Complementarity Formulations and Existence of Solutions of Dynamic Multi-Rigid-Body Contact Problems with Coulomb Friction

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Dedication. It gives us great pleasure to dedicate this paper to Professor Richard W. Cottle for his sixtieth birthday on 29 June 1994. Professor Cottle is the founder of the linear complementarity problem. This paper is motivated by an engineering application; like many such applications, complementarity plays a central role in the problem formulation, analysis, and solution. We are indebted to Professor Cottle for the fundamental contributions he has made in this field, for his constant encouragement, advice, and fruitful collaboration over many years. Without his help and guidance, this work would not have been possible.

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Abstract. In this paper, we study the problem of predicting the acceleration of a set of rigid, 3-dimensional bodies in contact with Coulomb friction. The nonlinearity of Coulomb's law leads to a nonlinear complementarity formulation of the system model. This model is used in conjunction with the theory of quasi-variational inequalities to prove for the first time that multi-rigid-body systems with all contacts rolling always has a solution under a feasibility-type condition. The analysis of the more general problem with sliding and rolling contacts presents difficulties that motivate our consideration of a relaxed friction law. The corresponding complementarity formulations of the multi-rigid-body contact problem are derived and existence of solutions of these models is established.

Key Words. Rigid-body contact problem, Coulomb friction, linear complementarity, quasi-variational inequality, set-valued mappings.

1 Introduction

One of the main goals of the robotics research community is to automate the planning of tasks to be carried out by robots. When tasks do not involve contact (*e.g.*, paint-spraying) planning is well understood and provably good planning algorithms exist for several classes of problems [16]. However, tasks involving contact between the robot and its surroundings (*e.g.*, grasping tasks) are more difficult to plan [9, 23]. When contact interactions (*e.g.*, rolling, sliding, or breaking) must change during task execution, the planning problems are even more difficult. (*e.g.*, mechanical assembly and dexterous manipulation tasks). The reason for this is that the "simple" dynamic model commonly used to predict the motion of the multi-rigid-body system is not well understood [4, 7, 8, 20].

Generally speaking, the multi-rigid-body contact problem is concerned with a number of passive rigid bodies of arbitrary shape (*objects*) that move dynamically due to frictional contact with a number of active rigid bodies. The active bodies can be viewed as the links of a *manipulator* joined by actuated revolute and prismatic joints (*e.g.*, a robotic hand). A model of such systems is fundamental to a robot task planner, because it can be used to predict how the robot and the objects it contacts will move from a given state when acted upon by a given input. The crux of the difficulty in planning, is that the multi-rigid-body contact model (for a given input) does not always have a solution, and when it does, it may not be unique [17, 3]. Therefore, plans generated using the model may not accomplish the goal when executed by corresponding real robot system.

As in several previous studies of this problem [17, 3], we focus our attention on the situation in which there are no impacts occurring at the time for which the contact problem is formulated (*i.e.*, the current time). Collisions are assumed to be dealt with as they arise by an appropriate impulse-momentum model, such as the ones studied by Seyfferth and Pfeiffer [30], Marques [21], Moreau [22], and Paoli [27]. Nonetheless, the reader should be aware that even though no impacts are occurring at the current time, our model could predict consistent impulsive contact forces. Indeed, as suggested by Baraff [3] in the 2-dimensional case, when a Lemke-like algorithm is applied to a linear complementarity formulation of the contact problem, termination on an unbounded ray could be interpreted as an impulse.

The objective of this paper is to introduce several useful complementarity formulations of the multi-rigid-body contact problem with Coulomb friction, to present a previously unknown existence result for the all-rolling problem using a theory of quasi-variational inequalities [5], and to discuss a complementarity constrained least-squares formulation for the general rolling-sliding problem. Our results provide strict theoretical guidelines for the applicability of the models, and thus are of great importance in automatic task planning, and more generally in the field of rigid body mechanics. The treatment in this paper is purely analytical; in the accompanying paper [31], we shall report the numerical results of solving the multi-rigid-body contact problem using the various complementarity formulations introduced herein.

2 The Multi-Rigid-Body Contact Model

We refer the reader to [31] for the detailed physical description of the multi-rigid-body contact model with Coulomb friction considered here. In this section, we shall summarize the governing equations of the model and indicate how one can replace Coulomb's nonlinear law with a linear friction pyramid law. Then, in Section 3, we shall present the corresponding complementarity systems.

For the purposes of this paper, we will assume that there is only one object, it may contact any number of manipulator links at any number of points, and the contact forces obey a Coulomb-like friction law (such a law will be made clear when it is introduced later). Our objective, given the actuator torques and forces applied at the joints of manipulator, is to determine the instantaneous accelerations of the object and the manipulator joints, the contact forces, and contact interaction (*e.g.*, rolling, sliding, or breaking) at each contact.

At the time instant for which the model is formulated (*i.e.*, the current time), there are n_c isolated contact points between the manipulator and the object. Let n_q be the object's number of degrees of motion freedom. In formulating the governing equations, it is convenient to define an inertial "world" frame whose origin coincides with the center of gravity of the object, and to assign a "contact" frame at each contact point. For each contact point $i = 1, \dots, n_c$, let the vector $c_i = [c_{in}\ c_{it}\ c_{io}]^T$ represent the (unknown) force applied to the object at contact point i with c_{in} being the component of the contact force in the direction of the contact normal and c_{it} and c_{io} being the components of the force in two orthogonal directions in the tangent plane of the contact; also let $W_i = [w_{in}\ w_{it}\ w_{io}]$ be the wrench matrix which transforms the contact force c_i into the world frame, where w_{in}, w_{it}, and w_{io} are (column) vectors of length n_q. Let W_n, W_t, and W_o be the $(n_q \times n_c)$ matrices with their i-columns equal to w_{in}, w_{it}, and w_{io} respectively; similarly, let c_n, c_t, and c_o be the n_c-vectors whose i-components are c_{in}, c_{it}, and c_{io}, respectively. Let g_{obj} and h_{obj} be the n_q-vectors defining the applied gravitational wrench (force and moment) and velocity product wrench experienced by the object, respectively. Summing all forces and moments in the world frame yields the following dynamic equation:

$$W_n c_n + W_t c_t + W_o c_o + g_{\text{obj}} + h_{\text{obj}} = M_{\text{obj}} \ddot{q}, \tag{1}$$

where M_{obj} is the $(n_q \times n_q)$ positive definite and symmetric inertia matrix of the object and $\ddot{q}$ is the (unknown) n_q-vector of linear and angular accelerations of the center of mass of the object expressed with respect to the world coordinate frame.

A similar dynamic equation holds for the manipulator:

$$\tau - \left[J_n^T c_n + J_t^T c_t + J_o^T c_o + g_{\text{man}} + h_{\text{man}} \right] = M_{\text{man}} \ddot{\theta}. \tag{2}$$

In this equation, τ (given) is the n_θ-vector of joint torques and forces supplied by the joint actuators, with n_θ being the number of joints of the manipulator; J_n^T, J_t^T, and J_o^T are the matrices which transform the contact forces, expressed in their respective contact frames, to equivalent joint torques and forces (torques for revolute joints and forces for prismatic joints); these matrices are of order $(n_\theta \times n_c)$; g_{man} and h_{man} are the n_θ-vectors of joint torques and forces induced by gravity and Coriolis and centripetal accelerations, respectively; M_{man} is the $(n_\theta \times n_\theta)$ positive definite and symmetric inertia matrix of the manipulator; and $\ddot{\theta}$ is the n_θ-vector of (unknown) joint accelerations.

The motion of the object is subject to kinematic acceleration constraints, the satisfaction of which implies that the bodies do not interpenetrate. For $i = 1, \ldots, n_c$, let $a_i = [a_{in}\ a_{it}\ a_{io}]^T$ denote the relative linear acceleration at the i-th contact point, expressed in terms of the i-th contact frame; similar to the definitions of c_n, c_t, and c_o, let a_n, a_t, and a_o denote the n_c-vectors with components a_{in}, a_{it}, and a_{io} respectively, for $i = 1, \ldots, n_c$. We have the following defining equations:

$$a_n = W_n^T \ddot{q} - J_n \ddot{\theta} + \dot{W}_n^T \dot{q} - \dot{J}_n \dot{\theta} \tag{3}$$

$$a_t = W_t^T \ddot{q} - J_t \ddot{\theta} + \dot{W}_t^T \dot{q} - \dot{J}_t \dot{\theta} \tag{4}$$

$$a_o = W_o^T \ddot{q} - J_o \ddot{\theta} + \dot{W}_o^T \dot{q} - \dot{J}_o \dot{\theta}, \tag{5}$$

where the object and joint velocities, $\dot{q}$ and $\dot{\theta}$, are given. The nonpenetration constraint mentioned before is given by the nonnegativity of the vector a_n, *i.e.*:

$$a_n \geq 0. \tag{6}$$

Note that there are no similar requirements on the vectors a_t and a_o.

The remaining constraints enforce Coulomb's law, which stipulates that each contact force lies within or on the boundary of its corresponding friction cone represented as follows:

$$c_{it}^2 + c_{io}^2 \leq \mu_i^2 c_{in}^2, \quad i = 1, \ldots, n_c, \tag{7}$$

where μ_i is the effective nonnegative coefficient of friction at the i-th contact point. Since the contact forces must be nontensile, we have:

$$c_{in} \geq 0, \quad i = 1, \ldots, n_c. \tag{8}$$

Incidentally, it is meaningful for one or more of the constants μ_i to be zero; nevertheless, a zero μ_i clearly renders the corresponding c_{it} and c_{io} equal to zero and the constraint (7) trivially satisfied. For this reason, we shall assume throughout this paper that each μ_i is positive.

