

Stability Characterizations of Rigid Body Contact Problems with Coulomb Friction*

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Abstract. This paper formally introduces several stability characterizations of systems of rigid bodies initially at rest and in unilateral contact with dry friction. These characterizations, *weak stability* and *strong stability* (and their complements), arise naturally from the dynamic model of the system, formulated as a complementarity problem. Using the tools of complementarity theory, these characterizations are studied in detail to understand their properties and to develop techniques to identify the stability classifications of general systems subjected to known external loads.

1 Introduction

Many useful mechanical systems are composed of a number of bodies that interact through multiple, unilateral frictional contacts. Examples include gears, cams, modular fixturing systems, and robot grippers¹. Designers of such systems rely heavily on the analyses of initial designs, which are often carried out under the rigid body assumption. Nonetheless, significant holes in both the relevant theory and computational tools remain. In this paper, motivated by applications in automated fixture synthesis, we attempt to close one of those holes through a rigorous study of the stability of a free rigid body (called a workpiece) initially at rest and in dry frictional contact with fixed rigid bodies (called fixels) from the perspective of multi-rigid-body dynamics and complementarity theory. Our primary objective is to develop a sound basis that will enable us to gain a thorough understanding of the main issues involved with stability. Our secondary objective is to derive theoretical results that will enable the development of tests that more accurately characterize stability than the overly conservative tests in use today. The main results are presented in the form of several new theorems which are illustrated with simple examples.

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¹In fact at a fine level of detail, every lower pair joint in every mechanism is actually implemented with a clearance, which leads to unilateral contacts in the joint interfaces.

1.1 Previous Work

There are two primary ways to stabilize a rigid workpiece. The first is known as form closure [10]. A workpiece is form-closed if it cannot move, even infinitesimally, without at least one fixel (a rigid fixture element) penetrating the workpiece. This sort of stability does not rely on friction and is easy to check (by solving a linear program [18]). A form-closed workpiece is completely constrained kinematically and thus will remain stationary in the face of any applied external load. Several automated design systems are based on form closure [1, 3, 4, 5, 7, 14, 21]². However, because form closure requires large numbers of contacts (at least 7 point contacts or 4 planar contacts on a single workpiece), it can sometimes be impossible to design form-closed fixtures that also provide the required access for machining tools or part insertions.

Recognizing the limitations of large numbers of contacts, Palmer [15] and several other authors studied stability without form closure [2, 9, 12, 16, 17, 22]. For such situations, the stability of the workpiece should be determined by examining the solution(s) to the dynamic model composed of the Newton-Euler equations for the workpiece, the relevant kinematic constraints, and a dry friction law. However, typically the dynamic equations are replaced by equilibrium equations, which can lead to false positive stability conclusions. In order to prevent this problem, the results in this paper are based on the dynamic equations.

Despite our beginning with a dynamic model, we do *not* adopt the usual stability definition for dynamic systems. The reason is that we allow sliding at the contacts which results in an irrecoverable loss of energy, and hence an arbitrarily perturbed workpiece will generally not return to its initial equilibrium configuration. Instead, we will adopt Fourier’s inequality [11], which can be stated as follows:

Definition 1: If the acceleration of the workpiece is zero (for all solutions of the dynamic model) for given fixel locations and applied load, then the workpiece is said to be *stable*. Equivalently, a workpiece is stable if the virtual work for every kinematically admissible virtual motion is nonpositive. *Note that for convenience, we will also refer to the load and the fixture as being stable when this condition is met.*

Palmer found that determining stability (which he referred to as “infinitesimal stability”) in the presence of friction was extremely difficult (co-NP complete [8]) due to solution nonuniqueness of the dynamic model, so he identified two other stability classifications that could be tested efficiently by linear programming methods [15]. These classifications were:

- Potential Stability – A set of contact forces exists that satisfies the equilibrium equations of the fixtured workpiece and Coulomb’s law at the contacts.
- Guaranteed Stability – A set of contact forces exists that satisfies the equilibrium equations of the fixtured workpiece when friction is neglected.

The primary problems with these stability characterizations are that they are overly conservative in one direction or the other, so their use in fixture design algorithms is limited. Figure 1 illustrates the problem. For a given fixture and workpiece configuration, let $SS(\boldsymbol{\mu})$ denote the set of *strongly stable* external loads (*i.e.*, those that satisfy the above stability Definition 1 in the presence of friction, where $\boldsymbol{\mu}$ is the vector of friction coefficients at the contact points). Similarly, let $SS(0)$

²For an excellent review and bibliography of the fixture stability and synthesis literature, see [4].

denote the set of loads that are strongly stable without friction (Palmer's "guaranteed stability") and let $WS(\boldsymbol{\mu})$ denote the set of *weakly stable* loads with friction (Palmer's "potential stability"). A load can be tested for membership in $WS(\boldsymbol{\mu})$ or $SS(0)$ using linear programming techniques, and as will be demonstrated, one can identify all the external loads in these sets for a given fixture. However, since there are loads in $WS(\boldsymbol{\mu})$ that have multiple dynamic model solutions, some of which correspond to instability (nonzero workpiece acceleration), fixture design using this set is not recommended. On the other hand, the set of loads $SS(0)$ is usually a small subset of $SS(\boldsymbol{\mu})$, so its use in design is also limited.

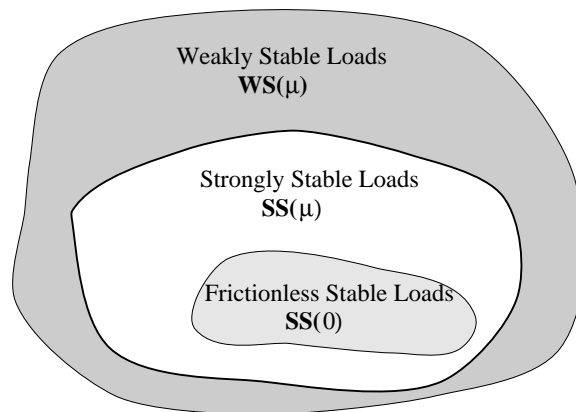


Figure 1: Important subsets of the set of all external loads.

Despite the limitations, Palmer's stability characterizations have been the best available for rigid fixture design without form closure. The results contained in this paper represent a significant step toward stability tests which are not conservative, and hence could lead to better fixture design and analysis tools.

2 Methodology

Our basic framework is the discrete-time dynamic model for multiple rigid bodies in contact presented in [20]. By setting the initial velocity of the free body (the workpiece) to zero and fixing the positions of the actuated bodies (the fixels), this model represents a fixtured workpiece. Three sets of conditions are imposed on the workpiece: (a) the Newton-Euler equation written in terms of the relative accelerations at the contacts, (b) contact conditions on the normal contact forces, and (c) dry friction constraints on the tangential contact forces. These are listed as follows. (See Appendix A for an explanation of the notation used throughout the paper.)

The Newton-Euler equation:

$$\begin{bmatrix} \mathbf{a}_n \\ \mathbf{a}_t \\ \mathbf{a}_o \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{c}_n \\ \mathbf{c}_t \\ \mathbf{c}_o \end{bmatrix} + \mathbf{b}, \quad (1)$$

where the subscripts n, t, o denote the normal (n) and two tangential directions (t, o) in the contact coordinate systems,

$$\mathbf{A} \equiv \mathcal{J}^T \mathcal{M}^{-1} \mathcal{J} \quad \text{and} \quad \mathbf{b} \equiv \mathcal{J}^T \mathcal{M}^{-1} \mathbf{g}_{\text{ext}}$$

with \mathcal{J} being the system Jacobian matrix and \mathcal{M} the system inertia matrix, the latter being symmetric positive definite, and \mathbf{g}_{ext} being the external load applied to the workpiece. The vector $\mathbf{a}_n = (a_{in})_{i=1}^{n_c}$ is composed of the relative normal accelerations at the contacts indexed by i , where n_c is the number of contact points among the bodies. The relative accelerations in the tangential directions, t and o , are defined analogously. The vectors of normal wrench intensities, \mathbf{c}_n , and frictional wrench intensities, \mathbf{c}_t and \mathbf{c}_o , are defined similarly. In the case of the fixture stability problem studied here, the system Jacobian matrix \mathcal{J} is composed of wrench matrices \mathbf{W}_n (in the normal direction), \mathbf{W}_t and \mathbf{W}_o (in the two tangential directions):

$$\mathcal{J} \equiv [\mathbf{W}_n \quad \mathbf{W}_t \quad \mathbf{W}_o].$$

These matrices simply map the contact forces into an inertial coordinate frame. The matrix \mathbf{A} can be written in partitioned form at follows:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{nn} & \mathbf{A}_{nt} & \mathbf{A}_{no} \\ \mathbf{A}_{tn} & \mathbf{A}_{tt} & \mathbf{A}_{to} \\ \mathbf{A}_{on} & \mathbf{A}_{ot} & \mathbf{A}_{oo} \end{bmatrix},$$

where for $a, b \in \{n, t, o\}$,

$$\mathbf{A}_{ab} \equiv \mathbf{W}_a^T \mathcal{M}^{-1} \mathbf{W}_b.$$

Similarly, the vector \mathbf{b} can be written in partitioned form:

$$\mathbf{b} = \begin{bmatrix} \mathbf{b}_n \\ \mathbf{b}_t \\ \mathbf{b}_o \end{bmatrix},$$

where for $a \in \{n, t, o\}$,

$$\mathbf{b}_a \equiv \mathbf{W}_a^T \mathcal{M}^{-1} \mathbf{g}_{\text{ext}}.$$

Normal contact conditions:

$$0 \leq \mathbf{a}_n \perp \mathbf{c}_n \geq 0, \quad (2)$$

where the notation \perp means perpendicularity. Note that this condition expresses the well-known complementarity between the normal contact load and acceleration at each unilateral contact. Specifically, if contact i breaks ($a_{in} > 0$) then the normal load c_{in} must be zero. Similarly, if a normal load is supported ($c_{in} > 0$), then the relative acceleration a_{in} must be zero.

Frictional constraints on tangential forces: for $i = 1, \dots, n_c$,

$$\begin{aligned} (c_{it}, c_{io}) \in \operatorname{argmin} \quad & c'_{it} a_{it} + c'_{io} a_{io} \\ \text{subject to} \quad & (c'_{it}, c'_{io}) \in \mathcal{F}(\mu_i c_{in}), \end{aligned} \quad (3)$$

where $\mathcal{F}(\cdot)$ is the Coulomb friction map and μ_i is the nonnegative friction coefficient at contact point i ; that is, for each nonnegative scalar $\zeta \geq 0$, $\mathcal{F}(\zeta)$ is a planar circular disk with center at the origin and radius ζ :

$$\mathcal{F}(\zeta) \equiv \{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq \zeta^2\}. \quad (4)$$

Note that in the context of the quadratic Coulomb law (4), the "argmin" condition in (3) implies that the friction force opposes the direction of impending slip (we recall that the system is initially at rest).

The results developed in this paper apply to more general friction laws (including some axisymmetric laws); nevertheless, for simplicity, we focus on the above standard Coulomb friction law. Later, we will also consider a variation of this law known as the friction pyramid law [20]

$$\mathcal{F}(\zeta) \equiv \{(a, b) \in \mathbb{R}^2 : \max(|a|, |b|) \leq \zeta\}. \quad (5)$$

With this friction law the friction force points toward the edge of the pyramid that yields the greatest power dissipation. Thus the relative motion is not exactly opposed, but energy is dissipated.

Definition 2: We call a solution $(\mathbf{c}_n, \mathbf{c}_t, \mathbf{c}_o)$ to the dynamic rigid body problem (system (1-3)) a *dynamic intensity*.

Every dynamic intensity $\mathbf{c} \equiv (\mathbf{c}_n, \mathbf{c}_t, \mathbf{c}_o)$ induces a vector of body accelerations $\ddot{\mathbf{q}}$, as follows:

$$\ddot{\mathbf{q}} \equiv \mathcal{M}^{-1}(\mathcal{J}\mathbf{c} + \mathbf{g}_{\text{ext}}). \quad (6)$$

Letting $\mathbf{a} \equiv (\mathbf{a}_n, \mathbf{a}_t, \mathbf{a}_o)$ denote the vector of relative accelerations at the contacts and using the fact that the workpiece is initially at rest, we see that

$$\mathbf{a} = \mathcal{J}^T \ddot{\mathbf{q}}.$$

Based on the above dynamic rigid body contact model, we redefine our two stability characterizations in terms of dynamic intensities and we introduce terminology for the complementary characterizations for three-dimensional bodies with Coulomb friction laws:

Definition 3: For a given external load \mathbf{g}_{ext} and fixel and workpiece configurations, the workpiece (and fixture and load) is said to be:

- weakly stable – if a dynamic intensity exists that induces zero body accelerations;
- strongly stable – if every dynamic intensity induces zero body accelerations;
- weakly unstable (Palmer's infinitesimal instability) – if it is not strongly stable; i.e., a dynamic intensity exists that induces a nonzero vector of body accelerations;
- strongly unstable (Palmer's guaranteed instability) – if it is not weakly stable; i.e., every dynamic intensity induces a nonzero vector of body accelerations.

