First-Order Stability Cells of Active Multi-Rigid-Body Systems *

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Abstract

A *stability cell* is a subset of the configuration space (C-space) of a set of actively controlled rigid bodies (*e.g.*, a whole-arm manipulator) in contact with a passive body (*e.g.*, a manipulated object) in which the contact state is guaranteed to be stable under the influence of Coulomb friction and external forces. A *first-order stability cell* is a subset of a stability cell with the following two properties: first, the state of contact uniquely determines the rate of change of the object's configuration given the rate of change of the manipulator's configuration; and second, the contact state cannot be altered by any infinitesimal variation in the generalized applied force. First-order

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stability cells can be used in planning whole-arm manipulation tasks in a manner analogous to the use of free-space cells in planning collision-free paths: a connectivity graph is constructed and searched for a path connecting the initial and goal configurations. A path through a free-space connectivity graph represents a motion plan that can be executed without fear of collisions, while a path through a stability-cell connectivity graph represents a whole-arm manipulation plan that can be executed without fear of "dropping" the object.

The main contribution of this paper is the conceptual and analytical development of first-order stability cells of three-dimensional, rigid-body systems as conjunctions of equations and inequalities in the C-space variables. Additionally, our derivation leads to a new quasistatic jamming condition that takes into account the planned motion and kinematic structure of the active bodies.

1 Introduction

The goal of robot planning research is to develop algorithms that can deduce sequences of control commands which cause one or more robots to accomplish desired high-level tasks, such as "Assemble the barbeque grill!" Besides imparting some degree of intelligence and autonomy to a robot, such algorithms can help to relieve human operators from the tedium of robot programming and debugging. Ultimately, these algorithms should be able to generate reliable plans despite uncertainty and to plan sensor usage to resolve ambiguities and recover from errors [8, 12, 23].

Notice that the command, "Assemble the barbeque grill!" requires the robot to perform a task involving contact with movable objects. The problem of planning tasks of this type has been termed the "manipulation planning problem," by Latombe [20]. Because solving this class of tasks requires the use of a (possibly complex) model of mechanics, it has been studied less than its counterpart, the motion planning problem, in which contact is expressly avoided. The planning problem considered here involves the quasistatic manipulation of a single movable object by a system of actively controlled bodies, which we refer to as a whole-arm manipulator (WAM). Henceforth, we will refer to this problem as the *whole-arm manipulation planning problem* (*WAM planning problem*). In the next three paragraphs, we will review the cell-decomposition approach to motion planning and compare it with our stability-cells approach to planning WAM tasks. Note that while we will be focusing on WAM applications, the theory could also be used to design mechanisms which have links that are connected to other links only through unilateral contacts.

Solving a motion planning problem without contacts amounts to finding a coordinated set of joint trajectories that defines a collision-free motion of a robot beginning from a given initial configuration and ending in a given goal configuration. Motion planning is usually performed in the system's configuration space (C-space), where the system is represented as a point (the C-point) and system motions are represented as continuous trajectories. One approach to motion planning requires the decomposition of C-space into free-space cells (cells corresponding to configurations in which the bodies are neither in contact nor overlapping), while creating a free-space connectivity graph. The nodes of the graph represent the free-space cells and the arcs connect nodes whose corresponding free-space cells share a boundary. Note that the arcs of the free-space connectivity graph are undirected.

A completed connectivity graph provides a global description of the system's free-space and thus provides data from which a solution can be determined in two stages. First, the graph can be searched for a path connecting the cells containing the initial and goal configurations. Second, a plan can be extracted from the graph as a continuous curve connecting the initial and goal configurations (and passing through the cells visited by the path through the graph). As long as the curve remains within the free-space cells, the planned manipulation can be performed with the assurance that no collisions will take place (assuming a static environment).

In contrast, WAM planning requires the generation of a continuous trajectory in contact space, the subset of C-space for which at least one contact exists and no bodies overlap. Roughly speaking, a *stability cell* (*S-cell*) is a subset of a smooth "facet" or "patch" of contact space for which an object can be manipulated in stable equilibrium by a whole-arm manipulator. Therefore, the S-cells constitute the subset of contact space in which paths may be constructed that satisfy the stability and kinematic constraints at every point. By generating an S-cell connectivity graph, WAM tasks can be planned by graph searching.

The approach is illustrated in Figure 1, which shows two smooth "facets" of contact space intersecting along a curve segment. The shaded regions represent S-cells. A trajectory of the C-point in the S-cells corresponds to stable manipulation of the object. Points in the unshaded regions are configurations which achieve contacts, but are unstable or are illegal because they represent configurations for which bodies overlap (in position away from the designated contact points). The bold path connecting the initial and goal configurations, I and G, represents a stable manipulation plan.

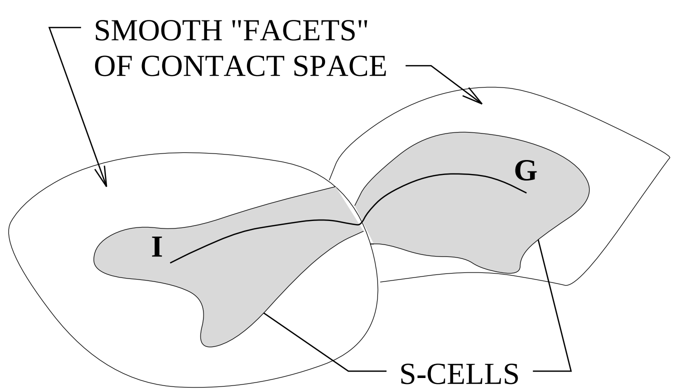

Figure 1: Two Intersecting "Facets" of Contact Space with S-cells

In the frictionless case, quasistatic motion is reversible, so the connectivity graph is undirected. As a result, the S-cell connectivity graph can be used in exactly the same manner as a free-space connectivity graph is used in motion planning. Any path connecting the initial and final configurations and lying entirely within the S-cells guarantees that the object will not be "dropped" during manipulation. However, when Coulomb friction is present, a valid path must remain within the S-cells *and* satisfy nonholonomic inequality constraints at every point.

This paper focuses on the special class of S-cells referred to as *first-order stability cells* (*FS-cells*). FS-cells are the subsets of S-cells with the following two properties: first, the state of contact is such that the velocity of the manipulator uniquely determines the velocity of the object; and second, the contact state cannot be altered by any infinitesimal variation in the generalized applied force. That is, if the contact state is to be altered by a force disturbance, then that disturbance must be of (nonzero) finite magnitude. Intuitively, first-order stability can be likened to a marble's stability when it is located in the bottom of a pyramid-shaped bowl: a (nonzero) finite force is required to dislodge the marble. In contrast, a marble at the bottom of a smooth bowl can be perturbed by an infinitesimal force. In this sense, manipulation plans composed of path segments through FS-cells are "more stable" than those traversing general S-cells. It is clear that the contact state can also be altered through kinematic sources, such as collisions and a vertex of one body sliding off the edge of a face of another, but those types of transitions are easily predicted. In this paper we are concerned with contact state transitions caused by variations in the applied forces.

The contributions of this paper include the conceptual and analytical development of FS-cells for WAM systems with and without Coulomb friction. We represent our FS-cells as conjunctions of equations and inequalities in the C-space variables, each of which can be expressed as a polynomial by a simple substitution if desired. In addition, we derive a new condition that describes all configurations for which a system moving quasistatically *cannot* jam. Unfortunately, some of the constraints relevant to systems with Coulomb friction and sliding contacts contain the joint velocity variables. This is unavoidable because the directions of the friction forces at the sliding contacts are determined by the directions of the corresponding relative velocities at the contacts, which in turn, are determined by the joint velocities. Note that if all the contacts are rolling, then the equations are independent of the velocity variables.