By definition, the normal component of the relative velocity at the i-th contact, v_{in}, is zero at the time the model is formulated. Among the n_c contact points, some are *rolling* and others *sliding*; no contacts are breaking at the current time. Specifically, with the tangential components of the relative velocity at the i-th contact, v_{it} and v_{io}, given, a contact point i is rolling if $v_{it}^2 + v_{io}^2 = 0$ and sliding otherwise. Let $\mathcal{R}$ and $\mathcal{S}$ denote respectively the subsets of $\{1, \ldots, n_c\}$ pertaining to the rolling and sliding contacts; $\mathcal{R}$ and $\mathcal{S}$ partition $\{1, \ldots, n_c\}$. For each sliding contact, the following restrictions apply:

$$\mu_i c_{in}(v_{it}, v_{io}) + \sqrt{v_{it}^2 + v_{io}^2}(c_{it}, c_{io}) = 0, \quad \text{for all } i \in \mathcal{S}, \tag{9}$$

For each rolling contact, similar restrictions apply:

$$\mu_i c_{in}(a_{it}, a_{io}) + \sqrt{a_{it}^2 + a_{io}^2}(c_{it}, c_{io}) = 0, \quad \text{for all } i \in \mathcal{R}. \tag{10}$$

Mathematically, the difference between the sliding constraint (9) and the rolling constraint (10) is that in the former, the velocities v_{it} and v_{io} are known, whereas in the latter, the accelerations a_{it} and a_{io} are unknown. Since $v_{it}^2 + v_{io}^2 > 0$ for $i \in \mathcal{S}$, (9) completely determines the tangential forces c_{it} and c_{io} for a sliding contact $i \in \mathcal{S}$ in terms of the corresponding normal force c_{in} via the expressions:

$$c_{it} = -\frac{\mu_i v_{it}}{\sqrt{v_{it}^2 + v_{io}^2}} c_{in} \quad \text{and} \quad c_{io} = -\frac{\mu_i v_{io}}{\sqrt{v_{it}^2 + v_{io}^2}} c_{in}. \tag{11}$$

Similar expressions exist for a rolling contact $i \in \mathcal{R}$ that has a nonzero tangential acceleration component. (This will occur if the contact interaction is about to change from rolling to sliding.) Notice that there is no additional restriction on the tangential components of the accelerations, a_{it} and a_{io}, for all $i \in \mathcal{S}$. Moreover, for each sliding contact $i \in \mathcal{S}$, the contact force lies on the boundary of the friction cone; that is, the triple, $(c_{in}\ c_{it}\ c_{io})$, satisfies the constraint (7) as an equation. The same conclusion holds for a rolling contact $i \in \mathcal{R}$ with a nonzero tangential acceleration component; equivalently stated, for any rolling contact i, the triple $(c_{in}\ c_{it}\ c_{io})$ lies in the interior of the friction cone only if $a_{it} = a_{io} = 0$.

To complete the formulation of the model, we must stipulate the following complementarity condition at all contact points:

$$c_{in} a_{in} = 0, \quad \text{for all } i = 1, \ldots, n_c. \tag{12}$$

To summarize, the dynamic 3-dimensional multi-rigid-body contact problem with Coulomb friction is to find vectors $c_n, c_t, c_o, a_n, a_t, a_o, \ddot{q}$, and $\ddot{\theta}$ satisfying conditions (1) to (10) and (12). A solution to this problem yields four possible types of contact interaction transitions classified as follows:

(i) rolling → rolling:

$$c_{in} \geq 0, \quad a_{in} = a_{it} = a_{io} = 0; \tag{13}$$

(ii) rolling → sliding:

$$c_{in} \geq 0, \quad a_{in} = 0, \quad a_{it}^2 + a_{io}^2 \neq 0, \tag{14}$$

$$c_{it} = -\frac{\mu_i a_{it}}{\sqrt{a_{it}^2 + a_{io}^2}} c_{in}, \quad c_{io} = -\frac{\mu_i a_{io}}{\sqrt{a_{it}^2 + a_{io}^2}} c_{in}; \tag{15}$$

(iii) sliding → sliding:

$$c_{in} \geq 0, \quad a_{in} = 0; \tag{16}$$

(iv) rolling or sliding → breaking:

$$a_{in} \geq 0, \quad c_{in} = 0, \quad (\Rightarrow c_{it} = c_{io} = 0). \tag{17}$$

The constraints on the normal components of the contact forces and accelerations, (6), (8), and (12), naturally lead one to suspect that the above model can be formulated as a certain complementarity problem. Nevertheless, the rolling constraints (10) cause complications. Indeed, as we shall see later, the resulting complementarity formulation is not of the standard type. Moreover, standard existence results from complementarity theory [24] are not applicable.

The friction pyramid model

The quadratic expressions (7) defining the friction cones and the corresponding relations (10) between the normal and tangential components of the forces at the rolling contacts cause the above model to be nonlinear. This fact motivates the second formulation in which the friction cones, at *only* the rolling contacts, are replaced by four-sided friction pyramids; as we shall see, the resulting model leads to a standard linear complementarity problem (LCP).

More specifically, the alternative, *linear*, dynamic multi-rigid-body contact model with Coulomb friction pyramid constraints is obtained by replacing the nonlinear friction constraints (7) and (10) by the following conditions:

$$\max(|c_{it}|, |c_{io}|) \leq \mu_i c_{in}, \quad \text{for all } i \in \mathcal{R}, \tag{18}$$

$$\mu_i c_{in}(|a_{it}|, |a_{io}|) + (a_{it} c_{it}, a_{io} c_{io}) = 0, \quad \text{for all } i \in \mathcal{R}. \tag{19}$$

The resulting problem is to find vectors $c_n, c_t, c_o, a_n, a_t, a_o, \ddot{q}$, and $\ddot{\theta}$ satisfying conditions (1) to (6), (8), (9), (12), (18), and (19). We stress that the usual nonlinear Coulomb friction Law is still applicable at the sliding contacts through equation (9).

A variation of the problem

In the above description of the multi-rigid-body problem, we have taken as input the entire vector of joint torques τ and treated the corresponding vector of joint accelerations $\ddot{\theta}$ as an unknown. In certain situations, only a subset $\alpha \subseteq \{1, \ldots, n_\theta\}$ of the torque components $\tau_i (i \in \alpha)$ and the complementary acceleration components $\ddot{\theta}_j (j \notin \alpha)$ are given, and the unknown torque components $\tau_j (j \notin \alpha)$ and acceleration components $\ddot{\theta}_i (i \in \alpha)$ then become part of the variables to be computed. This variant of the model can be treated in the same way as the original version; we will focus on the treatment of the latter only, (*i.e.*, the case for which $\alpha = \{1, \ldots, n_\theta\}$) .

3 Complementarity Formulations

It is known that many contact problems can be formulated as complementarity problems; see [1, 3, 11, 12, 14, 15, 17, 18, 30]. While the model with Coulomb friction cones described above does not lend itself to a standard nonlinear complementarity problem (NCP) formulation, the friction pyramid model can be formulated as a standard LCP [6].

The starting point in obtaining a complementarity formulation for both the friction cone and pyramid models is to eliminate the acceleration vectors of the object and the joints, $\ddot{q}$ and $\ddot{\theta}$, via the equations (1) and (2). Such elimination results in the following expressions:

$$\ddot{q} = M_{\text{obj}}^{-1} \left[W_n c_n + W_t c_t + W_o c_o + g_{\text{obj}} + h_{\text{obj}} \right],$$

$$\ddot{\theta} = M_{\text{man}}^{-1} \left[\tau - J_n^T c_n - J_t^T c_t - J_o^T c_o - g_{\text{man}} - h_{\text{man}} \right].$$

Substituting these expressions into equations (3), (4), and (5), we obtain:

$$\begin{bmatrix} a_n \\ a_t \\ a_o \end{bmatrix} = A \begin{bmatrix} c_n \\ c_t \\ c_o \end{bmatrix} + \begin{bmatrix} b_n \\ b_t \\ b_o \end{bmatrix}, \tag{20}$$

where

$$
\begin{aligned}
A &= \begin{bmatrix} A_{nn} & A_{nt} & A_{no} \\ A_{tn} & A_{tt} & A_{to} \\ A_{on} & A_{ot} & A_{oo} \end{bmatrix} \\
&\equiv \begin{bmatrix} W_n^T & J_n \\ W_t^T & J_t \\ W_o^T & J_o \end{bmatrix} \begin{bmatrix} M_{\text{obj}}^{-1} & 0 \\ 0 & M_{\text{man}}^{-1} \end{bmatrix} \begin{bmatrix} W_n & W_t & W_o \\ J_n^T & J_t^T & J_o^T \end{bmatrix},
\end{aligned}
\tag{21}
$$

$$
\begin{aligned}
\begin{bmatrix} b_n \\ b_t \\ b_o \end{bmatrix} &\equiv \begin{bmatrix} \dot{W}_n^T & \dot{J}_n \\ \dot{W}_t^T & \dot{J}_t \\ \dot{W}_o^T & \dot{J}_o \end{bmatrix} \begin{bmatrix} \dot{q} \\ -\dot{\theta} \end{bmatrix} \\
&+ \begin{bmatrix} W_n^T & J_n \\ W_t^T & J_t \\ W_o^T & J_o \end{bmatrix} \begin{bmatrix} M_{\text{obj}}^{-1} & 0 \\ 0 & M_{\text{man}}^{-1} \end{bmatrix} \begin{bmatrix} g_{\text{obj}} + h_{\text{obj}} \\ g_{\text{man}} + h_{\text{man}} - \tau \end{bmatrix}.
\end{aligned}
\tag{22}
$$

Notice that the matrix A is positive semidefinite and symmetric. Moreover, the single equation (20) is equivalent to the equations (1) to (5).

Next, we can use the expressions in (11) to eliminate the tangential forces corresponding to the sliding contacts, c_{it} and c_{io} for $i \in \mathcal{S}$. Since there are no additional restrictions on the corresponding accelerations, a_{it} and a_{io} for $i \in \mathcal{S}$, the equations in (20) that define these components can be dropped from the model without affecting its solvability. Carrying out this reduction results in:

$$
\begin{bmatrix} a_{\mathcal{S}n} \\ a_{\mathcal{R}n} \\ a_{\mathcal{R}t} \\ a_{\mathcal{R}o} \end{bmatrix} = \tilde{A} \begin{bmatrix} c_{\mathcal{S}n} \\ c_{\mathcal{R}n} \\ c_{\mathcal{R}t} \\ c_{\mathcal{R}o} \end{bmatrix} + \begin{bmatrix} b_{\mathcal{S}n} \\ b_{\mathcal{R}n} \\ b_{\mathcal{R}t} \\ b_{\mathcal{R}o} \end{bmatrix}, \tag{23}
$$

where

$$\tilde{A} = \begin{bmatrix} (\tilde{A}_{nn})_{\mathcal{SS}} & (A_{nn})_{\mathcal{SR}} & (A_{nt})_{\mathcal{SR}} & (A_{no})_{\mathcal{SR}} \\ (\tilde{A}_{nn})_{\mathcal{RS}} & (A_{nn})_{\mathcal{RR}} & (A_{nt})_{\mathcal{RR}} & (A_{no})_{\mathcal{RR}} \\ (\tilde{A}_{tn})_{\mathcal{RS}} & (A_{tn})_{\mathcal{RR}} & (A_{tt})_{\mathcal{RR}} & (A_{to})_{\mathcal{RR}} \\ (\tilde{A}_{on})_{\mathcal{RS}} & (A_{on})_{\mathcal{RR}} & (A_{ot})_{\mathcal{RR}} & (A_{oo})_{\mathcal{RR}} \end{bmatrix}. \tag{24}$$