We summarize these four concepts in the following diagram:

$$\begin{array}{ccc} \text{strongly stable} & \Rightarrow & \text{weakly stable} \\ \updownarrow \text{negation} & & \updownarrow \text{negation} \\ \text{weakly unstable} & \Leftarrow & \text{strongly unstable} \end{array}$$

2.1 Weak stability

Clearly, the load \mathbf{g}_{ext} is weakly stable if and only if there exists a contact force vector \mathbf{c} satisfying:

$$\begin{aligned}\mathcal{J}\mathbf{c} + \mathbf{g}_{\text{ext}} &= 0 \\ \mathbf{c} &\in \mathcal{F}(\boldsymbol{\mu}),\end{aligned}\tag{7}$$

where $\mathcal{F}(\boldsymbol{\mu})$ is the Coulomb friction cone; that is

$$\mathcal{F}(\boldsymbol{\mu}) \equiv \prod_{i=1}^{n_c} \{ (c_{in}, c_{it}, c_{io}) : c_{in} \geq 0, (c_{it}, c_{io}) \in \mathcal{F}(\mu_i c_{in}) \}$$

where $\boldsymbol{\mu} \equiv (\mu_i)$ is the vector of friction coefficients μ_i at the contacts and \prod represents the Cartesian product operation applied to the spaces of the contact forces. Note that the nonnegativity of the normal contact force c_n is guaranteed by the friction cone $\mathcal{F}(\boldsymbol{\mu})$. We should point out that for the standard Coulomb friction laws (4) and (5), (7) is a convex inequality system; as such, determining its consistency is in general not a difficult task. In particular, with the friction pyramid law (5), (7) is a system of linear inequalities and its consistency can therefore be checked by linear programming methods.

The above discussion suggests that the task of checking if a given applied load is weakly stable (or equivalently strongly unstable) is not particularly difficult. Indeed, this task can be accomplished in "polynomial time" by an interior point method [13], even in the case of the quadratic friction cone. Practically, this method is expected to be highly efficient when applied to the system (7).

Geometrically, the system (7) defines the cone of weakly stable forces:

$$\text{WS}(\boldsymbol{\mu}) \equiv \{ \mathbf{g}_{\text{ext}} : \text{the system (7) is consistent} \}.\tag{8}$$

Clearly, this cone is the image of the friction cone $\mathcal{F}(\boldsymbol{\mu})$ under the linear transformation defined by the negative of the system Jacobian matrix \mathcal{J} ; that is,

$$\text{WS}(\boldsymbol{\mu}) = -\mathcal{J}(\mathcal{F}(\boldsymbol{\mu})).$$

As will be seen, the cone $\text{WS}(\boldsymbol{\mu})$ will play a central role throughout our study. The complement of $\text{WS}(\boldsymbol{\mu})$ consists of the strongly unstable applied loads. Under a polyhedral friction law, such as the friction pyramid law (5), the cone $\text{WS}(\boldsymbol{\mu})$ is polyhedral. We illustrate this cone in two examples below.

Example 1: Consider a uniform laminar disk (the workpiece) of mass m and radius R in the plane in contact with two immovable fixels and external loading as shown in Figure 2. The fixels are located by the angles θ_1 and θ_2 measured in the counterclockwise directions about the origin of an inertial coordinate frame centered at the center of the disk. The components of the contact forces, c_{1n} and c_{2n} , are directed from the fixels toward the center of the disk. The corresponding friction force components are tangent to the disk, with positive values of c_{it} assumed to produce clockwise (negative) moments. We wish to examine conditions on the angles $\theta_i \in (0, \pi)$ and the friction coefficients μ_i so that an applied load $\mathbf{g}_{\text{ext}} \in \mathbb{R}^3$ is weakly stable.

The data for this problem are as follows. The problem is planar; thus there is only one tangential direction (no "o" direction) at each contact. There are two contact points; thus $n_c = 2$. Moreover

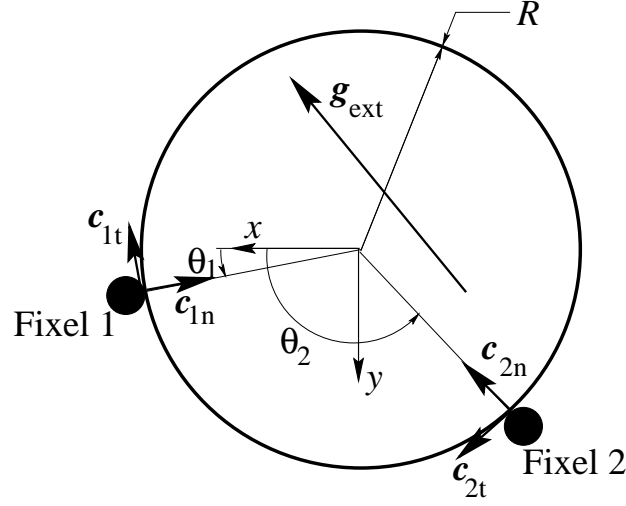


Figure 2: A loaded laminar disk in contact with immovable fixels.

we have:

$$\mathbf{W}_n \equiv \begin{bmatrix} -\cos \theta_1 & -\cos \theta_2 \\ -\sin \theta_1 & -\sin \theta_2 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{W}_t \equiv \begin{bmatrix} \sin \theta_1 & \sin \theta_2 \\ -\cos \theta_1 & -\cos \theta_2 \\ -R & -R \end{bmatrix}$$

$$\mathcal{M} = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & mR^2/2 \end{bmatrix}, \quad \mathbf{g}_{\text{ext}} = \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}.$$

Let $r \equiv \sin(\theta_2 - \theta_1)$ and $s \equiv \cos(\theta_2 - \theta_1)$. Note that $r = 0$ when the fixels are diametrically opposed or coincident. Since these special cases can be resolved easily, we will assume in what follows that $r \neq 0$. By reordering the fixels if necessary, we may further take r to be positive.

The condition for the weak stability of \mathbf{g}_{ext} is the existence of $\mathbf{c} = (c_{1n}, c_{2n}, c_{1t}, c_{2t})$ satisfying the following linear inequality system:

$$\mathcal{J}\mathbf{c} + \mathbf{g}_{\text{ext}} = 0 \quad (9)$$

$$|c_{it}| \leq \mu_i c_{in}, \quad i = 1, 2. \quad (10)$$

A contact force \mathbf{c} satisfying the static equilibrium equation (9) can be solved in terms of the friction force at the second contact:

$$\begin{bmatrix} c_{1n} \\ c_{2n} \\ c_{1t} \\ c_{2t} \end{bmatrix} = \begin{bmatrix} r^{-1}(g_1 \sin \theta_2 - g_2 \cos \theta_2 + sg_3/R) \\ r^{-1}(-g_1 \sin \theta_1 + g_2 \cos \theta_1 - g_3/R) \\ g_3/R \\ 0 \end{bmatrix} + \begin{bmatrix} r^{-1}(1-s) \\ r^{-1}(1-s) \\ -1 \\ 1 \end{bmatrix} c_{2t}. \quad (11)$$

Substituting this expression into the friction constraints (10) yields the following four inequalities:

$$-\mu_1 g_1 \sin \theta_2 + \mu_1 g_2 \cos \theta_2 + (r - \mu_1 s) g_3 / R \leq (r + \mu_1 (1 - s)) c_{2t} \quad (12)$$

$$-\mu_1 g_1 \sin \theta_2 + \mu_1 g_2 \cos \theta_2 - (r + \mu_1 s) g_3 / R \leq (-r + \mu_1 (1 - s)) c_{2t} \quad (13)$$

$$-\mu_2 g_1 \sin \theta_1 + \mu_2 g_2 \cos \theta_1 - \mu_2 g_3 / R \geq (r - \mu_2 (1 - s)) c_{2t} \quad (14)$$

$$-\mu_2 g_1 \sin \theta_1 + \mu_2 g_2 \cos \theta_1 - \mu_2 g_3 / R \geq -(r + \mu_2 (1 - s)) c_{2t}. \quad (15)$$

Thus the cone $WS(\boldsymbol{\mu})$ consists of all triples (g_1, g_2, g_3) for which there exists a scalar c_{2t} such that the above four inequalities hold. Figure 3 shows $WS(\boldsymbol{\mu})$ on the unit sphere centered at the origin of \mathbb{R}^3 for $\mu_1 = 0.2$, $\mu_2 = 0.5$, $\theta_1 = \pi/4$, and $\theta_2 = 2.5\pi/4$. The generators of $WS(\boldsymbol{\mu})$ are indicated by four dark gray bubbles, which delimit the edges of the convex cone of weakly stable external loads. Any external load passing through the interior or boundary of the "quadrilateral" formed by the segments of the great circle shown (by the small black bubbles) has weak stability. Note that the g_3 direction (the moment direction) is indicated by the big black bubble on the top of the sphere. The g_2 direction (the y -component direction of the external force) is marked by the black bubble inside the quadrilateral.

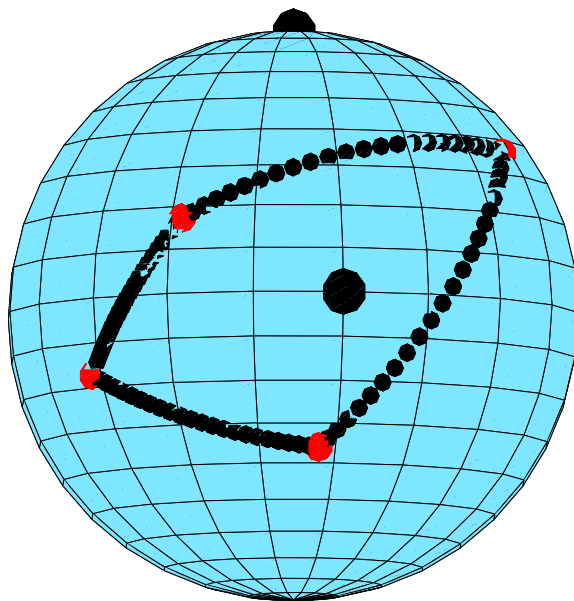


Figure 3: The set of weakly stable external loads in \mathbb{R}^3 . The friction coefficients are taken as: $\mu_1 = 0.2$ and $\mu_2 = 0.5$

Figure 4 shows the two dimensional slice of $WS(\boldsymbol{\mu})$ through the equator of the sphere shown in Figure 3, thus corresponding to $g_3 = 0$. With $g_3 = 0$ in this Figure, the external loads in question are those representable as pure forces passing through the center of the disk. Notice that the set of weakly stable external forces are those contained in the convex cone formed by the radii to the contact points.

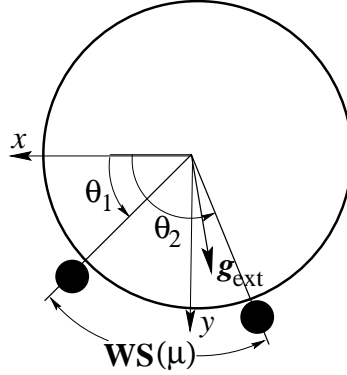


Figure 4: The set of weakly stable external loads in \mathfrak{R}^2 ; $g_3 = 0$. The friction coefficients are taken as: $\mu_1 = 0.2$ and $\mu_2 = 0.5$

Simplification of inequalities (12–15) is possible by considering the signs of the two scalars

$$\delta_1 \equiv -r + \mu_1(1 - s) \quad \text{and} \quad \delta_2 \equiv -r + \mu_2(1 - s)$$

which pertain to the geometry and friction coefficients of the problem and are independent of the load \mathbf{g}_{ext} . We illustrate their simplification in two cases; other cases can be analyzed analogously.

Case 1. Both δ_1 and δ_2 are positive; that is

$$\min(\mu_1, \mu_2) > \frac{r}{1 - s}.$$

In this case $\text{WS}(\boldsymbol{\mu}) = \mathfrak{R}^3$; that is, all loads \mathbf{g}_{ext} are weakly stable. The reader should note that δ_i is positive if and only if the friction cone at contact i contains the other contact point. Thus this case corresponds to the situation referred to as “force closed” by Nguyen [14]. Physically, the result that all loads are weakly stable implies that every \mathbf{g}_{ext} can be balanced by a combination of normal and friction loads, provided that the normal components of the contact forces can be made arbitrarily large. For this example, δ_1 and δ_2 are positive if and only if the smaller of the friction coefficients is greater than 1.4966. In terms of the unit sphere in Figure 3, increasing the friction coefficients corresponds to separating the generators (the dark gray bubbles). Once the values of both friction coefficients increase beyond 1.4966, the 4 generators positively span \mathfrak{R}^3 .

Case 2. δ_1 and δ_2 differ in sign, say

$$\mu_1 > \frac{r}{1 - s} \quad \text{and} \quad \mu_2 < \frac{r}{1 - s},$$

in other words, the friction cone at contact 1 can “see” contact 2, but the converse is not true. In this case, a load \mathbf{g}_{ext} is weakly stable if and only if

$$\max \left(\frac{-\mu_1 g_1 \sin \theta_2 + \mu_1 g_2 \cos \theta_2 + (r - \mu_1 s) g_3 / R}{-r + \mu_1(1 - s)}, \frac{-\mu_1 g_1 \sin \theta_2 + \mu_1 g_2 \cos \theta_2 + (r - \mu_1 s) g_3 / R}{r + \mu_1(1 - s)} \right)$$

$$\leq \min \left(\frac{\mu_2 g_1 \sin \theta_1 - \mu_2 g_2 \cos \theta_1 + \mu_2 g_3 / R}{r - \mu_2(1 - s)}, \frac{\mu_2 g_1 \sin \theta_1 - \mu_2 g_2 \cos \theta_1 + \mu_2 g_3 / R}{r + \mu_2(1 - s)} \right).$$

Figure 5 shows the set of weakly stable loads for $\mu_1 = 1.8$, $\mu_2 = 0.5$ with the other data remaining the same as above, $\theta_1 = \pi/4$ and $\theta_2 = 2.5\pi/4$. Again, the big black bubble on the top of the sphere indicates the g_3 direction while the one on the equator represents the y -component of the external load. Note that increasing μ_1 from 0.2 causes the two left-most generators in the "quadrilateral" shown in Figure 3 to separate following their great circle. When μ_1 reaches a value of 1.4966 the left-most generator from the original quadrilateral reaches the great circle defined by the two right-most generators, causing the "quadrilateral" to degenerate into a "triangle." Further increasing μ_1 to 1.8 yields the triangular set shown, with one of the original generators (shown as a gray dot) inside.