An interesting observation is that the dimension of an FS-cell (ignoring the possible relevant velocity variables), with or without friction, is equal to the number of manipulator joints. The reason for this is that as the number of contact constraints increases, the number of degrees of freedom of system motion in C-space decreases, and for each lost degree of motion freedom, the system gains one degree of freedom in the applicable joint efforts. The invariance in the dimension of FS-cells is perhaps unexpected and even disappointing, since sometimes artificial contacts are used to speed planning by reducing the dimension of the free-space cells [9, 16, 18]. However, on the positive side, the dimension of FS-cells, n_θ, is significantly less than that of the C-space, $n_\theta + 6$.

1.1 Previous Related Research

The work presented here formalizes and generalizes previous work in dexterous manipulation planning by Trinkle and Hunter [36] and Trinkle *et al.* [38]. In these papers, a particular dexterous manipulation system was studied. In the first paper, a planner was developed for the quasistatic frictionless case and a plan was generated to perform a particular dexterous manipulation task. In the second paper, that task was post-processed to determine its validity in the face of uncertainty in the coefficients of Coulomb friction. The development of FS-cells undertaken here will facilitate planning with friction, so that expensive post-processing can be avoided. The concept of FS-cells is not new, but our results are more general than previous results, as they are applicable to three-dimensional, whole-arm manipulation systems with multiple movable passive objects.[1]

While many researchers have planned robotic tasks involving sliding contact, few have done so for general multi-body mechanical systems. Most work has been limited to problems which could be planned without cell-decomposing the C-space. Two such problems were solved by Peshkin [29] and Brock [1]. Peshkin extended Mason's results on planar quasistatic "pushing" to plan spiral paths for a robot finger to localize disks in a horizontal plane and Brock planned rotations and translations of a soda can between two fingers and pushed by a third. Even though sliding was allowed, the geometries of the "grasps" were static, so the equilibrium equations and contact constraints had to be imposed on only one point of C-space. Jamming was not an issue. A somewhat more complex task, "baton twirling," was studied by Fearing. He considered a small number of primitive grasp geometries and control strategies for which sliding terminated by jamming in a fixed stable grasp. His stable grasp criterion was equivalent to Nguyen's frictional form-closure[2] result for two-contact, planar grasps with friction [27]. These cases were sufficient to plan and execute the twirling task.

Whitney's analysis of the peg-in-hole problem [44] was one of the first to consider sliding constraints in C-space. He did not pose the peg insertion task in C-space, but by formulating the problem symbolically, he analyzed the quasistatic mechanical model in contact space. His careful analysis of the jamming problem led to the development of the Remote Center of Compliance wrist, which passively implements a highly successful strategy to avoid jamming. An important aspect of Whitney's jamming analysis that was different from previous analyses (for example, the analysis of Simunovic [34]) was his use of compliance information. Our analysis of jamming for general three-dimensional, multi-rigid-body systems uses the compliance of the joints' position controllers in a similar way. We use the knowledge of the planned joint trajectories and the errors that would be induced by jamming to develop conditions in which jamming is *impossible*. As will be seen below, these conditions are a vital part of our first-order stability conditions.

[1] We study the specific case of only one manipulated object, but the extension to multiple objects is straight forward.

[2] Nguyen refers to frictional form closure[4] as "force closure." However, we reserve the use of the term "force closure" for situations originally identified by Reuleaux [32]

The most general formulation of the manipulation planning problem in which sliding was allowed, was published by Li and Canny [22]. However, in their formulation, they assumed that the manipulated object could only contact the "hand" on the most distal link of each "finger" and they have not yet developed a corresponding planning algorithm. They did not discuss the possibility of jamming.

We now discuss in more detail, the previous work in manipulation planning that involved the imposition of stability or other model constraints on C-space to plan tasks requiring sliding and/or rolling between the object and the manipulator.

Brost planned pushing, dropping, and grasping actions guaranteed to remove the uncertainty in the position and orientation of an object [2, 3]. In all these kinds of tasks, the object could slide or roll on surfaces of the manipulator (in the fixturing task, the fixture can be viewed as a stationary manipulator), and so, by our definition, these tasks were whole-arm manipulation tasks. In both situations, part of Brost's planning technique required the imposition of sufficient conditions for object stability (with Coulomb friction) in contact space to form stable regions analogous to our FS-cells. In fact, if friction and uncertainties had been neglected, Brost's regions of stability would have shrunk to become one-point FS-cells. While Brost was thorough in his inclusion of uncertainty into his models and planning techniques, his results were limited to planar systems.

The work presented here is applicable to spatial systems, but no uncertainty is considered. This is because FS-cells were developed with an alternative approach to planning in uncertain environments in mind. In future work, planning will be done in two stages: in Stage 1, plans for the nominal quasistatic system will be generated; and in Stage 2, the plans will be modified to make them robust to variations in the uncertain parameters as suggested by Xiao [45] and Dakin [6] and time scaling techniques will be used to allow high-speed execution [33]. Preliminary results on nominal plan modification when friction is uncertain can be found in [14, 38]. Note that a dynamic model could replace the quasistatic one used in Stage 1, but then nominal plan generation would become more time consuming.

Goldberg [17] planned orienting tasks for planar parts by a sequence of squeezing actions executed with a "frictionless" parallel jaw gripper. The initial orientation of the part was unknown, but the part was known to lie somewhere between the jaws. His work relied on imposing simple, purely geometric jamming conditions on the system's C-space. Here jamming meant that the part had achieved a stable orientation with respect to the jaws, thus preventing them from moving closer together. An algorithm for generating plans of minimum length (fewest number of grasp actions) was developed for generalized polygonal parts (*i.e.*, parts whose boundaries are composed of circular arcs and linear segments). While Goldberg's orienting algorithm is complete for generalized polygonal parts, its extension to three-dimensional parts appears to be extremely difficult. An approach based on our FS-cells could be applied to three-dimensional parts.

Mason developed a mechanical model to predict the quasistatic motion of planar parts pushed on a rough plane [26]. The part was free to slide against the "pusher." Peshkin applied Mason's model to the pushing of polygons by "fences" [31, 30]. He developed a technique to determine bounds on the pushing distance required for a part to achieve a stable orientation in contact with a fence. He used this technique to design a parts feeder which used a conveyor belt to drag the parts through a series of fences. The parts entered the fence system in any orientation and emerged in the desired orientation. If the conveyor belt were turned off and the fences were translated along the belt far enough, a part would emerge in the proper orientation. This process can be viewed as whole-arm manipulation and the planning aspect is in the design of the geometry of the manipulator (*i.e.*, the arrangement of the fences).

Donald and Pai [10] developed an algorithm to simulate the motion of simple, planar, compliant parts as they are inserted into a fixtured base part. Coulomb friction was assumed to act at the contacts, but not in the pawl joints. The objective of this work was to test candidate assembly plans and provide information to a designer who would use it to modify part designs. There was no attempt to create assembly plans, but the work could serve as the basis for such a planner, because the prediction of all types of motions of the parts was possible. In fact, the predictors, one of which detected jamming, were all written as functions of the C-space variables.