Here

$$\begin{bmatrix} (\tilde{A}_{nn})_{\mathcal{SS}} \\ (\tilde{A}_{nn})_{\mathcal{RS}} \\ (\tilde{A}_{tn})_{\mathcal{RS}} \\ (\tilde{A}_{on})_{\mathcal{RS}} \end{bmatrix} \equiv \begin{bmatrix} (A_{nn})_{\mathcal{SS}} \\ (A_{nn})_{\mathcal{RS}} \\ (A_{tn})_{\mathcal{RS}} \\ (A_{on})_{\mathcal{RS}} \end{bmatrix} - \begin{bmatrix} (A_{nt})_{\mathcal{SS}} \\ (A_{nt})_{\mathcal{RS}} \\ (A_{tt})_{\mathcal{RS}} \\ (A_{ot})_{\mathcal{RS}} \end{bmatrix} V_{\mathcal{S}t} - \begin{bmatrix} (A_{no})_{\mathcal{SS}} \\ (A_{no})_{\mathcal{RS}} \\ (A_{to})_{\mathcal{RS}} \\ (A_{oo})_{\mathcal{RS}} \end{bmatrix} V_{\mathcal{S}o},$$

$V_{\mathcal{S}t}$ and $V_{\mathcal{S}o}$ are, respectively, the diagonal matrices with diagonal entries given by $\mu_i v_{it}/\sqrt{v_{it}^2 + v_{io}^2}$ and $\mu_i v_{io}/\sqrt{v_{it}^2 + v_{io}^2}$ for $i \in \mathcal{S}$; and in general, for an $N \times N$ matrix M, if α and β are subsets of $\{1, \ldots, N\}$, $M_{\alpha\beta}$ denotes the submatrix of M consisting of rows and columns indexed by α and β respectively.

The model with the friction pyramid law

Given the development so far, it is clear that Coulomb's friction law has already been applied at all sliding contacts. The friction pyramid law will only be applied at the rolling contacts. To derive the linear complementarity formulation, we adopt the technique used in [26] for a planar quasistatic model under the same friction law. Specifically, for each rolling contact $i \in \mathcal{R}$, we define:

$$\begin{aligned} s_{it}^+ &\equiv \mu_i c_{in} + c_{it}, \quad s_{it}^- \equiv \mu_i c_{in} - c_{it}, \\ s_{io}^+ &\equiv \mu_i c_{in} + c_{io}, \quad s_{io}^- \equiv \mu_i c_{in} - c_{io}; \end{aligned} \tag{25}$$

$$\begin{aligned} a_{it}^+ &\equiv \max(0, a_{it}), \quad a_{it}^- \equiv \max(0, -a_{it}), \\ a_{io}^+ &\equiv \max(0, a_{io}), \quad a_{io}^- \equiv \max(0, -a_{io}). \end{aligned} \tag{26}$$

Note that (25) implies

$$s_{it}^+ + s_{it}^- = 2\mu_i c_{in} = s_{io}^+ + s_{io}^-. \tag{27}$$

The following lemma is key to the LCP formulation of the rigid body problem with the friction pyramid constraints.

Lemma 1 *If (18) and (19) are satisfied by some* $(c_{in}, c_{it}, c_{io}, a_{it}, a_{io})$*, then* $(s_{it}^{\pm}, s_{io}^{\pm}, a_{it}^{\pm}, a_{io}^{\pm})$*, defined by (25) and (26) satisfy*

$$\begin{aligned} (s_{it}^{\pm}, s_{io}^{\pm}, a_{it}^{\pm}, a_{io}^{\pm}) \geq 0, \\ a_{it}^{+} s_{it}^{+} = a_{it}^{-} s_{it}^{-} = a_{io}^{+} s_{io}^{+} = a_{io}^{-} s_{io}^{-} = 0. \end{aligned} \tag{28}$$

Conversely, if (28) and (25), but not necessarily (26), hold for some tuple $(c_{in}, c_{it}, c_{io}, a_{it}^{\pm}, a_{io}^{\pm}, s_{it}^{\pm}, s_{io}^{\pm})$*, then (18) and (19) hold for* $(c_{in}, c_{it}, c_{io}, a_{it}, a_{io})$*, where*

$$a_{it} \equiv a_{it}^{+} - a_{it}^{-}, \quad a_{io} \equiv a_{io}^{+} - a_{io}^{-}. \tag{29}$$

Proof. This is straightforward. Q.E.D.

We may now use the expressions (25) and (29) to replace the free variables c_{it}, c_{io}, a_{it}, and a_{io} by the nonnegative variables $(s_{it}^{\pm}, s_{io}^{\pm}, a_{it}^{\pm}, a_{io}^{\pm})$. Making this substitution into the equation (23), rearranging terms, and including (27), we obtain an equation that is equivalent to (23) and (25) combined:

$$\begin{bmatrix} a_{\mathcal{S}n} \\ a_{\mathcal{R}n} \\ a_{\mathcal{R}t}^{+} \\ a_{\mathcal{R}o}^{+} \\ s_{\mathcal{R}t}^{-} \\ s_{\mathcal{R}o}^{-} \end{bmatrix} = M \begin{bmatrix} c_{\mathcal{S}n} \\ c_{\mathcal{R}n} \\ s_{\mathcal{R}t}^{+} \\ s_{\mathcal{R}o}^{+} \\ a_{\mathcal{R}t}^{-} \\ a_{\mathcal{R}o}^{-} \end{bmatrix} + \begin{bmatrix} b_{\mathcal{S}n} \\ b_{\mathcal{R}n} \\ b_{\mathcal{R}t} \\ b_{\mathcal{R}o} \\ 0 \\ 0 \end{bmatrix}, \tag{30}$$

where

$$M \equiv \begin{bmatrix} (M_{nn})_{\mathcal{S}\mathcal{S}} & (M_{nn})_{\mathcal{S}\mathcal{R}} & (A_{nt})_{\mathcal{S}\mathcal{R}} & (A_{no})_{\mathcal{S}\mathcal{R}} & 0 & 0 \\ (M_{nn})_{\mathcal{R}\mathcal{S}} & (M_{nn})_{\mathcal{R}\mathcal{R}} & (A_{nt})_{\mathcal{R}\mathcal{R}} & (A_{no})_{\mathcal{R}\mathcal{R}} & 0 & 0 \\ (M_{tn})_{\mathcal{R}\mathcal{S}} & (M_{tn})_{\mathcal{R}\mathcal{R}} & (A_{tt})_{\mathcal{R}\mathcal{R}} & (A_{to})_{\mathcal{R}\mathcal{R}} & I & 0 \\ (M_{on})_{\mathcal{R}\mathcal{S}} & (M_{on})_{\mathcal{R}\mathcal{R}} & (A_{ot})_{\mathcal{R}\mathcal{R}} & (A_{oo})_{\mathcal{R}\mathcal{R}} & 0 & I \\ 0 & 2U_{\mathcal{R}} & -I & 0 & 0 & 0 \\ 0 & 2U_{\mathcal{R}} & 0 & -I & 0 & 0 \end{bmatrix}, \tag{31}$$

with $U_{\mathcal{R}}$ being the diagonal matrix with diagonal entries $\mu_i, i \in \mathcal{R}$, and

$$\begin{bmatrix} (M_{nn})_{\mathcal{SS}} & (M_{nn})_{\mathcal{SR}} \\ (M_{nn})_{\mathcal{RS}} & (M_{nn})_{\mathcal{RR}} \\ (M_{tn})_{\mathcal{RS}} & (M_{tn})_{\mathcal{RR}} \\ (M_{on})_{\mathcal{RS}} & (M_{on})_{\mathcal{RR}} \end{bmatrix} \equiv$$

$$\begin{bmatrix} (A_{nn})_{\mathcal{SS}} & (A_{nn})_{\mathcal{SR}} \\ (A_{nn})_{\mathcal{RS}} & (A_{nn})_{\mathcal{RR}} \\ (A_{tn})_{\mathcal{RS}} & (A_{tn})_{\mathcal{RR}} \\ (A_{on})_{\mathcal{RS}} & (A_{on})_{\mathcal{RR}} \end{bmatrix} - \begin{bmatrix} (A_{nt})_{\mathcal{SS}} & (A_{nt})_{\mathcal{SR}} \\ (A_{nt})_{\mathcal{RS}} & (A_{nt})_{\mathcal{RR}} \\ (A_{tt})_{\mathcal{RS}} & (A_{tt})_{\mathcal{RR}} \\ (A_{ot})_{\mathcal{RS}} & (A_{ot})_{\mathcal{RR}} \end{bmatrix} \begin{bmatrix} V_{\mathcal{S}t} & 0 \\ 0 & U_{\mathcal{R}} \end{bmatrix} -$$

$$\begin{bmatrix} (A_{no})_{\mathcal{SS}} & (A_{no})_{\mathcal{SR}} \\ (A_{no})_{\mathcal{RS}} & (A_{no})_{\mathcal{RR}} \\ (A_{to})_{\mathcal{RS}} & (A_{to})_{\mathcal{RR}} \\ (A_{oo})_{\mathcal{RS}} & (A_{oo})_{\mathcal{RR}} \end{bmatrix} \begin{bmatrix} V_{\mathcal{S}o} & 0 \\ 0 & U_{\mathcal{R}} \end{bmatrix}.$$

Together with the nonnegativity and complementarity conditions: (6), (8), (12), and (28), the equation (30) defines an LCP with the matrix M given by (31) and the constant vector as given in (30); see also (22).

We summarize the above derivation in the following result.

Theorem 1 *The dynamic 3-dimensional multi-rigid-body contact problem with the friction pyramid laws (18) and (19) applied at the rolling contacts is equivalent to the LCP defined by the equation (30) and the nonnegativity and complementarity of the variables: (6), (8), (12), and (28).*

It should be clarified that an arbitrary solution to the equivalent LCP need not satisfy $(a^+_{\mathcal{R}t})^T a^-_{\mathcal{R}t} = 0$ or $(a^+_{\mathcal{R}o})^T a^-_{\mathcal{R}o} = 0$; nevertheless, any such solution must yield a solution to the contact problem, according to the converse part of Lemma 1.

In principle, the LCP stated in Theorem 1 is amenable to numerical solution by the well-known Lemke's almost complementary pivotal algorithm. Nevertheless, the results pertaining to the ray termination of this algorithm as summarized in [6, Section 4.4] are not applicable. Indeed, the matrix M

of this LCP does not seem to belong to any known class of matrices existing in the linear complementarity literature. As a consequence, while Lemke's algorithm may find a solution, it is not guaranteed to do so whenever a solution exists. In the accompanying paper [31], we have obtained sufficient conditions on the data of the model which ensure the successful termination of Lemke's algorithm for computing a solution of this LCP.