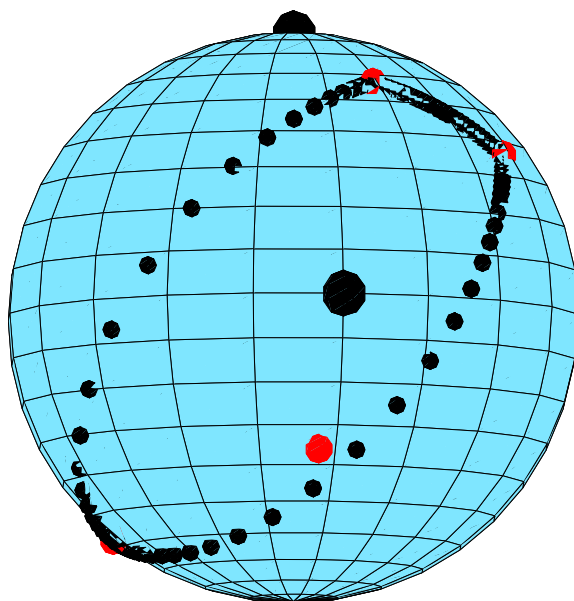


Figure 5: The set of weakly stable external loads in \mathcal{R}^3 . The friction coefficients are taken as: $\mu_1 = 1.8$ and $\mu_2 = 0.5$.

Figure 6 shows the slice through the sphere corresponding to $g_3 = 0$. Note that the set $WS(\mu)$ has grown (by 0.08 radians) to include loads outside the cone formed by the radii to the contact points. \square

Example 2. Figure 7 shows a block initially at rest on a fixed inclined plane. Representing the contact force as a pair of point loads at the ends of the line segment of contact yields the following wrench matrices:

$$\mathbf{W}_n = \begin{bmatrix} -\sin \theta & -\sin \theta \\ \cos \theta & \cos \theta \\ 1 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{W}_t = \begin{bmatrix} -\cos \theta & -\cos \theta \\ -\sin \theta & -\sin \theta \\ -1 & -1 \end{bmatrix},$$

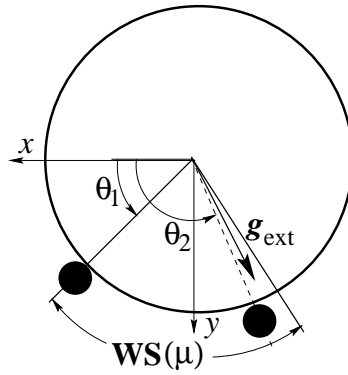


Figure 6: The set of weakly stable external loads in \mathcal{R}^2 ; $g_3 = 0$. The friction coefficients are taken as: $\mu_1 = 1.8$ and $\mu_2 = 0.5$.

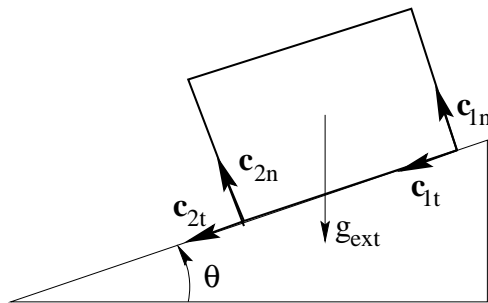


Figure 7: A block at rest on an inclined plane.

where $\theta \in (0, \pi/2)$. Proceeding as in Example 1, we see that for a force vector $\mathbf{c} = (c_{1n}, c_{2n}, c_{1t}, c_{2t})$ satisfying the static equilibrium equation (9), the following must hold:

$$\begin{aligned} c_{1n} &= \frac{1}{2} [g_1 (\sin \theta + \cos \theta) + g_2 (\sin \theta - \cos \theta) - g_3] \\ c_{2n} &= \frac{1}{2} [g_1 (\sin \theta - \cos \theta) - g_2 (\sin \theta + \cos \theta) + g_3] \\ c_{1t} &= g_1 \cos \theta + g_2 \sin \theta - c_{2t}. \end{aligned} \quad (16)$$

Substituting these equations into the friction constraints (10) yields the following four inequalities:

$$\begin{aligned} g_1 ((2 - \mu_1) \cos \theta - \mu_1 \sin \theta) + g_2 ((2 - \mu_1) \sin \theta + \mu_1 \cos \theta) + \mu_1 g_3 &\leq 2c_{2t} \\ g_1 ((2 + \mu_1) \cos \theta + \mu_1 \sin \theta) + g_2 ((2 + \mu_1) \sin \theta - \mu_1 \cos \theta) - \mu_1 g_3 &\geq 2c_{2t} \\ \mu_2 g_1 (\sin \theta - \cos \theta) - \mu_2 g_2 (\sin \theta + \cos \theta) + \mu_2 g_3 &\geq 2c_{2t} \\ -\mu_2 g_1 (\sin \theta - \cos \theta) + \mu_2 g_2 (\sin \theta + \cos \theta) - \mu_2 g_3 &\leq 2c_{2t}. \end{aligned}$$

From these inequalities, we deduce that a load $\mathbf{g}_{\text{ext}} = (g_1, g_2, g_3)$ is weakly stable if and only if

$$g_1 (\sin \theta + \cos \theta) + g_2 (\sin \theta - \cos \theta) - g_3 \geq 0 \quad (17)$$

$$g_1 (\sin \theta - \cos \theta) - g_2 (\sin \theta + \cos \theta) + g_3 \geq 0 \quad (18)$$

$$\begin{aligned} g_1 [(\mu_1 + \mu_2) \sin \theta + (\mu_1 - \mu_2 - 2) \cos \theta] + g_2 [(\mu_1 - \mu_2 - 2) \sin \theta - (\mu_1 + \mu_2) \cos \theta] \\ + (\mu_2 - \mu_1) g_3 \geq 0 \end{aligned} \quad (19)$$

$$\begin{aligned} g_1 [(\mu_1 + \mu_2) \sin \theta + (\mu_1 - \mu_2 + 2) \cos \theta] + g_2 [(\mu_1 - \mu_2 + 2) \sin \theta - (\mu_1 + \mu_2) \cos \theta] \\ + (\mu_2 - \mu_1) g_3 \geq 0. \end{aligned} \quad (20)$$

The physical interpretation of these inequalities is as follows. Inequalities (17) and (18), represent the constraint that the line of action of the external load must pass through the contact segment and must have a component opposite the outward normal of the ramp. This ensures that the block will not tip or spontaneously lift off the ramp. Inequalities (19) and (20), represent the constraints the external load to lie inside the composite friction cone making sliding impossible. \square

We close this section by noting that decreasing any coefficient of friction causes the set of weakly stable loads to contract monotonically:

$$\text{WS}(\boldsymbol{\mu}) \subseteq \text{WS}(\boldsymbol{\mu}') \quad \text{if } \boldsymbol{\mu} \leq \boldsymbol{\mu}'.$$

3 Main Results

In this section, we present the main results pertaining to the stability concepts defined in the last section. We begin with a preliminary result that gives an equivalent way of describing strong stability. In essence, this result asserts that strong stability (i.e., zero body accelerations, $\ddot{\mathbf{q}} = 0$) can be characterized as nonpositive virtual work (i.e., $\ddot{\mathbf{q}}^T \mathbf{g}_{\text{ext}} \leq 0$)³; this result is consistent with the asserted equivalence in Definition 1 of stability. A proof of the following proposition is given in Appendix B.

³Note that since the system begins at rest, the instantaneous acceleration $\ddot{\mathbf{q}}$ is proportional to the instantaneous velocity $\dot{\mathbf{q}}$ and hence the expression given is proportional to the virtual work.

Proposition 1 *Let \mathbf{g}_{ext} be a given applied load. The following statements are equivalent:*

- (a) \mathbf{g}_{ext} is strongly stable;
- (b) every dynamic intensity yields nonpositive virtual work.
- (c) \mathbf{g}_{ext} is weakly stable and every dynamic equilibrium intensity yields zero relative tangential accelerations, \mathbf{a}_t and \mathbf{a}_o .

The distinction between weak stability and strong stability is clearly due to the nonuniqueness of dynamic intensities. If the dynamic rigid body contact model has a unique solution, then these two stability concepts are equivalent. Based on a uniqueness result obtained in [20], we state a sufficient condition for this equivalence to hold. Subsequently, this result will be generalized.

Proposition 2 *Suppose that the Jacobian matrix \mathcal{J} has full column rank. There exists a scalar $\bar{\mu} > 0$ such that if $\mu_i \in [0, \bar{\mu}]$ for all $i = 1, \dots, n_c$, \mathbf{g}_{ext} is weakly (un)stable if and only if it is strongly (un)stable.*

The scalar $\bar{\mu}$ has to do with the preservation of the "P-property" of certain perturbations of the matrix \mathbf{A} (which is positive definite under the full rank assumption in the above proposition). For more discussion on this scalar, see [19].

3.1 The role of the frictionless stability

The frictionless problem corresponds to $\boldsymbol{\mu} = 0$. This case plays an important role in the frictional problem. For one thing, the frictionless case provides another instance where weak and strong stability are equivalent. This is part of the content of Theorem 1 below. Frictionless stability refers to (weak or strong) stability in the frictionless problem. Besides establishing the equivalence of weak and strong stability, this theorem also shows that frictionless stability is easy to check, namely, by solving a linear program; furthermore, frictionless stability is actually equivalent to (weak or strong) stability for all friction coefficients. Thus we see that frictionless stability is a very desirable property. Note that while many have previously conjectured that frictionless stability implies strong stability with friction, we were not aware of a formal proof until now (see Appendix B).

Unlike Proposition 2, the theorem below and all subsequent results do not require \mathcal{J} to have full column rank.

Theorem 1 *Let \mathbf{g}_{ext} be a given applied load. The following five statements are equivalent.*

- (a) There exists a vector $\hat{\mathbf{c}}_n$ satisfying

$$\mathbf{W}_n \hat{\mathbf{c}}_n + \mathbf{g}_{\text{ext}} = 0, \quad \hat{\mathbf{c}}_n \geq 0. \quad (21)$$

- (b) The load \mathbf{g}_{ext} is weakly stable for all friction coefficients.
- (c) The load \mathbf{g}_{ext} is strongly stable for all friction coefficients.
- (d) The load \mathbf{g}_{ext} is weakly stable for the frictionless problem.
- (e) The load \mathbf{g}_{ext} is strongly stable for the frictionless problem.

With the above result, it is natural to ask what happens if frictionless stability is absent. The next result asserts that if the workpiece possesses a certain "separation property", then frictionless instability implies strong instability in the case of small friction coefficients; thus in this situation, there must be sufficient friction at the contacts in order for strong, or even weak, stability to hold.

Theorem 2 *Suppose that there exists a vector \mathbf{u}_n satisfying $\mathbf{W}_n^T \mathbf{u}_n > 0$. The following two statements are equivalent:*

- (a) \mathbf{g}_{ext} is (weakly or strongly) unstable for the frictionless problem;
- (b) there exists a scalar $\bar{\mu} > 0$ such that if $\mu_i \in [0, \bar{\mu}]$ for all $i = 1, \dots, n_c$, $\mathbf{g}_{\text{ext}} \notin WS(\boldsymbol{\mu})$; that is, \mathbf{g}_{ext} is strongly unstable for the problem with $\boldsymbol{\mu} \equiv (\mu_i)$.

The physical interpretation of the assumption of Theorem 2 (that is, the existence of the vector \mathbf{u}_n) is as follows. If there exists a generalized acceleration (\mathbf{u}_n) of the fixtured workpiece that would cause all contacts to separate simultaneously, then the external load is strongly unstable for all friction coefficients sufficiently small if and only if it is strongly unstable when there is no friction. Notice that the existence of such a separating acceleration \mathbf{u}_n depends entirely on geometry and has nothing to do with the applied load. We say that the workpiece has the *separation property* if such an acceleration exists.

From Theorems 1 and 2, it becomes evident that the most difficult case for analyzing strong stability is when the load is not (strongly or weakly) stable in frictionless contact but becomes strongly stable when friction is present. A critical value of the friction coefficients where the transition from instability to (weak or strong) stability occurs (if it occurs at all) is unfortunately not known and is expected to be very difficult to determine in general. Nevertheless, such a value can be computed in special cases.

In order to illustrate Theorems 1 and 2, it will be useful to introduce the polyhedral cone defined by all nonnegative combinations of the columns of the matrix $-\mathbf{W}_n$; that is,

$$WS(0) \equiv -\text{pos}(\mathbf{W}_n) = \{ \mathbf{g}_{\text{ext}} : \text{the system (21) is consistent} \}.$$

Theorem 1 then says that this cone $WS(0)$ is precisely the set of all applied loads \mathbf{g}_{ext} that are strongly stable for all friction coefficients; moreover, Theorem 2 implies that if the workpiece has the separation property, then a load $\mathbf{g}_{\text{ext}} \notin WS(0)$ is weakly (not necessarily strongly) stable only if there is sufficient friction at the contacts. We illustrate Theorems 1 and 2 further using the two examples from the last section.

