \in Y$; moreover, this pair of vectors is orthogonal. Conversely, if $(z, w) \in X$ and $(u, v) \in Y$ satisfy $z^T u = w^T v = 0$, we claim that $z \in \{0, 1\}^n$. Indeed, if $z_i > 0$, then $u_i = 0$ which implies $v_i = 1$ which in turn implies that $w_i = 0$; thus $z_i = 1$. Q.E.D.

The above proposition shows that the UCP is in general not easy to solve. In the next section, we shall describe an algorithm for solving this problem and apply it to the quasistatic contact problem.

4 A Bilinear Programming Approach

Associated with the UCP defined by the pair of nonempty polyhedra (X, Y) in R^n_+, we can define the following "natural" bilinear program:

$$\begin{array}{ll} \text{minimize} & x^T y \\ \text{subject to} & (x, y) \in X \times Y. \end{array} \tag{13}$$

Since X and Y are both subsets of the nonnegative orthant of R^n, the objective function of this program is always nonnegative on its feasible region. Thus, by the Frank-Wolfe Theorem in quadratic programming, (13) will always have an optimal solution. Clearly, a pair (x, y) solves the UCP if and only if (x, y) is a globally optimal solution of (13) with a zero objective value. Thus, the UCP can be solved by finding a globally optimal solution to the bilinear program (13).

Section 1 in Chapter 9 of [4] contains a rather extensive treatment of a bilinear program; in particular, some basic properties and various algorithmic approaches for computing a solution are discussed. By Proposition IX.1 in the reference and the above discussion, it follows that the problem (13) must have an optimal solution $(\bar{x}, \bar{y})$ such that $\bar{x}$ and $\bar{y}$ are vertices of X and Y, respectively. Based on this fact, we describe a linear programming based algorithm for solving the UCP.

The algorithm

Step 0. (Initialization) Let (x^0, y^0) be an arbitrary vector in $X \times Y$, where the superscript indicates the number of iterations performed by the algorithm.

Step 1. (LP step) In general, given a non-complementary pair $(x^\nu, y^\nu) \in X \times Y$, we obtain $(x^{\nu+1}, y^{\nu+1}) \in X \times Y$ by applying the simplex method in a sequential fashion where $x^{\nu+1}$ is an optimal solution of the linear program in the variable x:

$$\begin{array}{ll} \text{minimize} & x^T y^\nu \\ \text{subject to} & x \in X, \end{array} \tag{14}$$

and with $x^{\nu+1}$ computed, $y^{\nu+1}$ is an optimal solution of the linear program in the variable y:

$$\begin{array}{ll} \text{minimize} & y^T x^{\nu+1} \\ \text{subject to} & y \in Y. \end{array} \tag{15}$$

We emphasize that both of these linear programs must have optimal solutions; furthermore, $(x^{\nu+1})^T y^{\nu+1} \leq (x^\nu)^T y^\nu$.

Step 2. If $(x^{\nu+1})^T y^{\nu+1} = 0$, stop; a desired complementary solution to the UCP is obtained. If $0 < (x^{\nu+1})^T y^{\nu+1} < (x^\nu)^T y^\nu$, return to Step 1 with ν replaced by $\nu + 1$. If

$$0 < (x^{\nu+1})^T y^{\nu+1} = (x^\nu)^T y^\nu, \tag{16}$$

then (x^ν, y^ν) is a stationary point of the bilinear program (13). Continue.

Step 3. We attempt to decrease the bilinear function $x^T y$ from its current value $(x^\nu)^T y^\nu$ by generating all adjacent extreme points of x^ν and y^ν in X and Y, respectively, This can be done by an algorithm described in [6, §3.7.1]. We denote the sets of adjacent extreme points of x^ν and y^ν in X and Y by $E(x^\nu)$ and $E(y^\nu)$, respectively. We search for a pair $(x, y) \in E(x^\nu) \times E(y^\nu)$ such that

$$x^T y < (x^\nu)^T y^\nu.$$

If such a pair exists, then we return to Step 1 with ν replaced by $\nu+1$ and (x^ν, y^ν) replaced by the pair (x, y) just identified. Otherwise, we randomly choose a pair $(x, y) \in E(x^\nu) \times E(y^\nu)$ and repeat Step 3 with $(x^\nu, y^\nu) \leftarrow (x, y)$ and $\nu \leftarrow \nu + 1$.