The primary shortcoming of Donald's and Pai's work was the narrowness of the class of applicable systems. The systems had to be planar with each pawl connected directly to the root body by a passive, compliant, revolute joint. The pawls could not be multi-jointed and the allowable insertion trajectories were limited to translations of the root body. The FS-cells developed here are applicable to systems with multi-link "pawls" with compliance provided by the joints' servo-controllers, and general trajectories are allowed. As our jamming criterion is more general than theirs, we have not been able to write it as a function of only the C-space variables; it also contains a subset of the joint velocity variables.

In the next Section, we will present our assumptions, define the two relevant configuration spaces, and present the applicable kinematic constraints and the equilibrium equations. In Section 3, our definition of first-order stability cells will be motivated from the point of view of frictionless systems and potential energy. The problems associated with directly extending that definition to systems with Coulomb friction will lead us to a modified definition based on Fourier's Inequality [19]. Then we will focus on the frictionless case (Section 4), in which the concepts of passive and active FS-cells will be introduced. In Section 5, the analysis will be extended to include Coulomb friction and jamming. Finally, in Section 6, we will conclude with a summary and suggestions for future work.

2 Preliminaries

The quasistatic model used in this paper is the same as the one used in [37]. For completeness, we will briefly present the relevant assumptions and equations below.

Consider a system of three-dimensional rigid bodies with n_c simultaneous contacts (Figure 2 shows a two-dimensional system, but we stress that our analysis is fully three-dimensional). Bodies that are immobile with respect to the inertial frame, as indicated by a series of short parallel lines on a boundary, comprise the base of the manipulator (bodies B_0 and B_1 in the Figure). The manipulator is composed of the base and the bodies connected to it by revolute or prismatic joints (bodies B_2-B_6 in the Figure). The remaining bodies will be referred to as objects (bodies B_7 and B_8). Since they are not directly actuated, they can only move in response to external and contact forces. Note that contact may occur anywhere on any body.

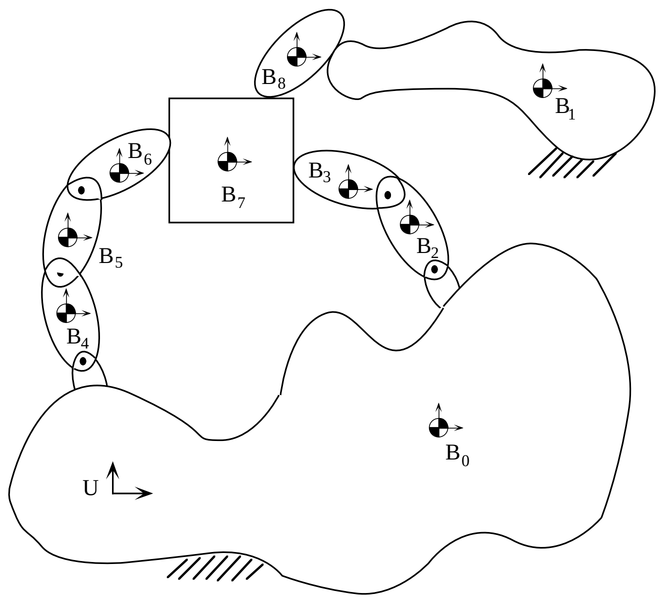

Figure 2: A Whole-Arm Manipulator in Contact with Two Objects

The analysis below will be based on what is commonly called a contact mode, which is a particular assignment of a contact interaction (*i.e.*, rolling, sliding, or breaking) to each contact. For the most part, we will be concerned with contact modes consisting of only rolling and sliding contact interactions, so $n_c = n_R + n_S$, where n_R and n_S are the numbers rolling and sliding contacts, respectively. In the frictionless case, rolling is impossible, so $n_R = 0$. Breaking contacts will be considered when appropriate.

2.1 Assumptions

1. The bodies in the system are rigid polyhedra.
2. Coulomb friction exists at the contacts.
3. The Denavit-Hartenberg parameters of the WAM are known.

4. Each joint may be either position- or effort-controlled (effort-control implies torque-control of revolute joints and force-control of prismatic joints).

5. Dynamic effects are negligible.

6. There is only one object.

7. No uncertainty exists.

Assumptions 1-5 are commonly made in analyses of robotic manipulation systems and experimental results support their validity, particularly in cases involving sliding contacts [5, 15, 26, 31, 38]. Assumption 6 is used to simplify the analysis presented here, but it may be removed by straight forward extension of the model. Assumption 7 yields a deterministic system model, thereby keeping the dimension of C-space as small as possible. The quasistatic assumption, Assumption 5, can be removed by the inclusion of inertial forces in the equilibrium equations (which then becomes D'Alembert's Principle), replacing the kinematic velocity constraints with the analogous acceleration constraints, using Dupont's results on jamming in dynamic systems [11], and augmenting C-space with the velocity variables. The polyhedral body assumption in Assumption 1 can be relaxed by replacing the holonomic rolling constraint with the proper nonholonomic constraints. The effect would be to raise the dimensions of the FS-cells with rolling contacts and correspondingly, the representations of those FS-cells would change. However, we point out that all other aspects of our analysis is valid for curved rigid bodies, because they are based on forces and instantaneous velocities.

2.2 Configuration Space and Contact Formation Cells

Let the 6-vector, $\mathbf{q}$, and the n_θ-vector, $\boldsymbol{\theta}$, viewed as column vectors, represent the configurations of the object and the manipulator, respectively. Here n_θ is the number of joints in the manipulator.[3] Note that in the planar case $\mathbf{q}$ would be a 3-vector. Together, a particular $\mathbf{q}$ and $\boldsymbol{\theta}$ represent a particular configuration of the entire system. All possible $(6+n_\theta)$-vectors, $(\mathbf{q}^T\ \boldsymbol{\theta}^T)^T$, comprise the system's C-space, $\mathcal{X}$. However, in situations calling for compliant motion, we need to augment $\mathcal{X}$ with the elements of the joint effort vector $\boldsymbol{\tau}$. This augmented C-space, denoted by $\mathcal{Y}$, is the product of $\mathcal{X}$ with Euclidean n_θ-space, R^{n_θ}, so has dimension $6 + 2n_\theta$. Thus, we have $\mathcal{Y} = \mathcal{X} \times \mathcal{R}^{n_\theta}$.

Our development of FS-cells depends on the notion of a "facet" of contact space, which we refer to as a contact formation cell (*CF-cell*). Recall that a "contact formation" is a specific set of elemental contacts [7]. The CF-cell is the "facet" of contact space corresponding to a given contact formation. This is the subset of C-space for which at least

[3]Referring to $\mathbf{q}$ and $\boldsymbol{\theta}$ as vectors is not strictly correct, because $\mathcal{X}$ is not a vector space. See [40] for more precise definitions.

one contact exists and no bodies overlap.[4] A CF-cell can be expressed as a conjunction of equations and inequalities in the system's configuration variables [20, 40]. We refer to these equations and inequalities as *geometric C-functions*; they include constraints which disallow interpenetration of the bodies and take into account their finite extent.

Let us collect all kineatic equations enforcing the maintenance of the current set of rolling and sliding contacts in the vector equation $\mathbf{f}_{geo}(\mathbf{q}, \boldsymbol{\theta}) = \mathbf{0}$. For each of the n_R contacts designated as rolling, two more constraint equations (one more in the plane) are added to mathematically prevent sliding. These constraints can be derived by adding two fictitious faces to the contacting bodies and adding the two corresponding fictitious elemental contacts (The intersection of the two fictitious faces and the actual face must be the contact point.). Therefore, the length of $\mathbf{f}_{geo}$ is $3n_R + n_S$ ($2n_R + n_S$ in the plane).