Example 1 (continued): Setting $\mu_1 = \mu_2 = 0$, we conclude from (11) that the cone $WS(0)$ consists of all loads (g_1, g_2, g_3) that satisfy $g_3 = 0$,

$$g_1 \sin \theta_2 - g_2 \cos \theta_2 \geq 0 \quad \text{and} \quad -g_1 \sin \theta_1 + g_2 \cos \theta_1 \geq 0. \quad (22)$$

This closed cone consists of all pure forces passing through the center of the disk and passing between the two contacts or through one of them. It is the same set illustrated in Figure 4 as $WS(\boldsymbol{\mu})$.

With $\mathbf{u}_n \equiv (0, -1, 0)$, we clearly have $\mathbf{W}_n^T \mathbf{u}_n > 0$. Thus the disk has the separation property. To illustrate Theorem 2, consider a load $\mathbf{g}_{\text{ext}} = (g_1, g_2, 0)$ that fails one of the two conditions in (22), and therefore lies outside of $WS(0)$. Such a load is illustrated in Figure 6. In order for this

load to be weakly stable in the frictional case, the previous analysis implies that there must exist a scalar c_{2t} such that the four inequalities (12–15) hold. It is not difficult to verify that if

$$0 \leq \max(\mu_1, \mu_2) < \frac{r}{1-s} = 1.4966 = \bar{\mu}$$

(i.e., neither contact's friction cone sees the other contact point), then there cannot exist any c_{2t} that balances $\mathbf{g}_{\text{ext}} \notin \text{WS}(0)$. Theorem 2 is therefore verified. \square

Example 2 (continued): Setting $\mu_1 = \mu_2 = 0$, we conclude from (16) that the cone $\text{WS}(0)$ consists of all loads (g_1, g_2, g_3) such that

$$g_1 \sin \theta - g_2 \cos \theta \geq |g_3| \quad \text{and} \quad g_1 \cos \theta + g_2 \sin \theta = 0. \quad (23)$$

These constraints preclude the possibility of initiating rotational motion and translational motion, respectively.

Clearly with $\mathbf{u}_n = -(1, t, 0)^T$ where $0 < t < \tan \theta$, we have $\mathbf{W}_n^T \mathbf{u}_n > 0$. Consider the vector $\mathbf{g}_{\text{ext}} = (0, -1, 0)^T$ which is easily seen to lie outside of $\text{WS}(0)$. It is not difficult to show by (17–20) that if

$$0 \leq \max(\mu_1, \mu_2) < \frac{1}{2} \tan \theta = \bar{\mu},$$

then this load \mathbf{g}_{ext} is not weakly stable for the frictional problem; again verifying Theorem 2. Note that if $\mu_1 = \mu_2 = \mu$, then one obtains the well-known result, $0 \leq \mu < \tan(\theta)$. The " $\frac{1}{2}$ " arises from the situations in which the friction coefficient at one contact is zero. \square

3.2 The WU_R sets

One of the primary goals of this paper is to identify the set of loads that are strongly stable (i.e., members of $\text{SS}(\boldsymbol{\mu})$) for a given friction coefficient. Since it is hard to identify such loads directly, we are interested in identifying loads that are weakly unstable ($\text{WU}(\boldsymbol{\mu})$) and therefore known to lie outside $\text{SS}(\boldsymbol{\mu})$. By Theorem 1, we know that loads lying outside of $\text{SS}(\boldsymbol{\mu})$ must also lie outside of the cone $\text{SS}(0)$. In order to motivate the main result in this subsection, Theorem 3, we state a preliminary result pertaining to the frictionless problem. The next result is inspired by the concept of a complementary cone in linear complementarity theory which we review in Appendix A.

Proposition 3 *If \mathbf{W}_n has full row rank, an applied load \mathbf{g}_{ext} is (weakly or strongly) unstable for the frictionless problem if and only if there exist a nonempty subset α of $\{1, \dots, n_c\}$ with complement $\bar{\alpha}$ and nonnegative vectors $\mathbf{a}_{n\alpha}$ and $\mathbf{c}_{n\bar{\alpha}}$ with $\mathbf{a}_{n\alpha} \neq 0$ such that*

$$\mathbf{W}_n^T \mathcal{M}^{-1} \mathbf{g}_{\text{ext}} = \mathbf{I}_{\cdot\alpha} \mathbf{a}_{n\alpha} - (\mathbf{A}_{nn})_{\cdot\bar{\alpha}} \mathbf{c}_{n\bar{\alpha}}.$$

Here α and $\bar{\alpha}$ are the index sets of the contacts that are to be separated and maintained, respectively. The dot subscript following \mathbf{A}_{nn} indicates that all rows of \mathbf{A}_{nn} are included. Notice that Proposition 3 depends on the full row rank assumption of \mathbf{W}_n to guarantee that if $\mathbf{a}_n = 0$, then $\bar{\mathbf{q}} = 0$ also. Since this rank condition is rather restrictive, this requirement is removed in the next proposition. Without this restriction, the phrase "and only if" must be removed.

Proposition 4 *An applied load \mathbf{g}_{ext} is (weakly or strongly) unstable for the frictionless problem if there exists a nonempty subset α of $\{1, \dots, n_c\}$ with complement $\bar{\alpha}$ and nonnegative vectors $\mathbf{a}_{n\alpha}$ and $\mathbf{c}_{n\bar{\alpha}}$ with $\mathbf{a}_{n\alpha} \neq 0$ such that*

$$\mathbf{W}_n^T \mathcal{M}^{-1} \mathbf{g}_{\text{ext}} = \mathbf{I}_{\cdot\alpha} \mathbf{a}_{n\alpha} - (\mathbf{A}_{nn})_{\cdot\bar{\alpha}} \mathbf{c}_{n\bar{\alpha}}.$$

The equation in the above propositions can be easily derived from the original Newton Euler equation (1) by setting friction forces to zero and removing the equations corresponding to the tangential components of the contact accelerations.⁴ Applying all subsets α of $\{1, \dots, n_c\}$ to the equation represents all possible combinations of breaking and maintained contacts. We will henceforth refer to each such combination as a "contact mode." Note that the set of external loads corresponding to any particular contact mode is a convex cone. Thus we see that the set of applied loads, denoted WU_{fl} , that are unstable for the frictionless problems can be described in terms of the union of finitely many polyhedra (the subscript "fl" denotes "frictionless").

Introducing friction into the problem, we define, for a given nonzero friction vector $\boldsymbol{\mu} \equiv (\mu_i)$, the set $\text{WU}_{\text{R}}(\boldsymbol{\mu})$ consisting of all load vectors \mathbf{g}_{ext} for which there exist a nonempty subset α of $\{1, \dots, n_c\}$ with complement $\bar{\alpha}$, nonnegative vectors $\mathbf{a}_{n\alpha}$ and $\mathbf{c}_{n\bar{\alpha}}$ with $\mathbf{a}_{n\alpha} \neq 0$, and (free) vectors $\mathbf{c}_{t\bar{\alpha}}$ and $\mathbf{c}_{o\bar{\alpha}}$ such that

$$\begin{aligned} \mathbf{W}_n^T \mathcal{M}^{-1} \mathbf{g}_{\text{ext}} &= \mathbf{I}_{\cdot\alpha} \mathbf{a}_{n\alpha} - (\mathbf{A}_{nn})_{\cdot\bar{\alpha}} \mathbf{c}_{n\bar{\alpha}} - (\mathbf{A}_{nt})_{\cdot\bar{\alpha}} \mathbf{c}_{t\bar{\alpha}} - (\mathbf{A}_{no})_{\cdot\bar{\alpha}} \mathbf{c}_{o\bar{\alpha}} \\ (\mathbf{W}_t^T \mathcal{M}^{-1} \mathbf{g}_{\text{ext}})_{\bar{\alpha}} &= -(\mathbf{A}_{tn})_{\cdot\bar{\alpha}} \mathbf{c}_{n\bar{\alpha}} - (\mathbf{A}_{tt})_{\cdot\bar{\alpha}} \mathbf{c}_{t\bar{\alpha}} - (\mathbf{A}_{to})_{\cdot\bar{\alpha}} \mathbf{c}_{o\bar{\alpha}} \\ (\mathbf{W}_o^T \mathcal{M}^{-1} \mathbf{g}_{\text{ext}})_{\bar{\alpha}} &= -(\mathbf{A}_{on})_{\cdot\bar{\alpha}} \mathbf{c}_{n\bar{\alpha}} - (\mathbf{A}_{ot})_{\cdot\bar{\alpha}} \mathbf{c}_{t\bar{\alpha}} - (\mathbf{A}_{oo})_{\cdot\bar{\alpha}} \mathbf{c}_{o\bar{\alpha}} \end{aligned}$$

and

$$(\mathbf{c}_{it}, \mathbf{c}_{io}) \in \mathcal{F}(\mu_i \mathbf{c}_{in}), \quad \forall i \in \bar{\alpha}.$$

These equations define the set of external loads for which all contacts either separate or roll. Sliding is not allowed, as indicated by the absence of $\mathbf{a}_{t\alpha}$ and $\mathbf{a}_{o\alpha}$.

Of particular interest among these $\text{WU}_{\text{R}}(\boldsymbol{\mu})$ sets is $\text{WU}_{\text{R}}(0)$; this is clearly a subset of WU_{fl} ; moreover, because the set $\text{WU}_{\text{R}}(\boldsymbol{\mu})$ does not include external loads corresponding to sliding contacts, we have

$$\text{WU}_{\text{R}}(\boldsymbol{\mu}) \subseteq \text{WU}_{\text{R}}(\boldsymbol{\mu}') \quad \text{if } \boldsymbol{\mu} \leq \boldsymbol{\mu}'. \quad (24)$$

In words, as the friction coefficient increases, the set of weakly unstable loads with no sliding contacts grows.

The role of the WU_{R} sets is formally established in the result below.

Theorem 3 *If $\mathbf{g}_{\text{ext}} \in \text{WU}_{\text{R}}(\hat{\boldsymbol{\mu}})$ for some friction vectors $\hat{\boldsymbol{\mu}}$, then \mathbf{g}_{ext} is weakly unstable for all friction vectors $\boldsymbol{\mu} \geq \hat{\boldsymbol{\mu}}$. In particular, if \mathbf{g}_{ext} lies in $\text{WU}_{\text{R}}(0)$, then \mathbf{g}_{ext} is weakly unstable (via a non sliding contact mode) for all friction coefficients.*

Example 1 (further continued). For convenience, we take the disk radius $R = \sqrt{2}$ and the mass $m = 1$; thus \mathcal{M} becomes the identity matrix. Recalling the quantities:

$$r = \sin(\theta_2 - \theta_1) \quad \text{and} \quad s = \cos(\theta_2 - \theta_1)$$

⁴Note that the "and only if" could be reinserted into Proposition 4 if one adds enough additional linearly independent equations corresponding to nonzero values of the relative translational and relative angular accelerations at the contacts. However, we restrict our attention here to the present propositions.

we have

$$\mathbf{A} = \left[\begin{array}{cc|cc} 1 & s & 0 & -r \\ s & 1 & r & 0 \\ \hline 0 & r & 3 & 2+s \\ \hline -r & 0 & 2+s & 3 \end{array} \right].$$

Omitting the algebraic manipulations, we can obtain a complete description of the set $\text{WU}_{\mathbf{R}}(\boldsymbol{\mu})$ as three convex cones. For this purpose, we define several vectors

$$\begin{aligned} g^1 &\equiv \begin{bmatrix} -\sin \theta_1 \\ \cos \theta_1 \\ 0 \end{bmatrix}, & g^2 &\equiv \begin{bmatrix} \sin \theta_2 \\ -\cos \theta_2 \\ 0 \end{bmatrix} \\ g^3 &\equiv \begin{bmatrix} \sin \theta_1 \\ -\cos \theta_1 \\ 1/\sqrt{2} \end{bmatrix}, & g^4 &\equiv \begin{bmatrix} -\sin \theta_2 \\ \cos \theta_2 \\ -1/\sqrt{2} \end{bmatrix} \\ g^5 &\equiv \begin{bmatrix} -\sin \theta_1 \\ \cos \theta_1 \\ \sqrt{2} \end{bmatrix}, & g^6 &\equiv \begin{bmatrix} -\sin \theta_2 \\ \cos \theta_2 \\ \sqrt{2} \end{bmatrix} \end{aligned}$$

and $u^3 = (0, 0, 1)^T$. We have

$$\text{WU}_{\mathbf{R}}(\boldsymbol{\mu}) = \text{WU}_{\mathbf{R}_1}(\boldsymbol{\mu}) \cup \text{WU}_{\mathbf{R}_2}(\boldsymbol{\mu}) \cup \text{WU}_{\mathbf{R}_{12}}$$

where

$$\begin{aligned} \text{WU}_{\mathbf{R}_1}(\boldsymbol{\mu}) &\equiv \{ \mathbf{g}_{\text{ext}} \in \mathfrak{R}^3 : \mathbf{g}_{\text{ext}} = x_1 g^3 + x_2 (s g^1 + g^2) + x_3 g^5 \\ &\quad \text{for some } (x_1, x_2, x_3) \text{ such that } x_1 > 0, |x_3| \leq \mu_1 x_2 \} \\ \text{WU}_{\mathbf{R}_2}(\boldsymbol{\mu}) &\equiv \{ \mathbf{g}_{\text{ext}} \in \mathfrak{R}^3 : \mathbf{g}_{\text{ext}} = x_1 g^4 + x_2 (g^1 + s g^2) + x_3 g^6 \\ &\quad \text{for some } (x_1, x_2, x_3) \text{ such that } x_1 > 0, |x_3| \leq \mu_2 x_2 \} \\ \text{WU}_{\mathbf{R}_{12}} &\equiv \{ \mathbf{g}_{\text{ext}} \in \mathfrak{R}^3 : \mathbf{g}_{\text{ext}} = -x_1 g^1 - x_2 g^2 + x_3 u^3 \\ &\quad \text{for some } (x_1, x_2, x_3) \text{ such that } 0 \neq (x_1, x_2) \geq 0 \}. \end{aligned}$$