There are several remarks pertaining to the algorithm described above. One is the fact that the algorithm attempts to maintain the inequality:

$$(x^{\nu+1})^T y^{\nu+1} < (x^\nu)^T y^\nu. \tag{17}$$

This goal of decreasing the objective values, although not always attained, qualifies our algorithm as one of minimizing $x^T y$. Another remark concerns the claim of stationarity of the pair (x^ν, y^ν) when (16) occurs. Indeed, when the latter happens, it is easy to show that the pair (x^ν, y^ν) must satisfy the inequality:

$$(x - x^\nu)^T y^\nu + (y - y^\nu)^T x^\nu \geq 0$$

for all $(x, y) \in X \times Y$. The above algorithm must, in a finite number of iterations (i.e., with a finite value of the iteration counter ν), identify either a complementary pair (x^ν, y^ν) satisfying $(x^\nu)^T y^\nu = 0$ or a non-complementary, but stationary pair (x^ν, y^ν). The reason for this finite identification is due to finite number of extreme points that the polyhedra X and Y possess and the fact that the bilinear objective value $x^T y$ can be decreased if (x^ν, y^ν) is not stationary (hence, not complementary either). Incidentally, in the article [1], Bennett and Mangasarian have described a variation of Step 1 to compute a stationary point of an uncoupled bilinear program (of which the UCP is a special case). They show that it is sufficient to let $(x^{\nu+1}, y^{\nu+1})$ be any pair of extreme points of $X \times Y$ such that (17) holds; in particular, it is not necessary to solve either program (14) or (15) to optimality. We also note that due to the independence of the sets X and Y, it is possible to obtain the pair $(x^{\nu+1}, y^{\nu+1})$ in parallel, rather than in the sequential fashion as we have stated. This variation is also discussed in the cited reference.

Step 3 of the algorithm attempts to improve the objective function $x^T y$ when stationarity (but non-complementarity) is reached. Since finding all adjacent extreme points of a given extreme point is a relatively easy task if the given point is nondegenerate, this step is expected to be computationally reasonable. Due to the intrinsic difficulty of the UCP, Step 3 is, in general, not guaranteed to return a pair of extreme points with an improved bilinear objective value. When no such improved extreme point is identified, the algorithm picks a new pair of points arbitrarily and proceed with this pair. Consequently, it is possible in theory for the algorithm to revisit the same pair of vectors; nevertheless, in the application of the algorithm for solving the quasistatic contact problem, the numerical results reported in the next section show that this cycling phenomenon occurs only in those cases for which no solutions to the problem exist.

5 Numerical Results

We have implemented the algorithm described in the last section for solving the quasistatic contact problem formulated as an UCP. The set of input data included $n_c, n_q, n_\theta, P, J_n, J_t, \dot{\theta}, d$ and μ which were all defined in Section 2. With these inputs, we generated the data for the polyhedra X and Y as described in Section 3. The algorithm was coded in C and a self-written code for the simplex method (with some linear algebraic subroutines taken from [7]) was used to solve (to optimality) the various linear programs (14) and (15). The experiments were conducted on a Sun SPARCStation IPX with 16 megabytes of memory and one CPU processor. The initial pair (x^0, y^0) was generated by solving two linear programs on the sets X and Y with a zero objective function. The termination rule was $x^T y \leq 10^{-10}$.

The output provided a final pair $(x, y) \in X \times Y$, the value $x^T y$, and the total number of intermediate pairs (x, y) found to reach the solution. The recovery of the force variables c_n and c_t, the velocity vector $\dot{q}$, and the vector of joint torques τ from the computed vectors (x, y) was easily done and also included in the computer code. Since we are more interested in the performance of the algorithm than in the actual output, we choose to report only the former and omit the latter.

We initially tested our code with three sets of data drawn from [11] (denoted Data Set 1,2 and 3 below). We successfully solved all of them with $x^T y = 0$ by finding no more than two (x, y) pairs (a total of 6 linear programs, including the two initial ones) before arriving at the complementary solution. We then further tested our code with fifty six different combinations of $\{\mu_i\}$ in Data Set 2. The results we obtained were consistent with those presented in [11]. Forty one of them were solved by exploring no more than two pairs of (x, y). Three other sets of data invoked Step 3 once which successfully identified the complementary solution from the adjacent extreme points of the current (x, y) pair. The remaining twelve data sets were taken from the "jamming" region; for these data sets, the algorithm terminated due to cycling and thus failed to yield (x, y) satisfying $x^T y = 0$. In fact, the problems with these data have no solutions. One such set of data is given as Data Set 4.