It is convenient to partition $\mathbf{f}_{geo}$ into two subvectors: $\mathbf{f}_{geoR}$ and $\mathbf{f}_{geoS}$ containing the C-functions associated with the rolling and sliding contacts, respectively. The C-functions corresponding to the breaking contacts are placed in the vector $\mathbf{f}_{geoB}$. Before these contacts break, $\mathbf{f}_{geoB}$ are equal to zero. The breaking of a contact is indicated by the corresponding element of $\mathbf{f}_{geoB}$ becoming positive. The inequalities ensuring that the elemental contacts remain within the finite bounds of the body features and disallowing interpenetration are collected in the vector inequality $\mathbf{h}_{geo}(\mathbf{q}, \boldsymbol{\theta}) \geq \mathbf{0}$.[5]

Our formal definition of CF-cells, $\mathcal{CF}_{\mathcal{X}}$ and $\mathcal{CF}_{\mathcal{Y}}$, in the spaces, $\mathcal{X}$ and $\mathcal{Y}$, are:

$$\mathcal{CF}_{\mathcal{X}} = \{(\mathbf{q}, \boldsymbol{\theta}) \in \mathcal{X} \mid \mathbf{f}_{geo} = \mathbf{0} \bigwedge \mathbf{h}_{geo} \geq \mathbf{0}\} \tag{1}$$

$$\mathcal{CF}_{\mathcal{Y}} = \mathcal{CF}_{\mathcal{X}} \times \mathcal{R}^{n_\theta}. \tag{2}$$

Note that in generic cases, the dimension of a CF-cell is $3n_R + n_S$ less than that of the ambient C-space. Recall that the vector $\mathbf{f}_{geo}$ does not include $\mathbf{f}_{geoB}$.

2.3 Kinematic Velocity Constraints

The kinematic velocity constraints, equilibrium equations, and Coulomb friction constraints are fundamental to quasistatic manipulation planning, so we introduce them briefly here and in the next subsection (for a detailed derivation with slightly different notation, see [37]). Let us choose a coordinate frame for every contact. The origin of the i^{th} frame coincides with the i^{th} contact point. The $\hat{\boldsymbol{n}}_i$ axis is normal to the tangent plane and is chosen to point inward with respect to the object. The $\hat{\boldsymbol{t}}_i$ and $\hat{\boldsymbol{o}}_i$ axes lie in the tangent plane and are orthogonal.

[4]We note that a CF-cell as defined in this paper is a subset of the corresponding CF-cell defined in a previous paper [13]. In that paper, the finite extent of the bodies was ignored and they were allowed to overlap.

[5]Note that it has been shown that $\mathbf{f}_{geo}(\mathbf{q}, \boldsymbol{\theta}) = \mathbf{0}$ is a manifold for any number of elemental contacts if the bodies in contact are polyhedral [21]. The same cannot be said if the bodies are curved.

The kinematic constraints enforcing the maintenance of the sliding and rolling contacts are contained in the vector equation $\mathbf{f}_{geo} = \mathbf{0}$. Maintaining the i^{th} rolling contact between two polyhedral bodies requires the following:

$$\frac{\partial(\mathbf{f}_{geoR})_i}{\partial \mathbf{q}}\mathbf{dq} + \frac{\partial(\mathbf{f}_{geoR})_i}{\partial \boldsymbol{\theta}}\mathbf{d}\boldsymbol{\theta} = 0 \tag{3}$$

where $(\mathbf{f}_{geoR})_i$ is the vector of length 3 containing the C-functions pertaining to the i^{th} rolling contact and $\mathbf{dq}$ and $\mathbf{d}\boldsymbol{\theta}$ are the total differentials of $\mathbf{q}$ and $\boldsymbol{\theta}$, respectively. The corresponding equation for the j^{th} sliding contact can be formed by replacing $(\mathbf{f}_{geoR})_i$ with $(\mathbf{f}_{geoS})_j$, where the length of $(\mathbf{f}_{geoS})_j$ is one.

The partial derivatives of the C-functions are more commonly known as wrenches and Jacobians in the robotics literature. Specifically, the partial derivatives of the usual C-function (viewed as a row vector) with respect to the object's configuration, $\mathbf{q}$, is proportional to the normal unit wrench of the i^{th} contact, $\mathbf{w}_{in}^T$. The partial derivative with respect to the manipulator's configuration is proportional to, $\mathbf{j}_{in}$, the row of the i^{th} contact's Jacobian matrix corresponding to velocity in the direction of the contact normal (see [39] for details). Similarly, if one fictitious plane used to enforce rolling at the i^{th} contact contains $\hat{\boldsymbol{n}}_i$ and $\hat{\boldsymbol{t}}_i$, and the other contains $\hat{\boldsymbol{n}}_i$ and $\hat{\boldsymbol{o}}_i$, then the derivatives of the corresponding C-functions are proportional to the unit wrenches, $\mathbf{w}_{io}^T$ and $\mathbf{w}_{it}^T$, and Jacobians, $\mathbf{j}_{io}$ and $\mathbf{j}_{it}$, in the $\hat{\boldsymbol{o}}_i$ and $\hat{\boldsymbol{t}}_i$ directions, respectively.

Differentiating $\mathbf{f}_{geo}$ with respect to time yields the kinematic velocity constraints as follows:

$$\mathbf{W}_A^T\dot{\mathbf{q}} - \mathbf{J}_A\dot{\boldsymbol{\theta}} = \mathbf{0} \tag{4}$$

where the $((3n_R + n_S) \times 6)$ matrix $\mathbf{W}_A^T$ has three rows, $\mathbf{w}_{in}^T$, $\mathbf{w}_{it}^T$, and $\mathbf{w}_{io}^T$, for each rolling contact and one row, $\mathbf{w}_{in}^T$, for each sliding contact, and the $((3n_R + n_S) \times n_\theta)$ matrix $\mathbf{J}_A$ consists of the corresponding rows, $\mathbf{j}_{in}$, $\mathbf{j}_{it}$, and $\mathbf{j}_{io}$.

For the breaking contacts, we require:

$$\mathbf{W}_{nB}^T\dot{\mathbf{q}} - \mathbf{J}_{nB}\dot{\boldsymbol{\theta}} > \mathbf{0} \tag{5}$$

where the $\mathbf{w}_{in}^T$ and $\mathbf{j}_{in}$ of the i^{th} breaking contact are the i^{th} rows of $\mathbf{W}_{nB}^T$ and $\mathbf{J}_{nB}$.

2.4 Equilibrium Constraints

Let $\mathbf{c}_i = [c_{in}\ c_{it}\ c_{io}]$ be the i^{th} contact force expressed in the i^{th} contact frame. The element c_{in} is the normal component of the contact force; c_{it} and c_{io} are the components of the friction force (these three components are also known as wrench intensities). Then the equilibrium equations of the object (6) and the manipulator (7), and the Coulomb friction constraints (8) and (9) are:

$$\mathbf{W}_n\mathbf{c}_n + \mathbf{W}_t\mathbf{c}_t + \mathbf{W}_o\mathbf{c}_o \;=\; -\mathbf{g}_{obj} \tag{6}$$

$$
\begin{aligned}
\mathbf{J}_n^T \mathbf{c}_n + \mathbf{J}_t^T \mathbf{c}_t + \mathbf{J}_o^T \mathbf{c}_o &= \boldsymbol{\tau} - \mathbf{g}_{man} && (7)\\
\mathbf{c}_i^T \mathbf{D}_i \mathbf{c}_i &\geq 0; \qquad i = 1, \ldots, n_c && (8)\\
\mathbf{c}_n &\geq \mathbf{0} && (9)
\end{aligned}
$$

where $\mathbf{c}_n$ is the normal wrench intensity vector of length n_c whose i^{th} component is c_{in} ($\mathbf{c}_t$ and $\mathbf{c}_o$ are defined analogously); $\mathbf{g}_{obj}$ is the external wrench vector of length 6 acting on the object; $\boldsymbol{\tau}$ is the n_θ-vector of joint efforts, with length equal to the number of joints, n_θ; $\mathbf{g}_{man}$ is the n_θ-vector of generalized gravity forces experienced by the joints of the manipulator; $\mathbf{0}$, is the zero vector, $\mathbf{D}_i = diag\{\mu_i^2, -1, -1\}$, with μ_i the effective coefficient of friction at the i^{th} contact (see [37] for the definitions of the $\mathbf{W}$ and $\mathbf{J}$ matrices).