The model with Coulomb's law

We consider applying Coulomb's law (*i.e.*, the friction cone law) at the rolling contacts. Thus equations (7) and (10) replace (18) and (19). With the cone law, the basic equation remains (23). We write:

$$\begin{aligned} a_{\mathcal{R}t} &= a^+_{\mathcal{R}t} - a^-_{\mathcal{R}t}, \quad a_{\mathcal{R}o} = a^+_{\mathcal{R}o} - a^-_{\mathcal{R}o}, \\ c_{\mathcal{R}t} &= c^+_{\mathcal{R}t} - c^-_{\mathcal{R}t}, \quad c_{\mathcal{R}o} = c^+_{\mathcal{R}o} - c^-_{\mathcal{R}o}, \end{aligned}$$

where the superscripts "+" and "−" refer, respectively, to the nonnegative and nonpositive parts of the variables. Substituting these expressions into (23) and rearranging terms, we obtain the following equation:

$$\begin{bmatrix} a_{\mathcal{S}n} \\ a_{\mathcal{R}n} \\ a^+_{\mathcal{R}t} \\ a^+_{\mathcal{R}o} \end{bmatrix} = B \begin{bmatrix} c_{\mathcal{S}n} \\ c_{\mathcal{R}n} \\ c^+_{\mathcal{R}t} \\ c^-_{\mathcal{R}t} \\ c^+_{\mathcal{R}o} \\ c^-_{\mathcal{R}o} \\ a^-_{\mathcal{R}t} \\ a^-_{\mathcal{R}o} \end{bmatrix} + \begin{bmatrix} b_{\mathcal{S}n} \\ b_{\mathcal{R}n} \\ b_{\mathcal{R}t} \\ b_{\mathcal{R}o} \end{bmatrix}, \qquad (32)$$

where B is the matrix given by

$$\begin{bmatrix} (\tilde{A}_{nn})_{\mathcal{S}\mathcal{S}} & (A_{nn})_{\mathcal{S}\mathcal{R}} & (A_{nt})_{\mathcal{S}\mathcal{R}} & -(A_{nt})_{\mathcal{S}\mathcal{R}} & (A_{no})_{\mathcal{S}\mathcal{R}} & -(A_{no})_{\mathcal{S}\mathcal{R}} & 0 & 0 \\ (\tilde{A}_{nn})_{\mathcal{R}\mathcal{S}} & (A_{nn})_{\mathcal{R}\mathcal{R}} & (A_{nt})_{\mathcal{R}\mathcal{R}} & -(A_{nt})_{\mathcal{R}\mathcal{R}} & (A_{no})_{\mathcal{R}\mathcal{R}} & -(A_{no})_{\mathcal{R}\mathcal{R}} & 0 & 0 \\ (\tilde{A}_{tn})_{\mathcal{R}\mathcal{S}} & (A_{tn})_{\mathcal{R}\mathcal{R}} & (A_{tt})_{\mathcal{R}\mathcal{R}} & -(A_{tt})_{\mathcal{R}\mathcal{R}} & (A_{to})_{\mathcal{R}\mathcal{R}} & -(A_{oo})_{\mathcal{R}\mathcal{R}} & -I & 0 \\ (\tilde{A}_{on})_{\mathcal{R}\mathcal{S}} & (A_{on})_{\mathcal{R}\mathcal{R}} & (A_{ot})_{\mathcal{R}\mathcal{R}} & -(A_{ot})_{\mathcal{R}\mathcal{R}} & (A_{oo})_{\mathcal{R}\mathcal{R}} & -(A_{oo})_{\mathcal{R}\mathcal{R}} & 0 & -I \end{bmatrix}.$$

The following lemma is key to the transformation of the friction cone constraints (7) and (10) into a complementarity system.

Lemma 2 *For each $i \in \mathcal{R}$, if $(c_{in}, c_{it}, c_{io}, a_{it}, a_{io})$, with $c_{in} \geq 0$, satisfies (7) and (10), then $(c_{in}, c_{it}^{\pm}, c_{io}^{\pm}, a_{it}^{\pm}, a_{io}^{\pm})$, with the superscript $\pm$ denoting the non-negative and nonpositive parts of the quantities, satisfies:*

$$\min\left(\mu_i^2 c_{in}^2 - (c_{it}^+ - c_{it}^-)^2 - (c_{io}^+ - c_{io}^-)^2,\ (a_{it}^+ - a_{it}^-)^2 + (a_{io}^+ - a_{io}^-)^2\right) = 0$$
$$(a_{it}^+ - a_{it}^-)(c_{io}^+ - c_{io}^-) - (a_{io}^+ - a_{io}^-)(c_{it}^+ - c_{it}^-) = 0$$
$$(a_{it}^+)c_{it}^+ = (a_{it}^-)c_{it}^- = (a_{io}^+)c_{io}^+ = (a_{io}^-)c_{io}^- = 0.$$

Conversely if arbitrary nonnegative numbers $(c_{in}, c_{it}^{\pm}, c_{io}^{\pm}, a_{it}^{\pm}, a_{io}^{\pm})$ satisfy the latter three equations, then $(c_{in}, c_{it}, c_{io}, a_{it}, a_{io})$, where

$$c_{it} \equiv c_{it}^+ - c_{it}^-, \quad c_{io} \equiv c_{io}^+ - c_{io}^-,$$
$$a_{it} \equiv a_{it}^+ - a_{it}^-, \quad a_{io} \equiv a_{io}^+ - a_{io}^-,$$

satisfies (7) and (10).

Proof. The equivalence clearly holds if $c_{in} = 0$ or $(a_{it}^+ - a_{it}^-)^2 + (a_{io}^+ - a_{io}^-)^2 = 0$. If c_{in} and $(a_{it}^+ - a_{it}^-)^2 + (a_{io}^+ - a_{io}^-)^2$ are both positive, the verification is not hard either. We omit the details. Q.E.D.

Based on the above lemma, we obtain an equivalent formulation of the dynamic 3-dimensional multi-rigid-body contact problem with Coulomb friction cone constraints. In the following theorem, we employ the notation of the Hadamard product of two vectors: $x \circ y$ is the vector whose i-th component is $x_i y_i$ for all i.

Theorem 2 *The dynamic 3-dimensional multi-rigid-body contact problem with Coulomb friction cone constraints is equivalent to the problem of finding vectors $(a_n, c_n, a_{\mathcal{R}t}^{\pm}, a_{\mathcal{R}o}^{\pm}, c_{\mathcal{R}t}^{\pm}, c_{\mathcal{R}o}^{\pm})$ satisfying (32), and*

$$\begin{aligned} \min(a_n, c_n) &= 0 \\ \min(a_{\mathcal{R}t}^+, c_{\mathcal{R}t}^+) &= \min(a_{\mathcal{R}o}^+, c_{\mathcal{R}o}^+) = 0 \\ \min(a_{\mathcal{R}t}^-, c_{\mathcal{R}t}^-) &= \min(a_{\mathcal{R}o}^-, c_{\mathcal{R}o}^-) = 0 \\ (a_{\mathcal{R}t}^+ - a_{\mathcal{R}t}^-) \circ (c_{\mathcal{R}o}^+ - c_{\mathcal{R}o}^-) &- (a_{\mathcal{R}o}^+ - a_{\mathcal{R}o}^-) \circ (c_{\mathcal{R}t}^+ - c_{\mathcal{R}t}^-) = 0 \end{aligned} \tag{33}$$

$$\min\left(\mu_{\mathcal{R}}^2 c_{\mathcal{R}n}^2 - (c_{\mathcal{R}t}^+ - c_{\mathcal{R}t}^-)^2 - (c_{\mathcal{R}o}^+ - c_{\mathcal{R}o}^-)^2,\ (a_{\mathcal{R}t}^+ - a_{\mathcal{R}t}^-)^2 + (a_{\mathcal{R}o}^+ - a_{\mathcal{R}o}^-)^2\right) = 0,$$

where the subscript $\mathcal{R}$ in the last equation means that this equation has to hold for all components $i \in \mathcal{R}$.

The system (33), including (32), is a complementarity problem of a special kind. It is not a standard NCP because of the last two equations in (33). In their present forms, (32) and (33) constitute a (square) system of nonsmooth equations involving the "min" function. As such, it is amenable to numerical solution by the (nonsmooth) Gauss-Newton method described in [25]. An extensive study of the computational aspects of solving the above nonlinear rigid body problem is currently under way.

4 The All-Rolling Case

So far, we have described two complementarity formulations of the dynamic 3-dimensional multi-rigid-body contact problem: one with Coulomb's law applying at all contacts and the other with Coulomb's friction cone approximated by a friction pyramid at the rolling contacts. We expect these formulations to be instrumental in the computation of a solution to the contact problem. In general, there is a difficulty in establishing the existence of a solution using these formulations. This is due to the presence of the sliding constraints (9). As an illustration of the difficulty, consider the matrix

$$\begin{bmatrix} (\tilde{A}_{nn})_{\mathcal{S}\mathcal{S}} & (A_{nn})_{\mathcal{S}\mathcal{R}} \\ (\tilde{A}_{nn})_{\mathcal{R}\mathcal{S}} & (A_{nn})_{\mathcal{R}\mathcal{R}} \end{bmatrix}$$

which occurs as a leading principal submatrix of $\tilde{A}$ associated with the normal forces c_n in (23). Although the matrix A in (21) is positive semidefinite, the above (2×2) block matrix does not seem to have any viable property that can be put to use.

A quasi-variational inequality formulation

As a first step toward the investigation of solution existence, we shall consider a special case of the multi-rigid-body model with Coulomb friction cones in which all contacts are initially rolling; that is, we assume $\mathcal{S} = \emptyset$. We call this the *all-rolling problem.* The analysis in this section makes use of some basic results from linear complementarity theory; the reader is referred to [6] for a comprehensive study of the linear complementarity problem. Results pertaining to the LCP with a symmetric positive semidefinite matrix are most relevant below; see Section 3.1 in the cited book for a summary of these background results.