These sets are illustrated on the unit sphere in \mathfrak{R}^3 in Figure 8 for $\mu_1 = 0.2$ and $\mu_2 = 0.5$. The big black bubble at the north pole is the positive g_3 axis, while the big black bubble at the lower right points in the y -direction. The 5/16 sector of the sphere toward the back corresponds to $\text{WU}_{\mathbf{R}_{12}}$ and is independent of the values of the friction coefficients. It becomes narrower as the contact points separate on the disk. The triangular set in the front delineates the loads in $\text{WU}_{\mathbf{R}_1}$. The short leg of the triangle widens along its present great circle as μ_1 increases, as predicted by Theorem 3. As expected from the symmetry of this example, there is also a triangular set emanating from the other side of $\text{WU}_{\mathbf{R}_{12}}$ with a leg that extends with increasing values of μ_2 . The leg dependent on μ_2 is indicated by the small black dots on the right. It is interesting to note that the quadrilateral formed by the convex combination of the two extensible legs of the triangular regions is exactly the set

$WS(\boldsymbol{\mu})$ shown in Figure 3 (as long as both friction coefficients are less than 1.4966). The remaining uncharted regions on the sphere correspond to external loads which induce contact modes with at least one sliding contact. This is true until one of the friction coefficients exceeds 1.4966. At this point, some loads correspond to more than one contact mode. \square

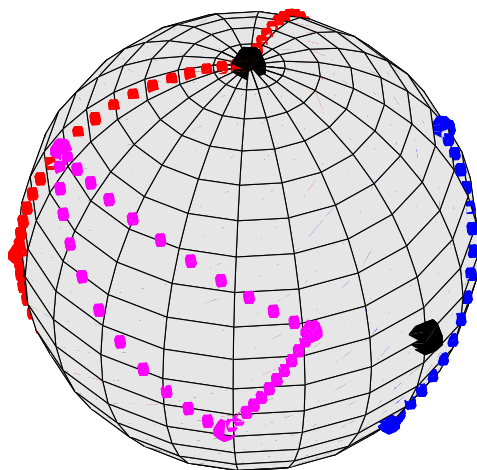


Figure 8: The set of weakly unstable external forces in \mathfrak{R}^3 . The friction coefficients are taken as: $\mu_1 = 0.2$ and $\mu_2 = 0.5$.

3.3 Strong stability for the friction pyramid law

Unlike the weakly stable loads, it is in general very difficult to test if an applied load \mathbf{g}_{ext} is strongly stable. This subsection is concerned with an important special case of the rigid body contact model with the friction pyramid (5). The main result herein, Theorem 4, identifies a class of rigid body systems for which a load \mathbf{g}_{ext} is strongly stable if and only if it is weakly stable. Thus for such a rigid body system, the task of identifying all the strongly stable loads becomes very easy.

The cornerstone of the main result in this subsection is the fact that rigid body contact problems with friction pyramid laws can be formulated as linear complementarity problems (LCPs); see [20, Section 3.2]. There are several such formulations; the basic one is as follows:

$$0 \leq \begin{bmatrix} \mathbf{c}_n \\ \mathbf{s}_t^+ \\ \mathbf{s}_o^+ \\ \mathbf{a}_t^- \\ \mathbf{a}_o^- \end{bmatrix} \perp \begin{bmatrix} \mathbf{a}_n \\ \mathbf{a}_t^+ \\ \mathbf{a}_o^+ \\ \mathbf{s}_t^- \\ \mathbf{s}_o^- \end{bmatrix} = M \begin{bmatrix} \mathbf{c}_n \\ \mathbf{s}_t^+ \\ \mathbf{s}_o^+ \\ \mathbf{a}_t^- \\ \mathbf{a}_o^- \end{bmatrix} + \begin{bmatrix} \mathbf{b}_n \\ \mathbf{b}_t \\ \mathbf{b}_o \\ 0 \\ 0 \end{bmatrix} \geq 0, \quad (25)$$

where

$$M \equiv \begin{bmatrix} M_{nn} & A_{nt} & A_{no} & 0 & 0 \\ M_{tn} & A_{tt} & A_{to} & I & 0 \\ M_{on} & A_{ot} & A_{oo} & 0 & I \\ 2 \operatorname{diag}(\boldsymbol{\mu}) & -I & 0 & 0 & 0 \\ 2 \operatorname{diag}(\boldsymbol{\mu}) & 0 & -I & 0 & 0 \end{bmatrix}$$

with

$$\begin{bmatrix} M_{nn} \\ M_{tn} \\ M_{on} \end{bmatrix} \equiv \begin{bmatrix} A_{nn} \\ A_{tn} \\ A_{on} \end{bmatrix} - \begin{bmatrix} A_{nt} + A_{no} \\ A_{tt} + A_{to} \\ A_{ot} + A_{oo} \end{bmatrix} \operatorname{diag}(\boldsymbol{\mu}).$$

In the above formulation, we have

$$\mathbf{a}_t^+ \equiv \max(0, \mathbf{a}), \quad \mathbf{a}_t^- \equiv \max(0, -\mathbf{a}),$$

$$\mathbf{s}_t^+ \equiv \operatorname{diag}(\boldsymbol{\mu}) \mathbf{c}_n + \mathbf{c}_t, \quad \mathbf{s}_t^- \equiv \operatorname{diag}(\boldsymbol{\mu}) \mathbf{c}_n - \mathbf{c}_t, \quad (26)$$

$$\mathbf{s}_o^+ \equiv \operatorname{diag}(\boldsymbol{\mu}) \mathbf{c}_n + \mathbf{c}_o, \quad \mathbf{s}_o^- \equiv \operatorname{diag}(\boldsymbol{\mu}) \mathbf{c}_n - \mathbf{c}_o. \quad (27)$$

The development that follows makes extensive use of LCP methodology; a summary of the relevant LCP results is contained in Appendix A.

By interchanging the roles of certain components of

$$\mathbf{a}_t^+, \mathbf{a}_o^+, \mathbf{s}_t^-, \mathbf{s}_o^-$$

with the corresponding components of their respective complementary partners,

$$\mathbf{s}_t^+, \mathbf{s}_o^+, \mathbf{a}_t^-, \mathbf{a}_o^-,$$

we obtain various equivalent formulations of the basic LCP (25). In LCP terminology, the latter formulations are "principal transforms" of (25) obtained by performing certain principal pivots corresponding to the above interchange of variables. The matrices defining the resulting equivalent LCPs have a structure similar to the above matrix M . In order to give the explicit expressions for these matrices, we define a "signed vector" $\boldsymbol{\sigma} \in \mathfrak{R}^{n_c}$ to be a vector such that for each $i = 1, \dots, n_c$, $\sigma_i \in \{1, -1\}$. For any such signed vector $\boldsymbol{\sigma}$, we let $\boldsymbol{\Sigma} \equiv \operatorname{diag}(\boldsymbol{\sigma})$; furthermore, for any two signed vectors $\boldsymbol{\sigma}_t$ and $\boldsymbol{\sigma}_o$ (and corresponding signed diagonal matrices $\boldsymbol{\Sigma}_t$ and $\boldsymbol{\Sigma}_o$), we define

$$M(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o) \equiv \begin{bmatrix} M_{nn}(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o) & A_{nt}\boldsymbol{\Sigma}_t & A_{no}\boldsymbol{\Sigma}_o & 0 & 0 \\ M_{tn}(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o) & \boldsymbol{\Sigma}_t A_{tt} \boldsymbol{\Sigma}_t & \boldsymbol{\Sigma}_t A_{to} \boldsymbol{\Sigma}_o & I & 0 \\ M_{on}(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o) & \boldsymbol{\Sigma}_o A_{ot} \boldsymbol{\Sigma}_t & \boldsymbol{\Sigma}_o A_{oo} \boldsymbol{\Sigma}_o & 0 & I \\ 2 \operatorname{diag}(\boldsymbol{\mu}) & -I & 0 & 0 & 0 \\ 2 \operatorname{diag}(\boldsymbol{\mu}) & 0 & -I & 0 & 0 \end{bmatrix}$$

with

$$\begin{bmatrix} M_{nn}(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o) \\ M_{tn}(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o) \\ M_{on}(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o) \end{bmatrix} \equiv \begin{bmatrix} A_{nn} \\ \boldsymbol{\Sigma}_t A_{tn} \\ \boldsymbol{\Sigma}_o A_{on} \end{bmatrix} - \begin{bmatrix} A_{nt}\boldsymbol{\Sigma}_t + A_{no}\boldsymbol{\Sigma}_o \\ \boldsymbol{\Sigma}_t A_{tt} \boldsymbol{\Sigma}_t + \boldsymbol{\Sigma}_t A_{to} \boldsymbol{\Sigma}_o \\ \boldsymbol{\Sigma}_o A_{ot} \boldsymbol{\Sigma}_t + \boldsymbol{\Sigma}_o A_{oo} \boldsymbol{\Sigma}_o \end{bmatrix} \operatorname{diag}(\boldsymbol{\mu}).$$

Each matrix $M(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o)$ corresponds to a principal transform of (25). For instance, the basic LCP (25) corresponds to $\sigma_{it} = \sigma_{io} = 1$ for all $i = 1, \dots, n_c$. The following transformed LCP:

$$0 \leq \begin{bmatrix} \mathbf{c}_n \\ \mathbf{s}_t^- \\ \mathbf{s}_o^- \\ \mathbf{a}_t^+ \\ \mathbf{a}_o^+ \end{bmatrix} \perp \begin{bmatrix} \mathbf{a}_n \\ \mathbf{a}_t^- \\ \mathbf{a}_o^- \\ \mathbf{s}_t^+ \\ \mathbf{s}_o^+ \end{bmatrix} = M(\hat{\boldsymbol{\sigma}}_t, \hat{\boldsymbol{\sigma}}_o) \begin{bmatrix} \mathbf{c}_n \\ \mathbf{s}_t^- \\ \mathbf{s}_o^- \\ \mathbf{a}_t^+ \\ \mathbf{a}_o^+ \end{bmatrix} + \begin{bmatrix} \mathbf{b}_n \\ -\mathbf{b}_t \\ -\mathbf{b}_o \\ 0 \\ 0 \end{bmatrix} \geq 0,$$

corresponds to $\hat{\sigma}_{it} = \hat{\sigma}_{io} = -1$ for all $i = 1, \dots, n_c$. In general, for an arbitrary pair of signed vectors $(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o)$, letting

$$\alpha_t^+ \equiv \{i : \sigma_{it} = 1\}, \quad \alpha_t^- \equiv \{i : \sigma_{it} = -1\}, \quad (28)$$

and

$$\alpha_o^+ \equiv \{i : \sigma_{io} = 1\}, \quad \alpha_o^- \equiv \{i : \sigma_{io} = -1\}, \quad (29)$$

we obtain the following equivalent LCP:

$$0 \leq \begin{bmatrix} \mathbf{c}_n \\ (\mathbf{s}_t^+)_{\alpha_t^+} \\ (\mathbf{s}_t^-)_{\alpha_t^-} \\ (\mathbf{s}_o^+)_{\alpha_o^+} \\ (\mathbf{s}_o^-)_{\alpha_o^-} \\ (\mathbf{a}_t^-)_{\alpha_t^+} \\ (\mathbf{a}_t^+)_{\alpha_t^-} \\ (\mathbf{a}_o^-)_{\alpha_o^+} \\ (\mathbf{a}_o^+)_{\alpha_o^-} \end{bmatrix} \perp \begin{bmatrix} \mathbf{a}_n \\ (\mathbf{a}_t^+)_{\alpha_t^+} \\ (\mathbf{a}_t^-)_{\alpha_t^-} \\ (\mathbf{a}_o^+)_{\alpha_o^+} \\ (\mathbf{a}_o^-)_{\alpha_o^-} \\ (\mathbf{s}_t^+)_{\alpha_t^+} \\ (\mathbf{s}_t^-)_{\alpha_t^-} \\ (\mathbf{s}_o^+)_{\alpha_o^+} \\ (\mathbf{s}_o^-)_{\alpha_o^-} \end{bmatrix} = M(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o) \begin{bmatrix} \mathbf{c}_n \\ (\mathbf{s}_t^+)_{\alpha_t^+} \\ (\mathbf{s}_t^-)_{\alpha_t^-} \\ (\mathbf{s}_o^+)_{\alpha_o^+} \\ (\mathbf{s}_o^-)_{\alpha_o^-} \\ (\mathbf{a}_t^-)_{\alpha_t^+} \\ (\mathbf{a}_t^+)_{\alpha_t^-} \\ (\mathbf{a}_o^-)_{\alpha_o^+} \\ (\mathbf{a}_o^+)_{\alpha_o^-} \end{bmatrix} + \begin{bmatrix} \mathbf{b}_n \\ (\mathbf{b}_t)_{\alpha_t^+} \\ -(\mathbf{b}_t)_{\alpha_t^-} \\ (\mathbf{b}_o)_{\alpha_o^+} \\ -(\mathbf{b}_o)_{\alpha_o^-} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \geq 0. \quad (30)$$

In order to state the main result, recall that $\text{SS}(\boldsymbol{\mu})$ denotes the set of all applied loads \mathbf{g}_{ext} that are strongly stable. Thus we have

$$\text{SS}(\boldsymbol{\mu}) \subseteq \text{WS}(\boldsymbol{\mu}). \quad (31)$$

We also let

$$\hat{M}(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o) \equiv \begin{bmatrix} M_{nn}(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o) & \mathbf{A}_{nt}\boldsymbol{\Sigma}_t & \mathbf{A}_{no}\boldsymbol{\Sigma}_o \\ M_{tn}(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o) & \boldsymbol{\Sigma}_t\mathbf{A}_{tt}\boldsymbol{\Sigma}_t & \boldsymbol{\Sigma}_t\mathbf{A}_{to}\boldsymbol{\Sigma}_o \\ M_{on}(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o) & \boldsymbol{\Sigma}_o\mathbf{A}_{ot}\boldsymbol{\Sigma}_t & \boldsymbol{\Sigma}_o\mathbf{A}_{oo}\boldsymbol{\Sigma}_o \end{bmatrix} \in \mathfrak{R}^{3n_c \times 3n_c}$$

denote the leading principal submatrix of $M(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o) \in \mathfrak{R}^{5n_c \times 5n_c}$ with the last two columns and rows deleted. It turns out these submatrices for various pairs of signed vectors $(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o)$ play a central role in Theorem 4 below which identifies a class of rigid body geometries for which equality holds in the expression (31). As one can expect, this class of "favorable" fixture geometries is rather complex;

yet the description provides a constructive (albeit not necessarily practically efficient) scheme for one to check the equality in (31). The basis of the theorem is the concept of a column quasi-adequate matrix introduced in Appendix A, particularly the inequality system (36) in Theorem 5. The proof of Theorem 4 is given in Appendix B.