The normal wrench matrix, $\mathbf{W}_n$, appearing in equation (6), transforms the normal components of the contact forces to the universal frame. It's columns are the unit wrenches, $\mathbf{w}_{in}$; for each contact. Similarly, $\mathbf{W}_t$ and $\mathbf{W}_o$ transform the frictional components of the contact forces and have columns $\mathbf{w}_{it}$ and $\mathbf{w}_{io}$. The normal Jacobian matrix, $\mathbf{J}_n^T$, has columns $\mathbf{j}_{in}^T$ and maps the normal components of the contact forces into joint efforts (forces at prismatic joints and torques at revolute joints). The Jacobian matrices, $\mathbf{J}_t^T$ and $\mathbf{J}_o^T$, have columns $\mathbf{j}_{it}^T$ and $\mathbf{j}_{io}^T$ and transform the frictional components of the contact forces. Note that equations (6) and (7) represent dynamic equations of motion if $\mathbf{g}_{obj}$ and $\mathbf{g}_{man}$ are redefined to include inertial, Coriolis, and centripetal forces and moments.

3 First-Order Stability Cells

The definition of first-order stability for a grasped frictionless object is given in [39] as follows: *A grasped frictionless object is first-order stable if the object's configuration corresponds to a stationary point* (*i.e.,* an equilibrium configuration) *of its constrained potential energy and if every* (kinematically) *feasible perturbation of the object away from the stationary configuration strictly increases its potential energy.* Equivalently, the virtual work of the gravitational force acting on the object for every possible nontrivial virtual displacement is negative. Thus it follows that the virtual work is negatively proportional to the change in the gravitational potential energy of the object.

It was shown in [39] that the object is first-order stable if and only if a certain linear program has a unique solution. For this to be true, there must be 6 linearly independent contacts (three in the plane) with strictly positive contact force magnitudes. This in turn, implies that those 6 contacts will be maintained during manipulation in the neighborhood of the equilibrium configuration and that a contact can only be broken by applying a disturbing force of (nonzero) finite magnitude to the object.[6] Moreover, when the object has first-order stability, the instantaneous joint velocities completely determine the object's instantaneous velocity (assuming that the joint velocities are kinematically admissible).

[6]This is a direct consequence of sensitivity theory in linear programming [24].

The natural extension of the definition of first-order stability to situations with Coulomb friction is through Fourier's Principle (also referred to as Fourier's Inequality) [19]. In the context of the dexterous manipulation problem studied here, Fourier's Principle is: *an object in an equilibrium configuration with Coulomb friction is stable if the maximum virtual work is zero.* Here the maximum is over all virtual motions (including the trivial virtual motion) with the corresponding contact forces in effect.

By applying Fourier's Inequality to the case of a frictionless particle at rest on a plane perpendicular to the gravitational force, we see that the particle is "stable" (nonaccelerating). However, in this situation the particle might be better classified as "marginally stable," because the virtual work of all virtual motions maintaining contact with the plane is zero. Our first-order stability condition is obtained if we change Fourier's Inequality to a strict inequality for all nontrivial virtual motions and allow satisfaction by equality only for the trivial virtual motion. This guarantees that the particle's position corresponds to a unique, nonsmooth local minimum of the potential energy, which implies that every nontrivial virtual displacement of the particle is resisted by a (nonzero) finite restoring force. Physically, this corresponds to the marble in the bottom of the pyramid-shaped bowl discussed earlier.

When Coulomb friction is present, the nonsmooth potential energy well can be replaced by the analogous (negative) virtual work well. Consider the particle on the plane again, but assume Coulomb friction is present. Then, in any position and for all nontrivial virtual motions, the virtual work is negative. Thus, we can view the particle as being at rest at the bottom of the (negative) virtual work well. The primary difference between this well and the potential energy well, is that after a nontrivial virtual displacement, the particle generally will not return to the bottom of the original (negative) virtual work well. Instead, it will come to rest at the bottom of a translated copy of the original well. For example, a book at rest on a table remains at rest until acted on be a force large enough to overcome friction or gravity. If such a force is applied and then removed, the book will eventually come to rest in a new location, but it will be at the bottom of another negative virtual work well.

Using the suggested strict form of Fourier's Inequality facilitates the unified treatment of the frictional and frictionless cases, but testing a configuration for first-order stability with friction would require the global solution of a "mathematical program with equilibrium constraints" [25]. These problems are known to be extremely difficult to solve in general. Therefore, we will use the following definition of first-order stability that is easier to test and guarantees the satisfaction of the stronger form of Fourier's Inequality. Sufficient conditions for first-order stability in both the frictionless and frictional cases will be derived in the next two sections.

Definition (First-Order Stability): *A grasped object will be referred to as* first-order stable *if the velocity of the manipulator completely determines the velocity of the object and if the contact mode for a given set of elemental contacts can only be altered by a (nonzero) finite* *disturbing force.*[7] [8]

Recall that "contact mode" refers to the set of elemental contacts and their interactions (*e.g.,* rolling, sliding, or breaking). Thus the contact mode is altered if the interaction of at least one contact has changed. It is possible that a previously separated pair of body features collide, giving rise to a new contact, or that a vertex slides off the face it has been in contact with. While these types of contact mode changes are important and are included in our analysis, they represent known boundaries of FS-cells. This means that it is not difficult to predict when such a mode change will occur. A more difficult problem that will be discussed later is that of preventing unwanted mode changes like the conversion of sliding to rolling when Coulomb friction is present. This problem is difficult, because the force at a sliding contact lies on the boundary of its friction cone, but Coulomb's Law allows the force at a rolling contact to be on the boundary of the cone too.

4 Frictionless First-Order Stability Cells

In this Section, we derive relationships in the variables, $\mathbf{q}$ and $\boldsymbol{\theta}$, that when imposed on a CF-cell, defines an FS-cell in the frictionless case. We also introduce two subclasses of FS-cells: *active FS-cells* and *passive FS-cells.* During system motion, the C-point can be forced to remain within an active FS-cell if the manipulator is operated under active compliant control (*e.g.*, some joints may be position-controlled while others are effort-controlled). In this case, we show that the relevant relationships, the *physical C-functions*, depend on $\boldsymbol{\tau}$ in addition to $\mathbf{q}$ and $\boldsymbol{\theta}$. In contrast, the C-point can remain in a passive FS-cell without the aid of compliant control; all joints may be position-controlled. Consequently, the physical C-functions defining passive FS-cells do not depend on $\boldsymbol{\tau}$.