In the all-rolling problem, we shall work with the full system (20) and take advantage of the special structure of the matrix A. This matrix is symmetric and positive semidefinite; as such A satisfies

$$x^T A x = 0 \Leftrightarrow Ax = 0;$$

indeed, we have

$$A \begin{bmatrix} c_n \\ c_t \\ c_o \end{bmatrix} = 0 \Leftrightarrow \begin{bmatrix} W_n & W_t & W_o \\ J_n^T & J_t^T & J_o^T \end{bmatrix} \begin{bmatrix} c_n \\ c_t \\ c_o \end{bmatrix} = 0.$$

In order to develop an existence theory for the all-rolling problem, we introduce one important assumption. First, formally define the friction cone:

$$\mathcal{F} \equiv \{(c_n, c_t, c_o) \in R_+^{n_c} \times R^{2n_c} : \text{ (7) holds}\}.$$

Also, let $\mathcal{N} \subseteq R^{3n_c}$ denote the null space of the matrix

$$\begin{bmatrix} W_n & W_t & W_o \\ J_n^T & J_t^T & J_o^T \end{bmatrix}, \tag{34}$$

and let $\mathcal{F}_{\mathcal{R}} \equiv \mathcal{F} \cap \mathcal{N}$. The assumption is:

$$\begin{bmatrix} \dot{q} \\ -\dot{\theta} \end{bmatrix}^T \begin{bmatrix} \dot{W}_n & \dot{W}_t & \dot{W}_o \\ \dot{J}_n^T & \dot{J}_t^T & \dot{J}_o^T \end{bmatrix} \begin{bmatrix} c_n \\ c_t \\ c_o \end{bmatrix} \geq 0, \quad \forall (c_n, c_t, c_0) \in \mathcal{F}_{\mathcal{R}}. \tag{35}$$

Equivalently, by the definition of the vector (b_n, b_t, b_o) in (22), this assumption can be rephrased as:

$$\begin{bmatrix} b_n \\ b_t \\ b_o \end{bmatrix}^T \begin{bmatrix} c_n \\ c_t \\ c_o \end{bmatrix} \geq 0, \quad \forall (c_n, c_t, c_0) \in \mathcal{F}_{\mathcal{R}}.$$

Inequality (35) holds if $\mathcal{F}_{\mathcal{R}} = \{0\}$. This special case includes the situation in which the matrix (34) has full column rank.

The assumption (35) is motivated by the frictionless case in which the friction coefficients μ_i are all zero. Indeed, in this case, any triple (c_n, c_t, c_o) in the friction cone $\mathcal{F}$ must have $c_t = c_o = 0$. Thus, the above assumption becomes

$$\left\{ c_n \geq 0, \quad \begin{bmatrix} W_n \\ J_n^T \end{bmatrix} c_n = 0 \right\} \Rightarrow \begin{bmatrix} \dot{q} \\ -\dot{\theta} \end{bmatrix}^T \begin{bmatrix} \dot{W}_n \\ \dot{J}_n^T \end{bmatrix} c_n \geq 0. \qquad (36)$$

Moreover, the rigid-body contact problem reduces to the standard LCP:

$$a_n = A_{nn} c_n + b_n \geq 0, \quad c_n \geq 0, \quad (a_n)^T c_n = 0, \qquad (37)$$

with the matrix A_{nn} and vector b_n given in (21) and (22) respectively, it follows from elementary LCP theory [6, Section 3.1] that (37) has a solution if and only if the implication (36) holds.

Returning to the all-rolling problem with positive friction coefficients, we note that the implication (36) remains valid under the assumption (35). Indeed, given any c_n satisfying the left-hand conditions in (36), the triple $(c_n, 0, 0) \in \mathcal{F}_{\mathcal{R}}$. Thus (36) follows from (35). In turn, (36) implies that for any pair $(c_t, c_o) \in R^{2n_c}$, the following LCP in the variable c_n:

$$a_n = A_{nn} c_n + A_{nt} c_t + A_{no} c_o + b_n \geq 0, \quad c_n \geq 0, \quad (a_n)^T c_n = 0,$$

has a nonempty convex solution set which we denote $\mathrm{SOL}(c_t, c_0) \subseteq R^{n_c}$. By the special structure of

$$A_{nn} = \begin{bmatrix} W_n^T & J_n \end{bmatrix} \begin{bmatrix} M_{\mathrm{obj}}^{-1} & 0 \\ 0 & M_{\mathrm{man}}^{-1} \end{bmatrix} \begin{bmatrix} W_n \\ J_n^T \end{bmatrix},$$

it is not difficult to show that for any two solutions $c_n, c_n' \in \mathrm{SOL}(c_t, c_o)$, we must have

$$\begin{bmatrix} W_n \\ J_n^T \end{bmatrix} (c_n - c_n') = 0.$$

This implies that

$$\begin{bmatrix} a_t \\ a_o \end{bmatrix} \equiv \begin{bmatrix} A_{tn} & A_{tt} & A_{to} \\ A_{on} & A_{ot} & A_{oo} \end{bmatrix} \begin{bmatrix} c_n \\ c_t \\ c_o \end{bmatrix} + \begin{bmatrix} b_t \\ b_o \end{bmatrix}$$

is independent of the solution $c_n \in \mathrm{SOL}(c_t, c_o)$; we let $F(c_t, c_o)$ denote this common vector. Thus F can be considered a mapping from R^{2n_c} into itself. We note that the set $\mathrm{SOL}(c_t, c_o)$ reduces to a singleton if A_{nn} is positive definite, or equivalently, if

$$\begin{bmatrix} W_n \\ J_n^T \end{bmatrix} \tag{38}$$

has linearly independent columns.

The following lemma summarizes two important properties of the mapping F.

Proposition 1 *The mapping F as defined above is continuous and monotone. Moreover, if the matrix A is positive definite, then F is strongly monotone.*

Proof. To show the continuity of F, let a sequence $\{(c_t^\nu, c_o^\nu)\}$ converge to (c_t^*, c_o^*). For each ν, let $c_n^\nu \in \mathrm{SOL}(c_t^\nu, c_o^\nu)$. Since the solution set of an LCP is a locally upper Lipschitzian multifunction of the constant vector [29], it follows that there exists a constant $L > 0$ such that for each ν sufficiently large, there exists $\tilde{c}_n^\nu \in \mathrm{SOL}(c_t^*, c_o^*)$ such that

$$\|c_n^\nu - \tilde{c}_n^\nu\| \leq L\|(c_t^\nu, c_o^\nu) - (c_t^*, c_o^*)\|.$$

Consequently, we have for all ν sufficiently large,

$$\|F(c_t^\nu, c_o^\nu) - F(c_t^*, c_o^*)\| \leq L'\|(c_t^\nu, c_o^\nu) - (c_t^*, c_o^*)\|,$$

where

$$L' \equiv L\left\|\begin{bmatrix} A_{tn} \\ A_{on} \end{bmatrix}\right\| + \left\|\begin{bmatrix} A_{tt} & A_{to} \\ A_{ot} & A_{oo} \end{bmatrix}\right\|.$$

To show the (strong) monotonicity of F, suppose there is a constant $\gamma \geq 0$ such that

$$x^T A x \geq \gamma x^T x, \quad \text{for all } x \in R^{3n_c}.$$

(Positive semidefiniteness of A corresponds to $\gamma = 0$; whereas positive definiteness corresponds to $\gamma > 0$.) Let (c_t^i, c_o^i), $i = 1, 2$, be two arbitrary pairs of vectors in R^{2n_c}; let $c_n^i \in \mathrm{SOL}(c_t^i, c_o^i)$ and

$$a_n^i = A_{nn}c_n^i + A_{nt}c_t^i + A_{no}c_o^i + b_n.$$

We have

$$(a_n^1 - a_n^2)^T(c_n^1 - c_n^2) = -(a_n^1)^T c_n^2 - (a_n^2)^T c_n^1 \leq 0.$$

Write

$$x^i \equiv \begin{bmatrix} c_n^i \\ c_t^i \\ c_o^i \end{bmatrix}, \quad \text{for } i = 1, 2.$$

Then we have

$$(x^1 - x^2)^T A(x^1 - x^2) =$$

$$(F(c_t^1, c_o^1) - F(c_t^2, c_o^2))^T \begin{bmatrix} c_t^1 - c_t^2 \\ c_o^1 - c_o^2 \end{bmatrix} + (a_n^1 - a_n^2)^T(c_n^1 - c_n^2);$$

which implies

$$\begin{aligned} (F(c_t^1, c_o^1) - F(c_t^2, c_o^2))^T \begin{bmatrix} c_t^1 - c_t^2 \\ c_o^1 - c_o^2 \end{bmatrix} &\geq \gamma \|x^1 - x^2\|^2 \\ &\geq \gamma \left(\|c_t^1 - c_t^2\|^2 + \|c_o^1 - c_o^2\|^2 \right). \end{aligned}$$

Thus the (strong) monotonicity of F follows. Q.E.D.

We now define a set-valued map K from R^{2n_c} into the family of nonempty convex subsets of R^{2n_c}. Specifically, for each $(c_t, c_o) \in R^{2n_c}$, $K(c_t, c_o)$ consists of all vectors $(c_t', c_o') \in R^{2n_c}$ for which there exists $c_n \in \text{SOL}(c_t, c_o)$ such that for all $i = 1, \ldots, n_c$,

$$(c_{it}')^2 + (c_{io}')^2 \leq \mu_i^2 c_{in}^2.$$

Clearly, $K(c_t, c_o)$ contains the origin of R^{2n_c}. Moreover, by the convexity of $\text{SOL}(c_t, c_o)$, it is not hard to show that $K(c_t, c_o)$ is convex.

As defined in [5], the quasi-variational inequality (QVI) associated with the pair (K, F) is the problem of finding a pair $(c_t, c_o) \in K(c_t, c_0)$ such that

$$F(c_t, c_0)^T \begin{bmatrix} c_t' - c_t \\ c_o' - c_o \end{bmatrix} \geq 0, \quad \text{for all } (c_t', c_o') \in K(c_t, c_o).$$

The relation between the QVI and the all-rolling problem is summarized in the following result.

Theorem 3 *If* (c_t, c_o) *solves the QVI* (K, F)*, then there is* (a_n, c_n, a_t, a_o) *such that* $(a_n, c_n, a_t, a_o, c_t, c_o)$ *solves the all-rolling dynamic 3-dimensional multi-rigid-body contact problem with Coulomb friction cones; conversely if the matrix (38) has full column rank and if* $(a_n, c_n, a_t, a_o, c_t, c_o)$ *solves the all-rolling contact problem, then* (c_t, c_o) *solves the QVI* (K, F)*.*

Proof. Let (c_t, c_o) solve the QVI (K, F). Since $(c_t, c_o) \in K(c_t, c_0)$, it follows that there exists $c_n \in \mathrm{SOL}(c_t, c_o)$ such that the friction cone constraints (7) are satisfied by the triple (c_n, c_t, c_o). Let $(a_t, a_o) = F(c_t, c_o)$, and

$$a_n \equiv A_{nn} c_n + A_{nt} c_t + A_{no} c_o + b_n.$$

In order for $(a_n, c_n, a_t, a_o, c_t, c_0)$ to solve the all-rolling problem, it remains to verify that (10) holds. Clearly, we need to consider only those contacts i for which $c_{in} > 0$.