Theorem 4 *Assume the friction pyramid law (5). The following two statements are equivalent.*

(a) *For every pair of signed vectors $(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o)$, and every nonempty subset $\gamma \subseteq \{1, \dots, 3n_c\}$ with complement $\bar{\gamma}$, the system of linear inequalities below is inconsistent:*

$$\hat{M}(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o)_\gamma \begin{bmatrix} \mathbf{u}_n - \mathbf{v}_n \\ \mathbf{u}_t - \mathbf{v}_t \\ \mathbf{u}_o - \mathbf{v}_o \end{bmatrix} > 0 \quad (32)$$

$$\hat{M}(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o)_{\bar{\gamma}} \begin{bmatrix} \mathbf{u}_n - \mathbf{v}_n \\ \mathbf{u}_t - \mathbf{v}_t \\ \mathbf{u}_o - \mathbf{v}_o \end{bmatrix} = 0 \quad (33)$$

$$\begin{bmatrix} \mathbf{u}_n \\ \mathbf{u}_t \\ \mathbf{u}_o \end{bmatrix}_\gamma = 0, \quad \begin{bmatrix} \mathbf{u}_n \\ \mathbf{u}_t \\ \mathbf{u}_o \end{bmatrix}_{\bar{\gamma}} \geq 0, \quad \begin{bmatrix} \mathbf{v}_n \\ \mathbf{v}_t \\ \mathbf{v}_o \end{bmatrix} \geq 0 \quad (34)$$

$$\begin{aligned} 2 \operatorname{diag}(\boldsymbol{\mu}) \mathbf{u}_n &\geq \max(\mathbf{u}_t, \mathbf{u}_o) \\ 2 \operatorname{diag}(\boldsymbol{\mu}) \mathbf{v}_n &\geq \max(\mathbf{v}_t, \mathbf{v}_o); \end{aligned} \quad (35)$$

(b) *equality holds in (31).*

Admittedly, the condition (a) in the above theorem is by no means easy to check. In fact the number of linear inequality systems involved is exponential in the number of contacts. More precisely, we need to examine all pairs of signed vectors; and for each such pair, we need to consider the inequality system (32–35) for each nonempty subset of $\{1, \dots, 3n_c\}$. Although the computational task of testing the inconsistency of all these systems is rather daunting, it allows us to make one of two important conclusions. If all systems are inconsistent, then we can safely conclude from the above theorem that the only strongly stable loads are those that are weakly stable. If a particular system is consistent, then we can obtain a load that is weakly stable but not strongly stable. The way to construct such a load is as follows. Suppose that for some pair of signed vectors $(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o)$ and some nonempty subset γ of $\{1, \dots, m\}$, the system (32–35) has a solution. We will use the vector \mathbf{v} to construct a desired weakly but not strongly stable load. Indeed define $\mathbf{c}_n \equiv \mathbf{v}_n$ and let α_t^\pm and α_o^\pm be given by (28) and (29) respectively. Furthermore, let

$$\mathbf{c}_{jt} \equiv \begin{cases} v_{jt} - \mu_j c_{jn} & \text{if } j \in \alpha_t^+ \\ \mu_j c_{jn} - v_{jt} & \text{if } j \in \alpha_t^- \end{cases} \quad \text{and} \quad \mathbf{c}_{jo} \equiv \begin{cases} v_{jo} - \mu_j c_{jn} & \text{if } j \in \alpha_o^+ \\ \mu_j c_{jn} - v_{jo} & \text{if } j \in \alpha_o^- \end{cases},$$

and set $\mathbf{g}_{\text{ext}} \equiv -\mathcal{J}\mathbf{c}$ where $\mathbf{c} \equiv (\mathbf{c}_n, \mathbf{c}_t, \mathbf{c}_o)$ is the so-defined intensity. The following result formally states that this load \mathbf{g}_{ext} is weakly but not strongly stable.

Proposition 5 *Let*

$$\begin{bmatrix} \mathbf{u}_n \\ \mathbf{u}_t \\ \mathbf{u}_o \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbf{v}_n \\ \mathbf{v}_t \\ \mathbf{v}_o \end{bmatrix}$$

be a solution pair of (32–35) corresponding to some pair of signed vectors $(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o)$ and a nonempty subset γ of $\{1, \dots, 3n_c\}$. The load \mathbf{g}_{ext} constructed above is weakly but not strongly stable.

We illustrate the application of Theorem 4 and Proposition 5 to the disk example with 2 contact points.

Example 1 (final analysis). As before we set

$$\theta_1 = \frac{\pi}{4} \quad \text{and} \quad \theta_2 = \frac{2.5\pi}{4}$$

which yield $r = 0.9239$, $s = 0.3827$ and $r/(1-s) = 1.4966$. There are four signed vectors (recall that this is a planar problem),

$$\boldsymbol{\sigma}_t \in \{(1, 1), (1, -1), (-1, 1), (-1, -1)\}.$$

For each one of these four vector $\boldsymbol{\sigma}_t$, there are 15 linear inequality systems (32–35) to be checked. We wrote a MATLAB program to check all 60 ($= 4 \times 15$) systems for many values of the friction coefficients μ_1 and μ_2 which we took to be the same, say μ . These systems are all inconsistent for μ not exceeding 1.4966, implying that all weakly stable loads are strongly stable corresponding to such friction coefficients. This experimental result agrees with Figure 5 which shows among other things that at the critical value of 1.4966, the two cone WS and SS do not exactly coincide, implying that there exist weakly stable loads that are not strongly stable.

Experimentally, we ran our MATLAB code with $\mu = 0.1 + r/(1-s)$ and with the value $\mu = 0.1 + r/(1-s)$ and with $\boldsymbol{\sigma}_t = (1, 1)$, of the 15 systems, 7 are consistent, implying that there are weakly but not strongly stable loads. For instance, with $\gamma = \{1, 2, 3, 4\}$, we obtain the following solution to the system (32–35):

$$\mathbf{u} = 0, \quad \mathbf{v} = \begin{bmatrix} 155.62764 \\ 155.19384 \\ 0 \\ 467.6318 \end{bmatrix}.$$

Setting

$$\mathbf{c}_n = \begin{bmatrix} 155.62764 \\ 155.19384 \end{bmatrix} \quad \text{and} \quad \mathbf{c}_t = \mathbf{v}_t - \mu \mathbf{c}_n = \begin{bmatrix} -234.46950 \\ 233.81593 \end{bmatrix},$$

we obtain

$$\mathbf{g}_{\text{ext}} = -\mathcal{J}\mathbf{c} = \begin{bmatrix} 0.4324728 \\ -1.8466864 \\ 0.9242900 \end{bmatrix}.$$

To see that this load is not strongly stable, it suffices to note that (recall that \mathcal{M} is the identity matrix with the above data)

$$\mathbf{b} = \mathcal{J}^T \mathbf{g}_{\text{ext}} = \begin{bmatrix} 1 \\ 1.871616 \\ 2.918752 \\ 1 \end{bmatrix}$$

which implies that the zero vector is a dynamic intensity with nonzero relative accelerations equal to the components of \mathbf{b} . \square

4 Conclusion

Motivated by the problem of fixture synthesis, we have studied the stability of a moveable rigid body (a workpiece) in dry frictional contact with several fixed rigid bodies (fixels). We have introduced the terms *weak stability* and *strong stability* to characterize two types of “stability” of a fixtured workpiece. These classifications are particularly relevant to the situation in which the contact forces of the workpiece cannot be uniquely determined from Newton’s Laws, the relevant kinematic constraints, and a dry friction law. Strong stability exists (for a given external load) when all admissible contact forces imply zero workpiece acceleration. This is the most desirable type of stability, because it provides absolute assurance that the workpiece will remain in place despite unknown internal stresses, however, it is difficult to test.

The primary contribution of this paper is new insight into the stability problem derived from four theorems that provide ways to test for strong stability. While we have focused on the case of one workpiece, the extension to multiple workpieces is trivial. Specifically, the dimensions of the vectors and matrices appearing in the various equations increase, but the results and conclusions still hold. The four theorems are summarized below and illustrated in Figure 9 in the context of a disk in the plane in contact with two fixels (Example 1). For simplicity, the figure only applies to external loads which are pure forces passing through the center of the disk.

1. Theorem 1 presents (for the first time known to the authors) a formal proof that if a workpiece is (weakly or strongly) stable without friction, then it is strongly stable for all (positive) values of the friction coefficients. For the disk example summarized in Figure 9, the external forces in the convex cone labeled “Theorem 1” are stable without friction. Theorem 1 implies that this cone is at least a subset of the set of all strongly stable loads for any (nonnegative) friction coefficients.
2. Theorem 2 implies that weak and strong instability are equivalent when the coefficients of friction are below some bound. In general, this bound is expected to be difficult to find, but in some special cases, it can be computed easily. Returning to Figure 9, all external forces lying strictly outside the cone identified by Theorem 1 are strongly and weakly unstable as long as the friction coefficients are less than 1.4966.
3. Theorem 3 indicates (perhaps counter-intuitively at first glance) that if for some external loading and friction coefficients, the workpiece has a solution with a nonzero acceleration with all contacts rolling or breaking, then the workpiece is guaranteed to have a solution

with the same contact mode if the friction coefficients are increased. Figure 9 shows external loads corresponding to Theorem 3 as two convex cones bounding the Theorem 1 cone. The cones as drawn correspond approximately to friction coefficients, $\mu_1 = 0.5$ and $\mu_2 = 0.8$. With no friction, the cones degenerate to the edges of the cone of Theorem 1. As the friction coefficients go to infinity, the edges of the cones move monotonically to the dashed lines (these are the edges of the normal cone of the $SS(0)$).

4. Theorem 4 provides an exponential-time algorithm to test the equivalence of the sets of strongly and weakly stable external loads using the friction pyramid law. Since the disk example is planar, the quadratic friction cone degenerates into a planar cone, thus making Theorem 4 applicable. It can be shown that the set of weakly stable loads is identical to the set of strongly stable loads as long as the friction coefficients are less than 1.4966. This result is complementary to the implication of Theorem 2.

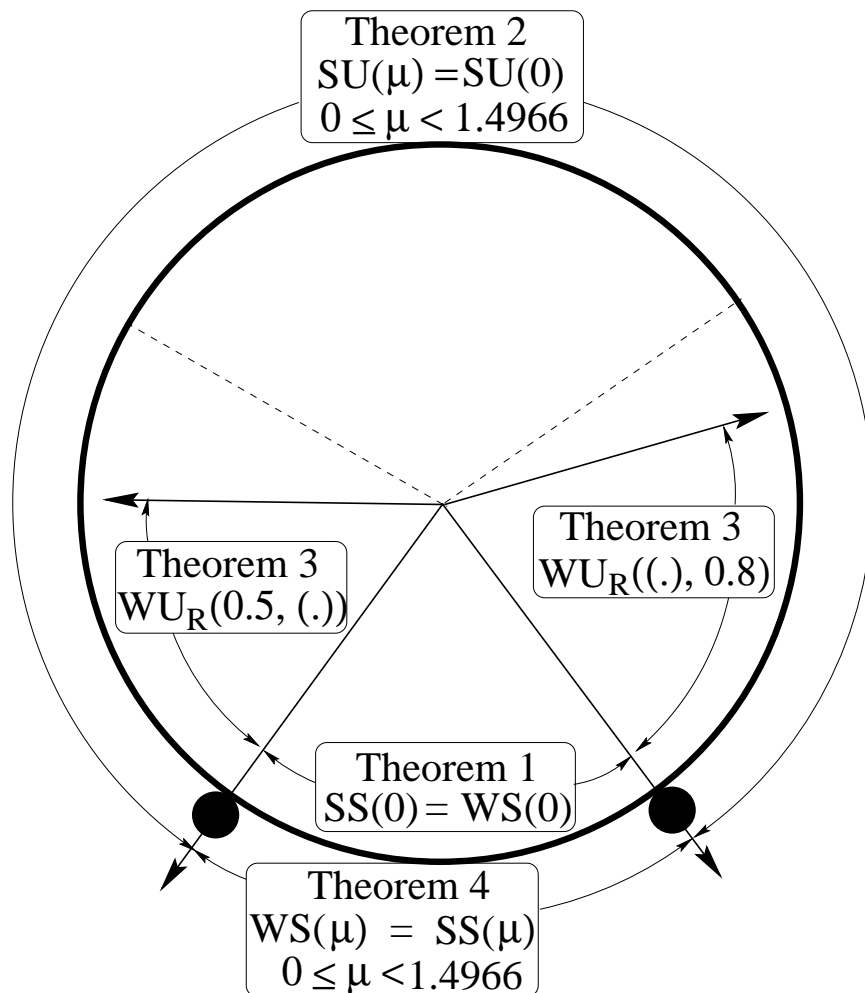


Figure 9: Summary of the stability sets of a disk in contact with two fixels.