The first requirement of an FS-cell is that the object velocity, $\dot{\mathbf{q}}$, be uniquely determined by the manipulator velocity, $\dot{\boldsymbol{\theta}}$. Given the desired contact mode (*i.e.*, the subset of contacts to be maintained), equation (4) and inequality (5) must be feasible for the given choice of $\dot{\boldsymbol{\theta}}$. Noting that in the frictionless case, $\mathbf{W}_A = \mathbf{W}_n$ and $\mathbf{J}_A^T = \mathbf{J}_n^T$, trivially extending the quasistatic model utility conditions given in [43] to three-dimensional systems, and specializing them to the frictionless case yields the following two conditions that guarantee the uniqueness of $\dot{\mathbf{q}}$:

$$Rank(\mathbf{W}_n) = 6 \tag{10}$$

$$\exists \ \mathbf{E}_P \ \ni \ \mathbf{P}_I \text{ is nonsingular} \tag{11}$$

where $\mathbf{P}_I = [\mathbf{W}_n^T \ \ -\mathbf{J}_n\mathbf{E}_P^T]$ and $\mathbf{E}_P^T$ is a selection matrix. Specifically, $\mathbf{E}_P^T$ is a matrix of zeros and ones that selects enough linearly independent columns of $\mathbf{J}_n$ to make $\mathbf{P}_I$ nonsingular. $\mathbf{E}_P^T$ is formed from an $(n_\theta \times n_\theta)$ identity matrix by removing the columns corresponding to the columns of $\mathbf{J}_n$ not used in $\mathbf{P}_I$. Note that $\mathbf{E}_P$ is generally not unique.

[7]Sufficient mathematical conditions will be given in Section 5.

[8]Note that this definition could be applied to dynamic systems with no modifications.

The second requirement is that the contact mode be maintained despite infinitesimal perturbations in the generalized applied force, $\begin{bmatrix} -\mathbf{g}_{obj} \\ \boldsymbol{\tau} - \mathbf{g}_{man} \end{bmatrix}$. Satisfying conditions (10) and (11) makes the maintenance of the contacts kinematically feasible; it does not make it physically possible. However, the contacts will be maintained despite infinitesimal perturbations in the generalized applied force if all the contact force magnitudes are positive, (*i.e.*, $\mathbf{c}_n > \mathbf{0}$). To see this, set the friction force components to zero in equations (6) and (7) yielding:

$$\begin{bmatrix} \mathbf{W}_n \\ \mathbf{J}_n^T \end{bmatrix} \mathbf{c}_n = \begin{bmatrix} -\mathbf{g}_{obj} \\ \boldsymbol{\tau} - \mathbf{g}_{man} \end{bmatrix}. \tag{12}$$

Then, let $\boldsymbol{\tau}_I = \mathbf{E}_P \boldsymbol{\tau}$, $\mathbf{g}_{manI} = \mathbf{E}_P \mathbf{g}_{man}$, and $\mathbf{g}_I = \begin{bmatrix} -\mathbf{g}_{obj} \\ \boldsymbol{\tau}_I - \mathbf{g}_{manI} \end{bmatrix}$, so equation (12) can be solved for the wrench intensities and a subset, $\boldsymbol{\tau}_{II}$, of the joint efforts as follows:

$$\mathbf{c}_n = Adj(\mathbf{P}_I^T)\mathbf{g}_I/d \tag{13}$$

$$\boldsymbol{\tau}_{II} = \mathbf{g}_{manII} + \mathbf{P}_{II}^T Adj(\mathbf{P}_I^T)\mathbf{g}_I/d \tag{14}$$

where Adj denotes the matrix adjoint operation, d is the determinant of $\mathbf{P}_I$, $\mathbf{P}_{II}^T$ is the matrix of dimension $(6+n_\theta-n_c \times n_c)$ whose columns are those of $\mathbf{J}_n$ not included in $\mathbf{P}_I$, and $\boldsymbol{\tau}_{II}$ and $\mathbf{g}_{manII}$ are the corresponding elements of $\boldsymbol{\tau}$ and $\mathbf{g}_{man}$. Since $\mathbf{c}_n$ depends linearly on the generalized applied force, it is clear that a nonzero finite perturbation of the generalized applied force is required to drive any element of $\mathbf{c}_n$ to zero (which is a necessary condition for breaking a contact).

Equation (13) indicates that $\mathbf{c}_n$ can be computed uniquely for any choice of joint efforts, $\boldsymbol{\tau}_I$. The efforts, $\boldsymbol{\tau}_{II}$, of the remaining joints must satisfy equation (14) if the manipulator is to maintain equilibrium. Thus the joints corresponding to $\boldsymbol{\tau}_I$ should be effort-controlled, while the others are position-controlled. If the position controllers maintain their set-points, then their efforts will satisfy equation (14) (see [43] for more discussion).

We summarize the frictionless first-order stability conditions in the following theorem:

Theorem 1 *An active, three-dimensional, multi-rigid-body system with frictionless contacts has first-order stability if equations (10) and (11) are satisfied, the joints corresponding to* $\boldsymbol{\tau}_I$ *are effort-controlled, the other joints are position-controlled, and the wrench intensity vector,* $\mathbf{c}_n$*, as defined by equation (13) is strictly positive.*

Proof: The development immediately preceding the statement of the theorem serves as a proof. . . . . *q.e.d.*

Corollary 1 *A first-order stability cell of an active, three-dimensional, multi-rigid-body system with frictionless contacts must have between 6 and* $n_\theta+6$ *contacts,* i.e., $6 \le n_c \le 6+n_\theta$.

Proof: This Corollary follows immediately from the equations (10) and (11). .... *q.e.d.*

As noted above, the conditions (10) and (11) and the inequalities of Corollary 1 are the quasistatic model utility conditions (specialized to the frictionless case) given in [43]. The two main differences between first-order stability and quasistatic model utility are that: for the former condition, all of the wrench intensities must be positive and some of the joints must be effort-controlled, while for the latter condition, only a linearly independent set of wrench intensities must be positive and there is no requirement on the joint control modes.

We are now in a position to define the physical C-functions for frictionless systems as functions of $\mathbf{q}$, $\boldsymbol{\theta}$, and $\boldsymbol{\tau}$. As stated above, during manipulation in a particular FS-cell, the contact force at each contact will be compressive. However, when a contact breaks (or a new contact is established), the corresponding contact force will be zero. These situations must be dealt with explicitly in order to plan changes in the contact mode. Therefore, the wrench intensity vector, $\mathbf{c}_n$, is partitioned into two vectors of physical C-functions, $\mathbf{h}_{phy}$ and $\mathbf{f}_{phy1}$, containing the positive and zero-valued elements, respectively. Also, equation (14) represents physical constraints on the efforts that may be applied at the effort-controller joints. Denote the corresponding vector of physical C-function as $\mathbf{f}_{phy2}$.