Data Set 1: $[n_c, n_q, n_\theta] = [3, 3, 6]$

$$P = \begin{bmatrix} 0.966 & 0.259 & 0.000 \\ 0.000 & 1.000 & 0.240 \\ -0.707 & 0.707 & 0.750 \\ -0.259 & 0.966 & -2.000 \\ -1.000 & 0.000 & -1.450 \\ -0.707 & -0.707 & -1.900 \end{bmatrix}$$

$$J_n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad J_t = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\dot{\theta} = [\ 0.1 \quad 0.995 \quad 0.1 \quad -0.995 \quad 0.1 \quad 0.995\ ]^T$$

$$d = [\ 0 \quad 0 \quad 0 \quad 0 \quad -1 \quad 0\ ]^T$$

$$\mu = [\ 0.5 \quad 0.5 \quad 0.3\ ]^T$$

Data Set 2: $[n_c, n_q, n_\theta] = [3, 3, 6]$

$$P = \begin{bmatrix} 0.804 & 0.595 & 1.040 \\ -0.707 & 0.707 & -3.500 \\ -0.973 & -0.232 & 14.000 \\ -0.595 & 0.804 & -20.900 \\ -0.707 & -0.707 & -17.900 \\ 0.232 & -0.973 & -7.630 \end{bmatrix}$$

$$J_n = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad J_t = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\dot{\theta} = [\ 0.92 \quad -0.391 \quad 0.9276 \quad -0.3747 \quad 0.9306 \quad -0.367\ ]^T$$

$$d = [\ 0 \quad 0 \quad 0 \quad 0 \quad -1 \quad 0\ ]^T$$

$$\mu = [\ 0.1 \quad 0.4 \quad 0.3\ ]^T$$

Data Set 3: Same as Data Set 1 except that

$$\dot{\theta} = [\ -0.161 \quad 0.987 \quad 0.995 \quad 0.1 \quad -0.774 \quad -0.633\ ]^T.$$

Data Set 4: Same as Data Set 2 except that

$$\mu = [\ 0.7 \quad 0.9 \quad 0.9\ ]^T$$

We tested our code with a data set of larger size ($n_c = 20, n_q = 3, n_\theta = 40$, and $|\alpha| = 23$; the full data set is too large to be included). A complementary pair (x, y) was identified after running a single iteration of the algorithm. Subsequently, we modified the external wrench (g_{obj}), the joint torques and forces induced by gravity (g_{man}) and the coefficients of friction μ

to generate three versions of the data set. The performance of our code on the modified data was similar to that of the original set. Only one iteration was needed to obtain a solution pair in all three cases. Moreover, the vector x in the solution pair (x, y) was identical to the one that solved the original data. The reason for the latter observation is two-fold. First, our modification of the data only concerned the polyhedron Y and thus rendered that vector x feasible in all three trials. Second, that vector x was rather degenerate with nine basic variables at value zero. Hence, by having fewer complementarity restrictions, it was more likely to obtain a feasible y to complement the given x as a solution pair.

We further tested our algorithm with some randomly generated data sets. The number of contacts n_c was allowed to vary, whereas n_θ and n_q were fixed at $2n_c$ and 3 respectively. We used a computer code in C written by Sandra Sudarsky to generate the matrices W and J and the vector τ that corresponded to a single workpiece in contact with several point fingers. The coefficients of friction μ_i were generated randomly with varying magnitude. Since we were interested only in problems with nonempty polyhedra X and Y, we generated the remaining data in the following way. First, we generated a random pair (c_n, c_t) that satisfied the Coulomb friction law (7), we then used equations (2) and (3) to determine g_{obj} and g_{man}. This ensured the nonemptiness of Y. Next we generated a random nonnegative vector v_n and determined $\dot{q}$ and $\dot{\theta}$ from equation (4). ¿From the resulting $\dot{q}$ and $\dot{\theta}$, we determined the set α so that $X \neq \emptyset$. By varying the value of n_c, the range of the coefficients of friction, and the size of the set α, twenty feasible problems were created to test our algorithm. The results are summarized in Table 1.