The results presented here leave several open questions for future study. Perhaps the most

important questions relate to the computation of the friction bound appearing in Theorem 2 and the use of all the results in this paper in an effective fixture design and analysis system. We intend to address these questions in future work.

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Appendix A : Notation and LCP Theory

Let $M \in \mathfrak{R}^{n \times n}$ be a given matrix. For any two subsets α and β of $\{1, \dots, n\}$, we let $M_{\alpha\beta}$ denote the submatrix of M whose rows and columns are indexed by α and β respectively. The rows of M indexed by α are denoted $M_{\alpha\cdot}$; similarly, the columns of M indexed by β are denoted $M_{\cdot\beta}$. For two vectors \mathbf{a} and \mathbf{b} of the same dimension, $\mathbf{a} \perp \mathbf{b}$ means that \mathbf{a} is perpendicular to \mathbf{b} ; $\mathbf{a} \circ \mathbf{b}$ denotes the Hadamard product of \mathbf{a} and \mathbf{b} , i.e., the i -th component of $\mathbf{a} \circ \mathbf{b}$ is equal to the product of the i -components of \mathbf{a} and \mathbf{b} . For an arbitrary matrix N , we denote by $\text{pos}(N)$ the nonnegative cone generated by the columns of N , that is,

$$\text{pos}(N) \equiv \{ N\mathbf{u} : \mathbf{u} \geq 0 \}.$$

For a vector \mathbf{a} , we let $\text{diag}(\mathbf{a})$ denote the diagonal matrix with diagonal entries given by the components of \mathbf{a} .

We give a summary of some LCP results [6] relevant to this study. To begin, given a vector $\mathbf{q} \in \mathfrak{R}^n$ and a matrix $M \in \mathfrak{R}^{n \times n}$, the linear complementarity problem, denoted LCP (\mathbf{q}, M) , is to find a vector $\mathbf{z} \in \mathfrak{R}^n$ such that

$$0 \leq \mathbf{z} \perp \mathbf{w} \equiv \mathbf{q} + M\mathbf{z} \geq 0.$$

If $M_{\alpha\alpha}$ is a nonsingular principal submatrix of M , where α is a subset of $\{1, \dots, n\}$ with complement $\bar{\alpha}$, then by "pivoting" on this submatrix (i.e., interchanging the roles of the (nonbasic) variables \mathbf{z}_α and the (basic) variables \mathbf{w}_α), we obtain the following principal transform of the LCP (\mathbf{q}, M) :

$$0 \leq \begin{bmatrix} \mathbf{w}_\alpha \\ \mathbf{z}_{\bar{\alpha}} \end{bmatrix} \perp \begin{bmatrix} \mathbf{z}_\alpha \\ \mathbf{w}_{\bar{\alpha}} \end{bmatrix} \equiv \hat{M} \begin{bmatrix} \mathbf{w}_\alpha \\ \mathbf{z}_{\bar{\alpha}} \end{bmatrix} \geq 0,$$

where \hat{M} is the "principal pivot transform" of M given by

$$\hat{M} \equiv \begin{bmatrix} (M_{\alpha\alpha})^{-1} & -(M_{\alpha\alpha})^{-1}M_{\alpha\bar{\alpha}} \\ M_{\bar{\alpha}\alpha}(M_{\alpha\alpha})^{-1} & M_{\bar{\alpha}\bar{\alpha}} - M_{\bar{\alpha}\alpha}(M_{\alpha\alpha})^{-1}M_{\alpha\bar{\alpha}} \end{bmatrix}.$$

In essence, this is how the equivalent LCP (30) is obtained.

Geometrically, the LCP (\mathbf{q}, M) can be described in terms of "complementary cones" that are defined as follows. For any subset α of $\{1, \dots, n\}$ with complement $\bar{\alpha}$, we let \mathcal{C}_α denote the nonnegative cone spanned by the columns $\mathbf{I}_{\cdot\alpha}$ and $-\mathbf{M}_{\cdot\bar{\alpha}}$; that is

$$\mathcal{C}_\alpha \equiv \text{pos}([\mathbf{I}_{\cdot\alpha} \quad -\mathbf{M}_{\cdot\bar{\alpha}}]).$$

Clearly, the union of all these cones \mathcal{C}_α for α ranging over all subsets of $\{1, \dots, n\}$ is equal to the set of vectors \mathbf{q} for which the LCP (\mathbf{q}, M) has a solution. For each nonempty $\alpha \subseteq \{1, \dots, n\}$, the subcone

$$\mathcal{C}'_\alpha \equiv \{ \mathbf{I}_{\cdot\alpha}\mathbf{w}_\alpha - \mathbf{M}_{\cdot\bar{\alpha}}\mathbf{z}_{\bar{\alpha}} : (\mathbf{w}_\alpha, \mathbf{z}_{\bar{\alpha}}) \geq 0, \mathbf{w}_\alpha \neq 0 \}$$

consists of all vectors \mathbf{q} for which the LCP (\mathbf{q}, M) has a solution \mathbf{z} with $\mathbf{w} \equiv \mathbf{q} + M\mathbf{z}$ not equal to the zero vector. In essence, the consideration of these subcones \mathcal{C}'_α lead to the definition of the WU_R sets.

Several classes of matrices M have played an important role in the study of rigid body contact; see [20]. In particular, the class of symmetric positive semidefinite matrices is especially relevant for the frictionless contact problem; the class of column adequate matrices serves as the prime motivation for the strongly stable results in Subsection 3.3. Specifically, a matrix M is *column adequate* if for each $\alpha \subseteq \{1, \dots, n\}$,

$$[\det M_{\alpha\alpha} = 0] \Rightarrow [M_{\cdot\alpha} \text{ has linearly dependent columns }].$$

It is known that the class of symmetric positive semidefinite matrices is a proper subclass of the column adequate matrices. Moreover, one can check if a given matrix M is column adequate in finite time. The following result presents an important property of the column adequate matrices relative to the solutions of the LCP.

Proposition 6 *Let $M \in \mathbb{R}^{n \times n}$ be a column adequate matrix. For every $q \in \mathbb{R}^n$, if z^1 and z^2 are any two solutions of the LCP (q, M) , then $Mz^1 = Mz^2$.*

It turns out that the class of column adequate matrices itself is not broad enough to treat the strong stability issue in rigid body contact. Instead we need to introduce a new class of matrices that is based on an equivalent definition of a column adequate matrix. Namely, a matrix M is column adequate if and only if

$$[z \circ Mz \leq 0] \Rightarrow [Mz = 0].$$

Definition: A real square matrix M is said to be column quasi-adequate if

$$\left. \begin{array}{l} z \circ Mz \leq 0 \\ Mz \geq 0 \end{array} \right\} \Rightarrow Mz = 0.$$

Clearly every column adequate matrix is column quasi-adequate; but as the following matrix shows, the converse is not true. Indeed the matrix:

$$M \equiv \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

is easily seen to be column quasi-adequate but not column adequate. The fundamental role of the column quasi-adequate matrices in LCP theory is contained in the following result. This result also shows that computationally the property of column quasi-adequacy can be checked in finite time. Since this matrix class has not appeared in the LCP literature, we give a complete proof of the result.

Theorem 5 *Let $M \in \mathbb{R}^{n \times n}$ be a given matrix. The following statements are equivalent.*

- (a) M is column quasi-adequate.
- (b) For every $q \in -\text{pos}(M)$, every solution z of the LCP (q, M) must satisfy $q + Mz = 0$.
- (c) For every nonempty index set $\alpha \subseteq \{1, \dots, n\}$ with complement $\bar{\alpha}$, the system of linear inequalities below is inconsistent:

$$\begin{array}{rcl} M_{\alpha\cdot} z & > & 0 \\ M_{\bar{\alpha}\cdot} z & = & 0 \\ z_{\alpha} & \leq & 0. \end{array} \tag{36}$$

Proof. (a) \Rightarrow (b). Suppose that \mathbf{M} is column quasi-adequate. Let \mathbf{q} be an element in $-\text{pos}(\mathbf{M})$ and let \mathbf{z} be an arbitrary solution of the LCP (\mathbf{q}, \mathbf{M}) . We have

$$0 \leq \mathbf{z} \perp \mathbf{w} = \mathbf{q} + \mathbf{M}\mathbf{z} \geq 0.$$

Let $\hat{\mathbf{z}} \geq 0$ be such that $\mathbf{q} + \mathbf{M}\hat{\mathbf{z}} = 0$. We have

$$\mathbf{w} = \mathbf{M}(\mathbf{z} - \hat{\mathbf{z}}) \geq 0, \quad \text{and} \quad (\mathbf{z} - \hat{\mathbf{z}}) \circ \mathbf{w} \leq 0.$$

By column quasi-adequacy, it follows that $\mathbf{M}\mathbf{z} = \mathbf{M}\hat{\mathbf{z}}$; thus $\mathbf{w} = 0$ and (b) holds.

(b) \Rightarrow (a). Suppose that (b) holds but \mathbf{M} is not column quasi-adequate. There exists a vector $\mathbf{z} \equiv (z_i)$ such that

$$\mathbf{z} \circ \mathbf{M}\mathbf{z} \leq 0 \quad \text{and} \quad 0 \neq \mathbf{M}\mathbf{z} \geq 0. \tag{37}$$

Let \mathbf{z}^+ and \mathbf{z}^- be the nonnegative and nonpositive part of \mathbf{z} respectively and define $\mathbf{q} \equiv -\mathbf{M}\mathbf{z}^-$. Then $\mathbf{q} \in -\text{pos}(\mathbf{M})$ and

$$0 \neq \mathbf{w} \equiv \mathbf{q} + \mathbf{M}\mathbf{z}^+ = \mathbf{M}\mathbf{z} \geq 0.$$

We claim that $\mathbf{z}^+ \perp \mathbf{w}$. Indeed if $z_i > 0$, then $0 \geq (\mathbf{M}\mathbf{z})_i$. Consequently, $w_i = 0$ and the claim is established. Since \mathbf{w} is nonzero, we obtain a contradiction to (b).

(a) \Rightarrow (c). This is obvious.

(c) \Rightarrow (a). If (c) holds but there exists a vector \mathbf{z} satisfying (37), let α be the (nonempty) index set consisting of those indices i for which $(\mathbf{M}\mathbf{z})_i > 0$. This index set α will contradict (c), thus completing the proof of the theorem. \square

Appendix B : Proofs of Main Stability Results.

Proposition 1. Both [(a) \Rightarrow (b)] and [(a) \Rightarrow (c)] are obvious. To show that (b) implies (a), let $\mathbf{c} \equiv (\mathbf{c}_n, \mathbf{c}_t, \mathbf{c}_o)$ be a dynamic intensity. By (b), we have $\mathbf{g}_{\text{ext}}^T \ddot{\mathbf{q}} \leq 0$. Thus

$$0 \geq \mathbf{a}^T \mathbf{c} = \ddot{\mathbf{q}}^T \mathcal{M}^{-1} \ddot{\mathbf{q}} - \mathbf{g}_{\text{ext}}^T \ddot{\mathbf{q}} \geq 0.$$

Thus equality holds throughout; in particular, it follows that

$$0 = \ddot{\mathbf{q}}^T \mathcal{M}^{-1} \ddot{\mathbf{q}}$$

which implies, by the positive definiteness of \mathcal{M}^{-1} , that $\ddot{\mathbf{q}} = 0$ as desired. Thus (a) holds.