Let $\mathbf{E}_S$ and $\mathbf{E}_B$ be the matrices that select the subsets of sliding and breaking contacts, respectively. Then $\mathbf{h}_{phy}$, $\mathbf{f}_{phy1}$, and $\mathbf{f}_{phy2}$ can be expressed as functions of $\mathbf{q}$, $\boldsymbol{\theta}$, and $\boldsymbol{\tau}$ as follows:

$$\begin{aligned}
\mathbf{h}_{phy}(\mathbf{q},\boldsymbol{\theta},\boldsymbol{\tau}) &= \mathbf{E}_S(Adj(\mathbf{P}_I^T(\mathbf{q},\boldsymbol{\theta}))\mathbf{g}_I(\mathbf{q},\boldsymbol{\theta},\boldsymbol{\tau})/d(\mathbf{q},\boldsymbol{\theta})) > \mathbf{0} && (15)\\
\mathbf{f}_{phy1}(\mathbf{q},\boldsymbol{\theta},\boldsymbol{\tau}) &= \mathbf{E}_B(Adj(\mathbf{P}_I^T(\mathbf{q},\boldsymbol{\theta}))\mathbf{g}_I(\mathbf{q},\boldsymbol{\theta},\boldsymbol{\tau})/d(\mathbf{q},\boldsymbol{\theta})) = \mathbf{0} && (16)\\
\mathbf{f}_{phy2}(\mathbf{q},\boldsymbol{\theta},\boldsymbol{\tau}) &= \boldsymbol{\tau}_{II} - \mathbf{g}_{manII}(\mathbf{q},\boldsymbol{\theta},\boldsymbol{\tau}) \\
&\quad - \mathbf{P}_{II}^T(\mathbf{q},\boldsymbol{\theta}))Adj(\mathbf{P}_I^T(\mathbf{q},\boldsymbol{\theta})))\mathbf{g}_I(\mathbf{q},\boldsymbol{\theta},\boldsymbol{\tau})/d(\mathbf{q},\boldsymbol{\theta})) = \mathbf{0} && (17)
\end{aligned}$$

where $\mathbf{f}_{phy1}$ has length $l \in \{0, 1, 2, ..., n_c\}$, $\mathbf{h}_{phy}$ has length $n_c - l$, and $\mathbf{f}_{phy2}$ has length $n_c - 6$. Note that when no contacts are breaking, $\mathbf{f}_{phy1}$ is degenerate, and when there are exactly 6 contacts, $\mathbf{f}_{phy2}$ is degenerate.

Finally, we must point out that $\mathbf{g}_{obj}$ and $\mathbf{g}_{man}$ are arbitrary generalized external forces. However, equations (15-17) are only functions of $\mathbf{q}$, $\boldsymbol{\theta}$, and $\boldsymbol{\tau}$, if $\mathbf{g}_{obj}$ and $\mathbf{g}_{man}$ are. One important case for which this is true, is when $\mathbf{g}_{obj}$ and $\mathbf{g}_{man}$ are due to gravity. Henceforth, we will assume that $\mathbf{g}_{obj}$ and $\mathbf{g}_{man}$ are functions of $\mathbf{q}$ and $\boldsymbol{\theta}$.

4.1 Passive Frictionless FS-cells

The solution of the normal wrench intensity vector, $\mathbf{c}_n$, in equation (13) suggests two distinct classes of FS-cells: those which depend on $\boldsymbol{\tau}$ and those which do not. We call the cells that do not depend on $\boldsymbol{\tau}$ *passive FS-cells*. For an FS-cell to be passive, there must be 6 contacts (*i.e.*, $n_c = 6$). When this is true, $\mathbf{E}_P$ is degenerate, so $\mathbf{P}_I^T = \mathbf{W}_n$ and $\mathbf{g}_I = -\mathbf{g}_{obj}$. In this situation, the object will be first-order stable if and only if $\mathbf{W}_n^{-1}$ exists and all the elements of $\mathbf{c}_n$ are positive. As a consequence, $\mathbf{f}_{phy1}$ is degenerate and $\mathbf{f}_{phy2}$ becomes an auxiliary condition for computing $\boldsymbol{\tau}$, so $\mathbf{g}_I$ and $\mathbf{h}_{phy1}$ lose their dependence on $\boldsymbol{\tau}$. Physically, these conditions imply that the external wrench alone will cause the object to comply with the manipulator as it moves. We refer to this type of system motion as *passively compliant*, because no active effort control is required at any joint to maintain the contact mode.

Let us denote the passive FS-cell for a given contact formation by $\mathcal{FS}_P$. This FS-cell can be formed by the intersection of the CF-cell in $\mathcal{X}$, $\mathcal{CF}_\mathcal{X}$, and the subset of $\mathcal{X}$ in which the contact forces are all compressive, $\mathcal{S}_P$. Thus we have:

$$\mathcal{FS}_P = \mathcal{CF}_\mathcal{X} \cap \mathcal{S}_P \tag{18}$$

where $\mathcal{S}_P = \{(\mathbf{q}, \boldsymbol{\theta}) \in \mathcal{X} \mid \mathbf{h}_{phy} > \mathbf{0}\}$. Recall that for passive FS-cells, $\mathbf{f}_{phy1}$ must be degenerate. The other physical C-function vector, $\mathbf{f}_{phy2}$, uniquely defines the joint efforts for each configuration in the passive FS-cell.

The elements of $\mathbf{W}_n$ are rational trigonometric functions of the elements of $\mathbf{q}$ and $\boldsymbol{\theta}$, and are independent of $\boldsymbol{\tau}$. Therefore, from equations equations (15-17), we see that the physical C-function of interest, $\mathbf{h}_{phy}$, is a rational trigonometric function of the elements of $\mathbf{q}$ and $\boldsymbol{\theta}$, and is independent of $\boldsymbol{\tau}$. However, some cell-decomposition algorithms require a polynomial representation. Fortunately, the set $\mathcal{S}_P$ can be rewritten as the union of two sets, $\mathcal{S}_P^+$ and $\mathcal{S}_P^-$, whose defining functions can be converted into polynomials by substituting $u_i = tan(\frac{\phi_i}{2})$ for each $i \in \{1, \ldots, n_c\}$ where ϕ_i is the i^{th} angular parameter in the elements of $\mathbf{q}$ and $\boldsymbol{\theta}$. Thus S_P is given as:

$$\mathcal{S}_P = \mathcal{S}_P^+ \cup \mathcal{S}_P^- \tag{19}$$

where the sets $\mathcal{S}_P^+$ and $\mathcal{S}_P^-$ are given by:

$$\mathcal{S}_P^+ = \{(\mathbf{q}, \boldsymbol{\theta}) \in \mathcal{X} \mid -\mathbf{E}_S Adj(\mathbf{W}_n)\mathbf{g}_{obj} > \mathbf{0} \bigwedge Det(\mathbf{W}_n) > 0\} \tag{20}$$

$$\mathcal{S}_P^- = \{(\mathbf{q}, \boldsymbol{\theta}) \in \mathcal{X} \mid -\mathbf{E}_S Adj(\mathbf{W}_n)\mathbf{g}_{obj} < \mathbf{0} \bigwedge Det(\mathbf{W}_n) < 0\}. \tag{21}$$

An alternative approach taken in [13], is to treat $cos(\phi_i)$ and $sin(\phi_i)$ as independent variables and add the constraint $cos^2(\phi_i) + sin^2(\phi_i) = 1$, for each $i \in \{1, \ldots, n_c\}$.

We reiterate that for passive FS-cells, there must be 6 contacts, all with positive contact force magnitudes. Thus, the number of breaking contacts, l, must be zero. The 6 contacts correspond to 6 geometric C-function equations, $\mathbf{f}_{geo} = \mathbf{0}$, written in the $6 + n_\theta$ variables of $\mathcal{X}$. Thus the dimension of passive FS-cells is typically n_θ (the inequalities $\mathbf{h}_{geo} > \mathbf{0}$ and $\mathbf{h}_{phy} > \mathbf{0}$ typically do not affect the dimension). In certain nongeneric situations, it is possible for the dimension of an FS-cell to be different from n_θ, but such situations are rare. If during manipulation in a passive FS-cell, l becomes positive, the corresponding l contacts are about to break. In this case, the object may still be stable, but stability can no longer be determined by the signs of the elements of $\mathbf{c}_n$; second- and/or high-order effects become important [35]. Also, in this case, the equations $\mathbf{f}_{geo} = \mathbf{0}$ and $\mathbf{f}_{phy1} = \mathbf{0}$ define a cell that represents part of the boundary between the FS-cell in question and portions of C-space for which the object is either higher-order stable or unstable [41].