Consider the following minimization problem in the variables (c'_t, c'_o), with $(c_n, a_t, a_o, c_t, c_o)$ fixed:

$$\begin{aligned} &\text{minimize} \quad (a_t)^T c'_t + (a_o)^T c'_o \\ &\text{subject to} \\ &(c'_{it})^2 + (c'_{io})^2 \le \mu_i^2 c_{in}^2, \quad \text{for all } i \text{ such that } c_{in} > 0. \end{aligned} \tag{39}$$

By the fact that (c_t, c_o) solves the QVI (K, F), it follows that (c_t, c_0) solves the above problem whose constraints satisfy the Slater constraint qualification. Thus by the Karush-Kuhn-Tucker (KKT) optimality conditions, there exists a multiplier $\sigma_i \ge 0$ for each i with $c_{in} > 0$ such that

$$a_{it} + \sigma_i c_{it} = 0, \quad a_{io} + \sigma_i c_{io} = 0$$

$$\sigma_i \left[(c_{it})^2 + (c_{io})^2 - \mu_i^2 c_{in}^2 \right] = 0.$$

It is easy to show that each

$$\sigma_i = \frac{\sqrt{a_{it}^2 + a_{io}^2}}{\mu_i c_{in}}.$$

Thus (10) follows.

Conversely, if (38) has full column rank, then $\mathrm{SOL}(c_t, c_o)$ is a singleton for all pairs (c_t, c_o). Hence the optimization problem (39) is equivalent to

$$\begin{aligned} &\text{minimize} \quad (a_t)^T c'_t + (a_o)^T c'_o \\ &\text{subject to} \quad (c'_t, c'_o) \in K(c_t, c_o). \end{aligned}$$

Since this is a convex program satisfying a constraint qualification, the KKT conditions are sufficient for optimality. Thus, by reversing the above argument, the converse assertion of the theorem follows. Q.E.D.

Besides establishing the claimed relationship between the QVI (K, F) and the all-rolling contact problem, the above proof also reveals an interesting interpretation of the rolling restriction (10) in a variational context. This interpretation is evidently not new as previous work [13, 14] has already made use of a variational inequality formulation of the Coulomb friction law.

Existence results

With the aid of Theorem 3, we can now proceed to derive some existence results for the all-rolling contact problem. We shall first treat the case where the matrix A is positive definite. As noted in the proof of Theorem 3, $\mathrm{SOL}(c_t, c_o)$ is a singleton, which we denote $\{c_n(c_t, c_0)\}$. According to Lemma 7.3.10 in [6], $c_n(c_t, c_o)$ is Lipschitz continuous in the argument (c_t, c_o).

For each positive scalar ρ, define the restricted set-valued map $K_\rho : R^{2n_c} \to R^{2n_c}$ as follows: for each $(c_t, c_o) \in R^{2n_c}$, $K_\rho(c_t, c_o)$ consists of all vectors $(c'_t, c'_o) \in R^{2n_c}$ such that for all $i = 1, \ldots, n_c$,

$$(c'_{it})^2 + (c'_{io})^2 \leq \min\left(\mu_i^2 c_{in}(c_t, c_o)^2, \rho^2\right).$$

For a reference on a comprehensive theory of set-valued maps, we cite [2]. The range of K_ρ is

$$\bigcup_{(c_t, c_o) \in R^{2n_c}} K_\rho(c_t, c_o).$$

Proposition 2 *If the matrix A is positive definite, then for every scalar $\rho > 0$, the set-valued map K_ρ has a compact range and is closed-valued and continuous.*

Proof. By definition, $K_\rho(c_t, c_o)$ is the Cartesian product of n_c 2-dimensional balls, each with center at the origin (in the plane) and the i-ball with radius $\min(\mu_i c_{in}(c_t, c_o), \rho)$. (Incidentally, if $c_{in}(c_t, c_o) = 0$ for some i, then the i-th ball degenerates to just the origin.) Consequently, the range K_ρ is compact. Since each set $K_\rho(c_t, c_o)$ is clearly closed, K_ρ is closed-valued.

Having a compact range and being closed-valued, the continuity of K_ρ at a pair (c_t, c_o) is therefore equivalent to two properties:

(a) K_ρ is closed at (c_t, c_o); that is

$$\left.\begin{array}{r} \{(c_t^\nu, c_o^\nu)\} \to (c_t, c_o) \\ \{(\hat{c}_t^\nu, \hat{c}_o^\nu)\} \to (\hat{c}_t, \hat{c}_o) \\ \forall \nu, (\hat{c}_t^\nu, \hat{c}_o^\nu) \in K_\rho(c_t^\nu, c_o^\nu) \end{array}\right\} \Rightarrow (\hat{c}_t, \hat{c}_o) \in K_\rho(c_t, c_o);$$

(b) K_ρ is lower semicontinuous at (c_t, c_0); that is

$$\left.\begin{array}{r} \{(c_t^\nu, c_o^\nu)\} \to (c_t, c_o) \\ (\hat{c}_t, \hat{c}_o) \in K_\rho(c_t, c_o) \end{array}\right\} \Rightarrow \exists \{(\hat{c}_t^\nu, \hat{c}_o^\nu)\} \to (\hat{c}_t, \hat{c}_o) \text{ and } \forall \nu, (\hat{c}_t^\nu, \hat{c}_o^\nu) \in K_\rho(c_t^\nu, c_o^\nu)$$

Property (a) is easily seen to be valid because of the continuity of the function $c_n(\cdot, \cdot)$. To show (b), let the left-hand side hold. For an arbitrary index i, we have

$$(\hat{c}_{it})^2 + (\hat{c}_{io})^2 \leq \min\left(\mu_i^2 c_{in}(c_t, c_o)^2, \rho^2\right). \tag{40}$$

If strict inequality holds for some index j, then clearly

$$(\hat{c}_{jt})^2 + (\hat{c}_{jo})^2 \leq \min\left(\mu_j^2 c_{jn}(c_t^\nu, c_o^\nu)^2, \rho^2\right)$$

for all ν sufficiently large. Thus for such an index j, we may define $(\hat{c}_{jt}^\nu, \hat{c}_{jo}^\nu) \equiv (\hat{c}_{jt}, \hat{c}_{jo})$ for all ν. Suppose equality holds in (40) for some index k. If $c_{kn}(c_t, c_o) = 0$, then $\hat{c}_{kt} = \hat{c}_{ko} = 0$; define for all ν,

$$(\hat{c}_{kt}^\nu, \hat{c}_{ko}^\nu) \equiv 0.$$

Then clearly

$$(\hat{c}_{kt}^\nu)^2 + (\hat{c}_{ko}^\nu)^2 \leq \min\left(\mu_k^2 c_{kn}(c_t^\nu, c_o^\nu)^2, \rho^2\right). \tag{41}$$

Suppose now $c_{kn}(c_t, c_o) > 0$. Write

$$c_{kn}(c_t^\nu, c_o^\nu) = c_{kn}(c_t, c_o) + \varepsilon_{k,\nu};$$

for all ν sufficiently large such that

$$|\varepsilon_{k,\nu}|^{1/2} < \min\left(\frac{1}{2}, \frac{\hat{c}_{kt}^2 + \hat{c}_{ko}^2}{\mu_k^2 c_{kn}(c_t, c_o)}\right),$$

define

$$(\hat{c}^{\nu}_{kt}, \hat{c}^{\nu}_{ko}) \equiv \sqrt{1 - 2|\varepsilon_{k,\nu}|^{1/2}}(\hat{c}_{kt}, \hat{c}_{ko}).$$

We have

$$\begin{aligned}
&\min\left(\mu_k^2 c_{kn}(c^{\nu}_t, c^{\nu}_o)^2, \rho^2\right) \\
&\geq \min\left(\mu_k^2 c_{kn}(c_t, c_o)^2, \rho^2\right) - 2|\varepsilon_{k,\nu}|\mu_k^2 c_{kn}(c_t, c_o) \\
&= (1 - 2|\varepsilon_{k,\nu}|^{1/2})(\hat{c}^2_{kt} + \hat{c}^2_{ko}) + 2|\varepsilon_{k,\nu}|^{1/2}\left(\hat{c}^2_{kt} + \hat{c}^2_{ko} - |\varepsilon_{k,\nu}|^{1/2}\mu_k^2 c_{kn}(c_t, c_o)\right) \\
&\geq (\hat{c}^{\nu}_{kt})^2 + (\hat{c}^{\nu}_{ko})^2.
\end{aligned}$$

Consequently, for any index i, we have defined $(\hat{c}^{\nu}_{it}, \hat{c}^{\nu}_{io})$ which converges to $(\hat{c}_{it}, \hat{c}_{io})$ as $\nu \to \infty$; moreover, (41) holds for all indices $k = 1, \ldots, n_c$ and all ν. Hence K_ρ is lower semicontinuous, and thus continuous, at (c_t, c_o). Q.E.D.

Applying Corollary 4.1 in [5], we have our first existence result for the all-rolling contact problem with Coulomb friction cones.

Theorem 4 *If the matrix A is positive definite, then the QVI (K, F) has a solution; thus so does the all-rolling 3-dimensional multi-rigid-body contact problem with Coulomb friction cone constraints.*

Proof. In order to apply the cited corollary, it suffices to observe three things: (i) for all $(c_t, c_o) \in R^{2n_c}$, the set $K(c_t, c_o)$ contains the origin, (ii) the function $F(c_t, c_o)$ is continuous and strongly monotone, and (iii) the set-valued map K_ρ is continuous for all $\rho > 0$. This corollary implies that the QVI (K, F) has a solution which, by Theorem 3, is a desired solution of the all-rolling problem. Q.E.D.

Our next goal is to show that the same existence conclusion about the contact problem is valid when the positive definiteness assumption on A is replaced by the weaker assumption (35). To this end, we point out that in Theorem 4, the existence of a solution to the QVI (K, F) does not depend on the particular structure (21) of the matrix A; instead, it is the positive definiteness of A that matters.