To show that (c) implies (a), let $\mathbf{c} \equiv (\mathbf{c}_n, \mathbf{c}_t, \mathbf{c}_o)$ be a dynamic intensity. By (c), we have $\mathbf{a}_t = \mathbf{a}_o = 0$. Thus,

$$\begin{bmatrix} \mathbf{a}_n \\ 0 \\ 0 \end{bmatrix} = \mathbf{A} \mathbf{c} + \mathbf{b}.$$

Since \mathbf{g}_{ext} is weakly stable, let $\mathbf{c}' \geq 0$ be such that $\mathcal{J} \mathbf{c}' + \mathbf{g}_{\text{ext}} = 0$. It follows that

$$\begin{bmatrix} \mathbf{a}_n \\ 0 \\ 0 \end{bmatrix} = \mathbf{A}(\mathbf{c} - \mathbf{c}').$$

Premultiplying this equation by

$$\mathbf{c} - \mathbf{c}' = \begin{bmatrix} \mathbf{c}_n - \mathbf{c}'_n \\ \mathbf{c}_t - \mathbf{c}'_t \\ \mathbf{c}_o - \mathbf{c}'_o \end{bmatrix}^T,$$

and using the fact that $\mathbf{c}_n^T \mathbf{a}_n = 0$ and $(\mathbf{c}'_n, \mathbf{a}_n) \geq 0$ and the positive semidefiniteness of \mathbf{A} , we deduce

$$0 \geq \begin{bmatrix} \mathbf{c}_n - \mathbf{c}'_n \\ \mathbf{c}_t - \mathbf{c}'_t \\ \mathbf{c}_o - \mathbf{c}'_o \end{bmatrix}^T \begin{bmatrix} \mathbf{a}_n \\ 0 \\ 0 \end{bmatrix} = (\mathbf{c} - \mathbf{c}')^T \mathbf{A}(\mathbf{c} - \mathbf{c}') \geq 0.$$

Thus equality holds throughout the last expression. By the special structure of \mathbf{A} , it then follows that

$$\mathcal{J}(\mathbf{c} - \mathbf{c}') = 0,$$

or equivalently, $\mathcal{J} \mathbf{c} + \mathbf{g}_{\text{ext}} = 0$, as desired. □

Theorem 1. (a) \Rightarrow (b). Let $\hat{\mathbf{c}}_n$ satisfy (21) and let $\mathbf{c} \equiv (\mathbf{c}_n, \mathbf{c}_t, \mathbf{c}_o)$ be an arbitrary dynamic intensity. It suffices to show that $\mathcal{J} \mathbf{c} + \mathbf{g}_{\text{ext}} = 0$. We have

$$\begin{bmatrix} \mathbf{a}_n \\ \mathbf{a}_t \\ \mathbf{a}_o \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{c}_n - \hat{\mathbf{c}}_n \\ \mathbf{c}_t \\ \mathbf{c}_o \end{bmatrix}.$$

Premultiplying this equation by

$$\begin{bmatrix} \mathbf{c}_n - \hat{\mathbf{c}}_n \\ \mathbf{c}_t \\ \mathbf{c}_o \end{bmatrix}^T,$$

using the following facts:

- (i) $(\mathbf{a}_n, \mathbf{c}_n, \hat{\mathbf{c}}_n) \geq 0$ and $\mathbf{a}_n^T \mathbf{c}_n = 0$,
- (ii) $\mathbf{c}_t^T \mathbf{a}_t + \mathbf{c}_o^T \mathbf{a}_0 \leq 0$ because the friction cone contains the origin, and
- (iii) \mathbf{A} is positive semidefinite,

and proceeding as in the proof of Proposition 1, we easily conclude $\mathcal{J}\mathbf{c} + \mathbf{g}_{\text{ext}} = 0$, as desired. Thus (b) holds.

The following implications are all obvious:

$$[(b) \Rightarrow (c) \text{ and } (d)]; \quad [(c) \text{ or } (d) \Rightarrow (e) \Leftrightarrow (a)].$$

Combining these implications with (a) \Rightarrow (b), we conclude that all five statements are equivalent. \square

Theorem 2. It suffices to prove (b) \Rightarrow (a). Assume by way of contradiction that there exists a sequence of nonnegative friction coefficients $\{\mu^k\}$ converging to zero such that for each k , \mathbf{g}_{ext} is weakly stable for the rigid body problem corresponding to $\mu^k \equiv (\mu_i^k)$. Hence for each k , there exists $(\mathbf{c}_n^k, \mathbf{c}_t^k, \mathbf{c}_o^k)$ satisfying

$$\begin{aligned} \mathbf{W}_n \mathbf{c}_n^k + \mathbf{W}_t \mathbf{c}_t^k + \mathbf{W}_o \mathbf{c}_o^k + \mathbf{g}_{\text{ext}} &= 0 \\ \mathbf{c}_n^k &\geq 0 \\ (\mathbf{c}_{it}^k, \mathbf{c}_{io}^k) &\in \mathcal{F}(\mu_i^k \mathbf{c}_{in}^k), \quad \text{for } i = 1, \dots, n_c. \end{aligned}$$

There are two cases: either the sequence $\{\mathbf{c}_n^k\}$ is bounded or it is unbounded. In the former case, we must have

$$\lim_{k \rightarrow \infty} (\mathbf{c}_t^k, \mathbf{c}_o^k) = (0, 0)$$

and every accumulation point of the sequence $\{\mathbf{c}_n^k\}$ (at least one of which must exist), say \mathbf{c}_n^∞ , must satisfy

$$\mathbf{W}_n \mathbf{c}_n^\infty + \mathbf{g}_{\text{ext}} = 0 \quad \text{and} \quad \mathbf{c}_n^\infty \geq 0.$$

By Theorem 1, this implies that \mathbf{g}_{ext} is stable for the frictionless problem, a contradiction. In the latter case, that is, the sequence $\{\mathbf{c}_n^k\}$ is unbounded, the normalized sequence $\{\mathbf{c}_n^k / \|\mathbf{c}_n^k\|\}$ must have at least one accumulation point and any such point, say \mathbf{c}_n^* , must satisfy

$$\mathbf{W}_n \mathbf{c}_n^* = 0 \quad \text{and} \quad 0 \neq \mathbf{c}_n^* \geq 0.$$

This contradicts the existence of the vector \mathbf{u}_n satisfying $\mathbf{W}_n^T \mathbf{u}_n > 0$. \square

Theorem 3. From the definition of the elements in $\text{WU}_R(\hat{\boldsymbol{\mu}})$, it is easy to see that if \mathbf{g}_{ext} is an such an element, then \mathbf{g}_{ext} is weakly unstable corresponding to $\hat{\boldsymbol{\mu}}$. This observation together with the inclusion (24) implies that if \mathbf{g}_{ext} belongs to $\text{WU}_R(\hat{\boldsymbol{\mu}})$, then \mathbf{g}_{ext} is weakly unstable corresponding to all $\boldsymbol{\mu} \geq \hat{\boldsymbol{\mu}}$. The assertion regarding $\text{WU}_R(0)$ is obvious. \square

Theorem 4. (a) \Rightarrow (b). Suppose that for every pair of signed vectors $(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o)$ and every nonempty index subset γ of $\{1, \dots, 3n_c\}$, the displayed system of linear inequalities (32–35) is inconsistent. Assume by way of contradiction that some load vector $\mathbf{g}_{\text{ext}} \in WS(\boldsymbol{\mu})$ is not strongly stable. By Proposition 1, it follows that there exists a dynamic intensity $(\hat{\mathbf{c}}_n, \hat{\mathbf{c}}_t, \hat{\mathbf{c}}_o)$ with induced accelerations $(\hat{\mathbf{a}}_n, \hat{\mathbf{a}}_t, \hat{\mathbf{a}}_o)$ such that $(\hat{\mathbf{a}}_t, \hat{\mathbf{a}}_o) \neq 0$. Let $(\hat{\mathbf{s}}_t^\pm, \hat{\mathbf{s}}_o^\pm)$ be the slack variables in the friction constraints; see (26) and (27) for the definition of these variables. Define several index sets:

$$\begin{aligned} \alpha_t &\equiv \{i : \hat{a}_{it} > 0\} & \text{and} & & \bar{\alpha}_t &\equiv \{i : \hat{a}_{it} \leq 0\}, \\ \alpha_o &\equiv \{i : \hat{a}_{io} > 0\} & \text{and} & & \bar{\alpha}_o &\equiv \{i : \hat{a}_{io} \leq 0\}, \end{aligned} \quad (38)$$

and a pair $(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o)$ of signed vectors as follows: for $j = 1, \dots, n_c$,

$$\sigma_{it} = \begin{cases} 1 & \text{if } i \in \alpha_t \\ -1 & \text{if } i \in \bar{\alpha}_t \end{cases} \quad \text{and} \quad \sigma_{io} = \begin{cases} 1 & \text{if } i \in \alpha_o \\ -1 & \text{if } i \in \bar{\alpha}_o. \end{cases}$$

Consider the LCP (30) corresponding to the index sets (??); setting the variables

$$(\mathbf{a}_t^-)_{\alpha_t^+}, (\mathbf{a}_t^+)_{\alpha_t^-}, (\mathbf{a}_o^-)_{\alpha_o^+}, (\mathbf{a}_o^+)_{\alpha_o^-}$$

equal to zero, we obtain the following linear system:

$$\begin{aligned} 0 \leq \begin{bmatrix} \mathbf{c}_n \\ (\mathbf{s}_t^+)_{\alpha_t} \\ (\mathbf{s}_t^-)_{\bar{\alpha}_t} \\ (\mathbf{s}_o^+)_{\alpha_o} \\ (\mathbf{s}_o^-)_{\bar{\alpha}_o} \end{bmatrix} \perp \begin{bmatrix} \mathbf{a}_n \\ (\mathbf{a}_t^+)_{\alpha_t} \\ (\mathbf{a}_t^-)_{\bar{\alpha}_t} \\ (\mathbf{a}_o^+)_{\alpha_o} \\ (\mathbf{a}_t^-)_{\bar{\alpha}_o} \end{bmatrix} &= \hat{\mathbf{M}}(\boldsymbol{\sigma}_t, \boldsymbol{\sigma}_o) \begin{bmatrix} \mathbf{c}_n \\ (\mathbf{s}_t^+)_{\alpha_t} \\ (\mathbf{s}_t^-)_{\bar{\alpha}_t} \\ (\mathbf{s}_o^+)_{\alpha_o} \\ (\mathbf{s}_o^-)_{\bar{\alpha}_o} \end{bmatrix} + \begin{bmatrix} \mathbf{b}_n \\ (\mathbf{b}_t)_{\alpha_t} \\ -(\mathbf{b}_t)_{\bar{\alpha}_t} \\ (\mathbf{b}_o)_{\alpha_o} \\ -(\mathbf{b}_o)_{\bar{\alpha}_o} \end{bmatrix} \geq 0 \\ 2 \operatorname{diag}(\boldsymbol{\mu}) \mathbf{c}_n &\geq \max \left(\begin{bmatrix} (\mathbf{s}_t^+)_{\alpha_t} \\ (\mathbf{s}_t^-)_{\bar{\alpha}_t} \end{bmatrix}, \begin{bmatrix} (\mathbf{s}_o^+)_{\alpha_o} \\ (\mathbf{s}_o^-)_{\bar{\alpha}_o} \end{bmatrix} \right). \end{aligned}$$

On the one hand, since \mathbf{g}_{ext} is weakly stable, the above LCP has a solution with

$$((\mathbf{a}_t^+)_{\alpha_t}, (\mathbf{a}_t^-)_{\bar{\alpha}_t}, (\mathbf{a}_o^+)_{\alpha_o}, (\mathbf{a}_t^-)_{\bar{\alpha}_o})$$

equal to zero. On the other hand, the same LCP has a solution

$$(\hat{\mathbf{c}}_n, (\hat{\mathbf{s}}_t^+)_{\alpha_t}, (\hat{\mathbf{s}}_t^-)_{\bar{\alpha}_t}, (\hat{\mathbf{s}}_o^+)_{\alpha_o}, (\hat{\mathbf{s}}_o^-)_{\bar{\alpha}_o})$$

that induces a nonzero $((\hat{\mathbf{a}}_t^+)_{\alpha_t}, (\hat{\mathbf{a}}_t^-)_{\bar{\alpha}_t}, (\hat{\mathbf{a}}_o^+)_{\alpha_o}, (\hat{\mathbf{a}}_t^-)_{\bar{\alpha}_o})$. These two solutions provide the vectors

$$\begin{bmatrix} \mathbf{u}_n \\ \mathbf{u}_t \\ \mathbf{u}_o \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \mathbf{v}_n \\ \mathbf{v}_t \\ \mathbf{v}_o \end{bmatrix}$$

that violate the inequality system in the assumption of the theorem. A contradiction is thus obtained and (b) therefore must hold.

(b) \Rightarrow (a). This follows from Proposition 5 whose proof is given next. □

Proposition 5. Let \mathbf{c} and \mathbf{g}_{ext} be as defined. By (35), it follows that for all $j = 1, \dots, n_c$,

$$\max(|c_{jt}|, |c_{ij}|) \leq \mu_j c_{jn}.$$

Thus \mathbf{g}_{ext} is weakly stable. To show that \mathbf{g}_{ext} is not strongly stable, we use the vector \mathbf{u} to define a dynamic intensity $\hat{\mathbf{c}}$ with nonzero relative accelerations. The definition of $\hat{\mathbf{c}}$ is similar to that of \mathbf{c} from the vector \mathbf{v} . Specifically, let $\hat{\mathbf{c}}_n \equiv \mathbf{u}_n$ and

$$\hat{c}_{jt} \equiv \begin{cases} u_{jt} - \mu_j \hat{c}_{jn} & \text{if } j \in \alpha_t^+ \\ \mu_j \hat{c}_{jn} - u_{jt} & \text{if } j \in \alpha_t^- \end{cases} \quad \text{and} \quad \hat{c}_{jo} \equiv \begin{cases} u_{jo} - \mu_j \hat{c}_{jn} & \text{if } j \in \alpha_o^+ \\ \mu_j \hat{c}_{jn} - u_{jo} & \text{if } j \in \alpha_o^-; \end{cases}$$

or equivalently,

$$\hat{\mathbf{c}}_t \equiv \boldsymbol{\Sigma}_t(\mathbf{u}_t - \text{diag}(\boldsymbol{\mu})\mathbf{u}_n), \quad \text{and} \quad \hat{\mathbf{c}}_o \equiv \boldsymbol{\Sigma}_o(\mathbf{u}_o - \text{diag}(\boldsymbol{\mu})\mathbf{u}_n).$$

The verification of the so-defined $\hat{\mathbf{c}}$ being a dynamic intensity with nonzero relative accelerations is fairly straightforward; it amounts to a backward derivation from the transformed LCP (30) to the basic LCP (25). We omit the algebraic details. □