Initially, our algorithm solved all but seven of the test problems. These unsolved problems seemed difficult by construction as they either had relatively large coefficients of friction or a relatively large cardinality of the set $\bar{\alpha}$, or even both. To explore these seven problems further, we applied a modified form of our algorithm, in which Step 3 was slightly changed. In both the original and the modified algorithms, Step 3 was only executed when the algorithm reached a stationary point. Then, if an adjacent extreme point could not be found which strictly reduced the product, $x^T y$, one was chosen at random. In the original algorithm, Step 3 was repeated with a randomly chosen pair $(x', y') \in E(x^\nu) \times E(y^\nu)$. The modification of the algorithm that we implemented returned the algorithm to Step 1 in this situation, rather than repeating Step 3. Intuitively, instead of repeatedly searching for "better" adjacent extreme points only, the algorithm initiated the solution of another sequence of linear programs (14) and (15) with possibly an initial increase in the objective value $x^T y$. Three of seven problems not solved by the original algorithm were solved under this new scheme. The * in Table 1 indicates the three problems that were solved by the modified algorithm.

The strategy of restarting the search from an adjacent extreme point proved to be useful in several instances. The solution of Problem 1 was found only after such a restart. Similarly, in Problem 11, this strategy identified a non-complementary pair that yielded a strict decrease in the objective value. Problems 3, 7 and 14 also restarted at adjacent extreme points to reach their solutions. No solutions were found for problems 17 to 20 even though Step 1 was reinvoked 20 times, after which the algorithm was set to terminate. However, we believe that the difficulty in solving Problem 17 was probably a result of numerical inaccuracy in our code.

Problem No.	n_c	Range of μ	$\|\alpha\|$	$\|\bar{\alpha}\|$	No. of pairs (x, y) found	Best $x^T y$
1	3	(0, 1.0)	6	0	2	0
2	10	(0, 1.0)	20	0	1	0
∗3	10	(0, 2.0)	20	0	5	0
4	15	(0, 1.2)	29	1	2	0
5	18	(0, 0.8)	27	9	1	0
6	20	(0, 1.0)	40	0	1	0
∗7	20	(0, 1.5)	40	0	4	0
8	20	(0, 1.2)	40	0	2	0
9	20	(0, 1.0)	30	10	2	0
10	20	(0, 1.0)	30	10	2	0
11	20	(0, 1.0)	40	0	4	0
12	30	(0, 1.0)	60	0	2	0
13	30	(0, 0.5)	45	15	1	0
∗14	30	(0, 0.6)	45	15	24	0
15	40	(0, 1.0)	80	0	1	0
16	40	(0, 1.0)	80	0	1	0
17	10	(0, 1.3)	15	5	52	0.00001
18	15	(0, 1.2)	24	6	39	14.61551
19	30	(0, 0.6)	45	15	49	24.32773
20	40	(0, 1.0)	60	20	52	1.23990

Table 1 : Random problems

In summary, the algorithm and its modification solved sixteen out of the twenty random test problems. At this time we are not certain whether the four unsolved problems actually have a solution (although we suspect that problem 17 might). Overall, the algorithm has been rather successful in solving the quasistatic multi-rigid-body contact problem with Coulomb friction.

The other existing algorithm for solving this problem [11] is an enumerative scheme that tests certain candidate solutions (potentially many) which satisfy some constraints derived from practical engineering concerns. This enumerative scheme has been applied only to simple cases with very small numbers of contacts. For example, in solving problems with $n_c = 8$ contacts and $|\alpha| = 2n_c$, the enumerative algorithm tests 1107 (x, y) pairs. When n_c is increased to 20, the number of pairs tested jumps to 377,379,369. The largest problem solved here had 40 contacts and would require the testing of approximately 10^{17} pairs. The advantage of our solution algorithm discussed here is that a relatively small number of pairs are tested. However, the disadvantages are that our algorithm might not find a solution even if one exists, and it cannot be easily modified to find more than one solution if the solution is not unique. The enumerative algorithm will find multiple (and possibly all) solutions to every problem, but is clearly limited to problems with small numbers of contacts.

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