Theorem 5 *Under assumption (35), the all-rolling 3-dimensional multi-rigid-body contact problem with Coulomb friction cone constraints has a solution.*

Proof. The matrix A is positive semidefinite; hence for every scalar $\varepsilon > 0$, $A + \varepsilon I$ is positive definite. Choose an arbitrary sequence of positive scalars $\{\varepsilon_\nu\}$ that converges to zero. By Theorem 4, the subsequent remark, and the proof of Theorem 3, it follows that for each ν, the following system has a solution $(a_n^\nu, c_n^\nu, a_t^\nu, a_o^\nu, c_t^\nu, c_0^\nu)$:

$$\begin{bmatrix} a_n \\ a_t \\ a_o \end{bmatrix} = (A + \varepsilon_\nu I) \begin{bmatrix} c_n \\ c_t \\ c_o \end{bmatrix} + \begin{bmatrix} b_n \\ b_t \\ b_o \end{bmatrix}$$

$$(a_n, c_n) \geq 0, \quad (a_n)^T c_n = 0$$

$$\left.\begin{aligned} \mu_i c_{in}(a_{it}, a_{io}) + \sqrt{a_{it}^2 + a_{io}^2}(c_{it}, c_{io}) = 0 \\ c_{it}^2 + c_{io}^2 \leq \mu_i^2 c_{in}^2 \end{aligned}\right\} \quad \forall i = 1, \ldots, n_c.$$

If the sequence

$$\{ \, (a_n^\nu, c_n^\nu, a_t^\nu, a_o^\nu, c_t^\nu, c_0^\nu) \, \}$$

is bounded, then any one of its accumulation points can be shown to be a desired solution of the all-rolling problem. So let us assume that this sequence is unbounded. Without loss of generality, we may assume that the normalized sequence

$$\left\{ \frac{(a_n^\nu, c_n^\nu, a_t^\nu, a_o^\nu, c_t^\nu, c_0^\nu)}{\|(a_n^\nu, c_n^\nu, a_t^\nu, a_o^\nu, c_t^\nu, c_0^\nu)\|} \right\}$$

converges to a nonzero vector $(a_n^*, c_n^*, a_t^*, a_o^*, c_t^*, c_0^*)$. Clearly, this limit vector satisfies the homogenized system:

$$\begin{bmatrix} a_n^* \\ a_t^* \\ a_o^* \end{bmatrix} = A \begin{bmatrix} c_n^* \\ c_t^* \\ c_o^* \end{bmatrix}$$

$$(a_n^*, c_n^*) \geq 0, \quad (a_n^*)^T c_n^* = 0$$

$$\left.\begin{aligned} \mu_i c_{in}^*(a_{it}^*, a_{io}^*) + \sqrt{(a_{it}^*)^2 + (a_{io}^*)^2}(c_{it}^*, c_{io}^*) = 0 \\ (c_{it}^*)^2 + (c_{io}^*)^2 \leq \mu_i^2 (c_{in}^*)^2 \end{aligned}\right\} \quad \forall i = 1, \ldots, n_c.$$

Clearly, we have

$$\max(c_{it}^* a_{it}^*, c_{io}^* a_{io}^*) \leq 0,$$

for all i. Consequently,

$$0 \geq \begin{bmatrix} c_n^* \\ c_t^* \\ c_o^* \end{bmatrix}^T \begin{bmatrix} a_n^* \\ a_t^* \\ a_o^* \end{bmatrix} = \begin{bmatrix} c_n^* \\ c_t^* \\ c_o^* \end{bmatrix}^T A \begin{bmatrix} c_n^* \\ c_t^* \\ c_o^* \end{bmatrix} \geq 0.$$

Hence it follows that $(c_n^*, c_t^*, c_o^*) \in \mathcal{N}$. Since $(c_n^*, c_t^*, c_o^*) \in \mathcal{F}$, by assumption (35), we deduce that

$$\begin{bmatrix} c_n^* \\ c_t^* \\ c_o^* \end{bmatrix}^T \begin{bmatrix} b_n \\ b_t \\ b_o \end{bmatrix} \geq 0. \tag{42}$$

We claim that the following relations hold for all i and all ν sufficiently large:

$$c_{in}^* a_{in}^\nu = 0, \quad c_{in}^* c_{in}^\nu \geq 0$$
$$c_{it}^* a_{it}^\nu \leq 0, \quad c_{it}^* c_{it}^\nu \geq 0$$
$$c_{io}^* a_{io}^\nu \leq 0, \quad c_{io}^* c_{io}^\nu \geq 0.$$

Indeed, suppose $c_{in}^* > 0$. Then we must have $c_{in}^\nu > 0$ for all ν sufficiently large, which implies, by complementarity, that $a_{in}^\nu = 0$. Thus the first two relations hold. The other four relations can be proved in a similar way. Consequently, we have,

$$0 \geq \begin{bmatrix} c_n^* \\ c_t^* \\ c_o^* \end{bmatrix}^T \begin{bmatrix} a_n^\nu \\ a_t^\nu \\ a_o^\nu \end{bmatrix} = \begin{bmatrix} c_n^* \\ c_t^* \\ c_o^* \end{bmatrix}^T \left((A + \varepsilon_\nu I) \begin{bmatrix} c_n^\nu \\ c_t^\nu \\ c_o^\nu \end{bmatrix} + \begin{bmatrix} b_n \\ b_t \\ b_o \end{bmatrix} \right). \tag{43}$$

Since $(c_n^*, c_t^*, c_o^*) \in \mathcal{N}$, it follows that

$$\begin{bmatrix} c_n^* \\ c_t^* \\ c_o^* \end{bmatrix}^T \begin{bmatrix} b_n \\ b_t \\ b_o \end{bmatrix} \leq 0.$$

Consequently, by (42), we obtain

$$\begin{bmatrix} c_n^* \\ c_t^* \\ c_o^* \end{bmatrix}^T \begin{bmatrix} b_n \\ b_t \\ b_o \end{bmatrix} = 0.$$

Hence from (43), we deduce

$$0 \geq \begin{bmatrix} c_n^* \\ c_t^* \\ c_o^* \end{bmatrix}^T \begin{bmatrix} c_n^\nu \\ c_t^\nu \\ c_o^\nu \end{bmatrix},$$

which yields, by a normalization followed by a limit argument, $(c_n^*, c_t^*, c_o^*) = 0$. This in turn implies $(a_n^*, a_t^*, a_o^*) = 0$. But this contradicts the fact that $(a_n^*, c_n^*, a_t^*, a_o^*, c_t^*, c_0^*)$ is a nonzero vector. Q.E.D.

Corollary 1 *When*

$$\tilde{b} \equiv \begin{bmatrix} \dot{W}_n^T & \dot{J}_n \\ \dot{W}_t^T & \dot{J}_t \\ \dot{W}_o^T & \dot{J}_o \end{bmatrix} \begin{bmatrix} \dot{q} \\ -\dot{\theta} \end{bmatrix} \tag{44}$$

is an element of the column space of A, then the all-rolling 3-dimensional multi-rigid-body contact problem with Coulomb friction cone constraints has a solution.

Proof. Indeed, under the stated assumption of the vector $\tilde{b}$, condition (35) is trivially satisfied. Specifically, the only vectors, $c \equiv (c_n, c_t, c_o)$, that are of interest in condition (35) lie in the null space of A. Thus $\tilde{b}^T c = 0$. Q.E.D.

An important consequence of this Corollary is that a solution always exists when the system is initially at rest (*i.e.*, $\dot{q} = \dot{\theta} = 0$). One application area where this consequence is relevant is automated fixture planning. In automated fixture planning, one is provided with the positions and orientations of a set of parts in contact. The objective is to determine where to place fixture elements (each of which provides an additional contact on a part), so that when the parts are assembled in the fixture and released, they do not collapse. If one can show that every solution to the corresponding multi-rigid-body contact problem has $\ddot{q} = 0$ (by definition, fixture elements are not actuated, so $\ddot{\theta} = 0$), then the fixture stabilizes the parts and thereby represents a valid fixture.

By suitably modifying the above argument, in particular, by recognizing that (19) is related to the pyramid constraint (18) just like (10) is to the cone constraint (7), we can establish the following existence result for the friction pyramid model. The details are omitted.

Theorem 6 *Under assumption (35) with $\mathcal{F}$ referring to the friction pyramid defined by the constraints*

$$\max(|c_{it}|, |c_{io}|) \leq \mu_i c_{in}, \quad \textit{for all } i = 1, \ldots, n_c, \tag{45}$$

the all-rolling 3-dimensional multi-rigid-body contact problem with Coulomb friction pyramid constraints has a solution.

Besides yielding an existence result for the contact problem, Theorem 6 is interesting from the point of view of LCP theory. Indeed, as we saw in the last section, the all-rolling 3-dimensional multi-rigid-body contact problem with Coulomb friction pyramid constraints can be formulated as a standard LCP. Yet the proof of Theorem 6 relies on QVI theory; this is different from the way many known LCP existence results are derived, namely, from the theory of variational inequalities [6, Chapter 3].

5 The Rolling-Sliding Case

When some contacts are initially sliding, that is, when $\mathcal{S} \neq \emptyset$, the multi-rigid-body contact problem, with either the friction cone or friction pyramid constraints, need not have a solution. Examples can be constructed for the one-contact problem with sliding ($n_c = 1$ and $\mathcal{R} = \emptyset$) to illustrate this no-solution phenomenon [17]. This is a failing of the model, because any physical system and situation that we are interested in has a solution in the real world. The non-existence of a solution is usually attributed to the simultaneous assumptions that the bodies are rigid and that the friction forces obey Coulomb's Law. However, our study suggests that a possible culprit is the restrictiveness of Coulomb's Law for sliding contact. Specifically, the source of the difficulty appears to be the constraints (9), which define the magnitude of each friction force to be $\mu_i c_{in}$ and its direction to be exactly opposite the relative sliding velocity (or acceleration, if the contact is initially rolling) at the contact.

In what follows, we propose a relaxed friction model similar to those used by Lötstedt and Baraff [3], and study the question of existence of solution. In particular, the relaxed model is formed by replacing equation (9), by the inequality: $v_{it}c_{it} + v_{io}c_{io} \leq 0$. This inequality ensures that the friction forces in the relaxed model cannot produce energy, and it provides some latitude in the direction and magnitude of the friction force at each sliding contact.

The relaxed friction model can be formulated as the following constrained least-squares problem:

$$\begin{array}{ll} \text{minimize} & f(c_{\mathcal{S}n}, c_{\mathcal{S}t}, c_{\mathcal{S}o}) \\ \text{subject to} & (20), (6), (7), (8), (10), (12), \\ \text{and} & v_{it}c_{it} + v_{io}c_{io} \leq 0, \quad \text{for all } i \in \mathcal{S}, \end{array} \tag{46}$$

where

$$f(c_{\mathcal{S}n}, c_{\mathcal{S}t}, c_{\mathcal{S}o}) \equiv$$
$$\textstyle\sum_{i\in\mathcal{S}} \left((\mu_i v_{it} c_{in} + \sqrt{v_{it}^2 + v_{io}^2}\, c_{it})^2 + (\mu_i v_{io} c_{in} + \sqrt{v_{it}^2 + v_{io}^2}\, c_{io})^2 \right)$$

is the residual of the sliding constraints (9). Notice that the 3-dimensional multi-rigid-body contact problem with Coulomb friction cone constraints has a solution if and only if the least-squares problem has an optimal solution with zero objective value. When the objective value is non-zero, then the contact forces lie inside or on the half cones defined by inequalities (7) and the new constraints in the least-squares problem (46). Since the objective function represents the error between the friction forces of the relaxed friction model and Coulomb's friction law, the solution found is as close to obeying Coulomb's law as possible.

In the problem (46), we may replace the friction cone constraints (7) and the rolling restrictions (10) by, respectively, the friction pyramid constraints (45) and the modified rolling conditions (19). It turns out the resulting least-squares friction pyramid model will always have a solution, provided that it is feasible.

Theorem 7 *If the least-squares friction pyramid problem:*

$$\begin{array}{ll} \textit{minimize} & f(c_{\mathcal{S}n}, c_{\mathcal{S}t}, c_{\mathcal{S}o}) \\ \textit{subject to} & (20), (6), (45), (8), (19), (12), \\ \textit{and} & v_{it}c_{it} + v_{io}c_{io} \leq 0, \quad \textit{for all } i \in \mathcal{S}, \end{array} \tag{47}$$

has a feasible solution, then it has an optimal solution.

Proof. Define the matrix $\tilde{M}$ by

$$\begin{bmatrix} (A_{nn})_{\mathcal{SS}} & (\tilde{M}_{nn})_{\mathcal{SR}} & (A_{nt})_{\mathcal{SR}} & (A_{no})_{\mathcal{SR}} & 0 & 0 & (A_{nt})_{\mathcal{SS}} & (A_{no})_{\mathcal{SS}} \\ (A_{nn})_{\mathcal{RS}} & (\tilde{M}_{nn})_{\mathcal{RR}} & (A_{nt})_{\mathcal{RR}} & (A_{no})_{\mathcal{RR}} & 0 & 0 & (A_{nt})_{\mathcal{RS}} & (A_{no})_{\mathcal{RS}} \\ (A_{tn})_{\mathcal{RS}} & (\tilde{M}_{tn})_{\mathcal{RR}} & (A_{tt})_{\mathcal{RR}} & (A_{to})_{\mathcal{RR}} & I & 0 & (A_{tt})_{\mathcal{RS}} & (A_{to})_{\mathcal{RS}} \\ (A_{on})_{\mathcal{RS}} & (\tilde{M}_{on})_{\mathcal{RR}} & (A_{ot})_{\mathcal{RR}} & (A_{oo})_{\mathcal{RR}} & 0 & I & (A_{ot})_{\mathcal{RS}} & (A_{oo})_{\mathcal{RS}} \\ 0 & 2U_{\mathcal{R}} & -I & 0 & 0 & 0 & 0 & 0 \\ 0 & 2U_{\mathcal{R}} & 0 & -I & 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$\begin{bmatrix} (\tilde{M}_{nn})_{\mathcal{SR}} \\ (\tilde{M}_{nn})_{\mathcal{RR}} \\ (\tilde{M}_{tn})_{\mathcal{RR}} \\ (\tilde{M}_{on})_{\mathcal{RR}} \end{bmatrix} \equiv \begin{bmatrix} (A_{nn})_{\mathcal{SR}} \\ (A_{nn})_{\mathcal{RR}} \\ (A_{tn})_{\mathcal{RR}} \\ (A_{on})_{\mathcal{RR}} \end{bmatrix} - \begin{bmatrix} (A_{nt})_{\mathcal{SR}} \\ (A_{nt})_{\mathcal{RR}} \\ (A_{tt})_{\mathcal{RR}} \\ (A_{ot})_{\mathcal{RR}} \end{bmatrix} U_{\mathcal{R}} - \begin{bmatrix} (A_{no})_{\mathcal{SR}} \\ (A_{no})_{\mathcal{RR}} \\ (A_{to})_{\mathcal{RR}} \\ (A_{oo})_{\mathcal{RR}} \end{bmatrix} U_{\mathcal{R}}.$$

According to Lemma 1, by introducing the variables $(s_{it}^{\pm}, s_{io}^{\pm}, a_{it}^{\pm}, a_{io}^{\pm})$ for

$i \in \mathcal{R}$, the problem (47) can be equivalently stated as:

$$
\begin{array}{ll}
\text{minimize} & f(c_{\mathcal{S}n}, c_{\mathcal{S}t}, c_{\mathcal{S}o}) \\
\text{subject to} &
\end{array}
$$

$$
\begin{bmatrix} a_{\mathcal{S}n} \\ a_{\mathcal{R}n} \\ a^{+}_{\mathcal{R}t} \\ a^{+}_{\mathcal{R}o} \\ s^{-}_{\mathcal{R}t} \\ s^{-}_{\mathcal{R}o} \end{bmatrix} = \tilde{M} \begin{bmatrix} c_{\mathcal{S}n} \\ c_{\mathcal{R}n} \\ s^{+}_{\mathcal{R}t} \\ s^{+}_{\mathcal{R}o} \\ a^{-}_{\mathcal{R}t} \\ a^{-}_{\mathcal{R}o} \\ c_{\mathcal{S}t} \\ c_{\mathcal{S}o} \end{bmatrix} + \begin{bmatrix} b_{\mathcal{S}n} \\ b_{\mathcal{R}n} \\ b_{\mathcal{R}t} \\ b_{\mathcal{R}o} \\ 0 \\ 0 \end{bmatrix} \tag{48}
$$

$$
\begin{array}{ll}
& (a_n, c_n) \geq 0, \quad (a_n)^T c_n = 0 \\
& (s^{\pm}_{\mathcal{R}t}, s^{\pm}_{\mathcal{R}o}, a^{\pm}_{\mathcal{R}t}, a^{\pm}_{\mathcal{R}o}) \geq 0 \\
& (s^{+}_{\mathcal{R}t})^T a^{+}_{\mathcal{R}t} = (s^{-}_{\mathcal{R}t})^T a^{-}_{\mathcal{R}t} = (s^{+}_{\mathcal{R}o})^T a^{+}_{\mathcal{R}o} = (s^{-}_{\mathcal{R}o})^T a^{-}_{\mathcal{R}o} = 0 \\
\text{and} & v_{it} c_{it} + v_{io} c_{io} \leq 0, \quad \text{for all } i \in \mathcal{S}.
\end{array}
$$

Note that the equations that correspond to the variables $a_{\mathcal{S}t}$ and $a_{\mathcal{S}o}$ have been removed from the above formulation. The constraints of the problem (48) consist of linear inequalities and the following complementarity conditions:

$$
(a_n)^T c_n = (s^{+}_{\mathcal{R}t})^T a^{+}_{\mathcal{R}t} = (s^{-}_{\mathcal{R}t})^T a^{-}_{\mathcal{R}t} = (s^{+}_{\mathcal{R}o})^T a^{+}_{\mathcal{R}o} = (s^{-}_{\mathcal{R}o})^T a^{-}_{\mathcal{R}o} = 0.
$$

As such, the feasible region of (48) is the union of finitely many convex polyhedra. Since the objective function f is always nonnegative, it follows from the well-known Frank-Wolfe theorem of quadratic programming [10] that (48) must have an optimal solution, provided that it has a feasible solution. Q.E.D.

The problem (48) belongs to the class of mathematical programs with equilibrium constraints (MPEC). The recent paper [19] develops a comprehensive theory for an optimization problem of this type.

The feasibility issue

Theorem 7 raises the question of when the problem (47) or (46) has a feasible solution. Consistent with our treatment in Section 4, we shall treat this issue only for the latter problem.

Define the cone

$$\mathcal{F}_{\mathcal{RS}} \equiv \{(c_n, c_t, c_o) \in \mathcal{F}_{\mathcal{R}} : v_{it}c_{it} + v_{io}c_{io} \leq 0, \text{ for all } i \in \mathcal{S}\}.$$

We postulate that

$$\begin{bmatrix} \dot{q} \\ -\dot{\theta} \end{bmatrix}^T \begin{bmatrix} \dot{W}_n & \dot{W}_t & \dot{W}_o \\ \dot{J}_n^T & \dot{J}_t^T & \dot{J}_o^T \end{bmatrix} \begin{bmatrix} c_n \\ c_t \\ c_o \end{bmatrix} \geq 0, \quad \forall (c_n, c_t, c_0) \in \mathcal{F}_{\mathcal{RS}}. \qquad (49)$$

Proposition 3 *Under assumption (49), the problem (46) has a feasible solution.*

Proof. We claim that (46) has a feasible solution with $c_{\mathcal{S}t} = c_{\mathcal{S}o} = 0$. With these sliding force variables set at zero (and ignoring the equations corresponding to the sliding accelerations $a_{\mathcal{S}t}$ and $a_{\mathcal{S}o}$), we can define a modified QVI whose solution will yield a desired feasible solution satisfying the remaining constraints in (46). In what follows, we give a very brief definition of this modified QVI and skip the rest of the proof, which is similar to that of Theorem 5.

The previous QVI (K, F) was defined in the space R^{2n_c} with the full set of variables (c_t, c_o). The modified QVI is defined exactly as before with the following changes: (i) the sliding variables $(c_{\mathcal{S}t}, c_{\mathcal{S}o})$ are set equal to zero in defining SOL, (ii) the function F is defined without the $(a_{\mathcal{S}t}, a_{\mathcal{S}o})$ components, and (iii) the set-valued map K is defined without the $(c_{\mathcal{S}t}, c_{\mathcal{S}o})$ components.

We leave the remaining details to the reader. Q.E.D.

6 Concluding Remarks

In this paper, we have introduced several complementarity formulations of the dynamic 3-dimensional multi-rigid-body contact problem with Coulomb friction. We have studied in some detail the issue of the existence of solution in the all-rolling case and proposed a least-squares model for the rolling-sliding case. The tool for the existence proof was QVI theory. We have also used a friction pyramid model and have shown that the least-squares problem with Coulomb friction pyramid constraints always has an optimal solution if feasible.

We point out that existence results subsume all previous results known to us. Specifically, Baraff has proved that a 3-dimensional system with one contact point that is rolling has a solution [3]. Also, specializing our results to the planar case yields results consistent with and more general than the results given by Rajan *et al.* [28] and Erdmann [8] who studied a single moving object in contact with immovable objects. Also, our results are less restrictive than the existence results given by Lötstedt for multiple moving objects.

Several theoretical questions remain open. One of these is the existence of an optimal solution to the least-squares model (46). In this paper, we have not discussed any numerical aspect of the models; these and other engineering issues as well as further existence results are discussed in the paper [31].

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