4.2 Active Frictionless FS-cells

When the number of contacts, n_c, is greater than 6, then the elements of the wrench intensity vector, $\mathbf{c}_n$, are linearly dependent on the joint efforts, $\boldsymbol{\tau}$ (see equation (12)). As discussed earlier, maintaining the current contact formation requires "active" effort-control of $n_c - 6$ joints. Therefore, we refer to the class of FS-cells as *active FS-cells*. One apparent difficulty with active FS-cells is that the wrench intensity and joint effort vectors cannot be uniquely represented in C-space, $\mathcal{X}$, as they can be for passive FS-cells. Therefore, active FS-cells must be defined in the augmented C-space, $\mathcal{Y}$, of all possible values of $\mathbf{q}$, $\boldsymbol{\theta}$, and $\boldsymbol{\tau}$.

The definition of an active FS-cell, denoted by $\mathcal{FS}_A$, is given as:

$$\mathcal{FS}_A = \mathcal{CF}_\mathcal{Y} \cap \mathcal{S}_A \tag{22}$$

where the set $\mathcal{CF}_\mathcal{Y}$ is the CF-cell in $\mathcal{Y}$ corresponding to the given contact formation, the set $\mathcal{S}_A$, is defined as: $\mathcal{S}_A = \{(\mathbf{q}, \boldsymbol{\theta}, \boldsymbol{\tau}) \in \mathcal{Y} \mid \mathbf{f}_{phy} = \mathbf{0} \bigwedge \mathbf{h}_{phy} > \mathbf{0}\}$, with $\mathbf{f}_{phy}$ formed by vertically concatenating $\mathbf{f}_{phy1}$ and $\mathbf{f}_{phy2}$. As $\mathcal{S}_P$ was partitioned into two subsets based on the sign of $Det(\mathbf{W}_n)$, so we can partition $\mathcal{S}_A$ based on the sign of $Det(\mathbf{P}_I)$.

Despite the fact that the definition of active FS-cells depends on $\mathbf{q}$, $\boldsymbol{\theta}$, and $\boldsymbol{\tau}$, their dimension is still only n_θ in the generic case if no contacts are breaking or forming (*i.e.,* $l = 0$). This can be seen by noting that while the dimension of the augmented C-space, $\mathcal{Y}$, is $6 + 2n_\theta$, the numbers of geometric and physical C-function equations grow to n_c (the number of elements of $\mathbf{f}_{geo}$) plus $n_\theta - (n_c - 6)$, (the number of elements of $\mathbf{f}_{phy2}$). Physically, this result corresponds to the well-known fact that in rigid body systems, the position and effort of a joint cannot be controlled simultaneously.

5 Frictional First-Order Stability Cells

When friction is present, the conditions for first-order stability become more complex. This is partly because contacts may roll or slide, but the most difficult problem is that Coulomb's Law is ambiguous when a contact force lies on the boundary of the friction cone. This means that a sliding contact may unexpectedly convert to rolling without any change in the generalized applied force. This would violate our requirement that only a (nonzero) finite perturbation of the generalized applied force can cause a change in the contact mode. Nonetheless, as is shown below, it is still possible to derive conditions under which conversion from sliding to rolling is impossible. However, the conditions are nonholonomic, as they depend on a subset of the system's velocity variables.

The kinematic constraints for a given contact mode are again given by equation (4) and inequality (5). Thus for $\dot{\mathbf{q}}$ to be uniquely determined by $\dot{\boldsymbol{\theta}}$, we have:

$$\begin{aligned} Rank(\mathbf{W}_A) &= 6 && (23)\\ \exists\ \mathbf{E}_P &\ni \mathbf{P}_I \text{ is nonsingular} && (24) \end{aligned}$$

where the selection matrix $\mathbf{E}_P$ now has dimension $((3n_R + n_S - 6) \times n_\theta)$ and $\mathbf{P}_I$ is now given by $\mathbf{P}_I = [\mathbf{W}_A^T \quad -\mathbf{J}_A \mathbf{E}_P^T]$. Again, it is implied that the subset of joints corresponding to $\boldsymbol{\theta}_I = \mathbf{E}_P \boldsymbol{\theta}$ should be effort-controlled.

The second condition required for first-order stability is that the contact mode be maintained despite infinitesimal perturbations in the generalized applied force. As in the frictionless case, a contact will not break if the normal component of its contact force is positive. Similarly, rolling contacts will continue rolling if their contact forces lie strictly within their friction cones. Therefore, we require:

$$\mathbf{c}_i^T \mathbf{D}_i \mathbf{c}_i > 0, \qquad \forall i \in \mathcal{R} \tag{25}$$

$$c_{in} > 0, \qquad \forall i \in \{1, \ldots, n_c\} \tag{26}$$

where $\mathcal{R}$ is the subset of indices, $\{1, \ldots, n_c\}$, corresponding to the rolling contacts.

For every sliding contact, notice that Coulomb's Law allows us to eliminate the unknowns c_{it} and c_{io} using the following equations:

$$c_{it} = -c_{in} v_{it} \mu_i / \sqrt{v_{it}^2 + v_{io}^2}, \qquad \forall i \in \mathcal{S} \tag{27}$$

$$c_{io} = -c_{in} v_{io} \mu_i / \sqrt{v_{it}^2 + v_{io}^2}, \qquad \forall i \in \mathcal{S} \tag{28}$$

where $v_{it} = \mathbf{w}_{it}^T \dot{\mathbf{q}} - \mathbf{j}_{it} \dot{\boldsymbol{\theta}}$, $v_{io} = \mathbf{w}_{io}^T \dot{\mathbf{q}} - \mathbf{j}_{io} \dot{\boldsymbol{\theta}}$, and $\mathcal{S}$ is the subset of $\{1, \ldots, n_c\}$ corresponding to the sliding contacts.

Substituting equations (27) and (28) into equations (6) and (7) yields (see [43] for details):

$$\begin{bmatrix} \mathbf{W}_{A\mu} \\ \mathbf{J}_{A\mu}^T \end{bmatrix} \mathbf{c}_A = \begin{bmatrix} -\mathbf{g}_{obj} \\ \boldsymbol{\tau} - \mathbf{g}_{man} \end{bmatrix}. \tag{29}$$

where $\mathbf{c}_A$, the applicable wrench intensity vector, has length $3n_R + n_S$. This vector contains three wrench intensities (c_{in}, c_{it}, and c_{io}) for each rolling contact, $i \in \mathcal{R}$, but only one wrench intensity, c_{in} for each sliding contact, $i \in \mathcal{S}$. Note that the dimension of $\mathbf{W}_{A\mu}$ is the same as that of $\mathbf{W}_A$. However, the columns of $\mathbf{W}_{A\mu}$ corresponding to the sliding contacts are wrenches corresponding to contact forces on the boundaries of the friction cones, while those in $\mathbf{W}_A$ correspond to the contact normal directions. The matrices $\mathbf{J}_{A\mu}$ and $\mathbf{J}_A$ are similarly related. Also, notice that through the substitution of equations (27) and (28) just performed, the system's velocity variables have just entered the system equilibrium equation, equation (29).

In the frictionless case, the existence of $\mathbf{P}_I^{-1}$ facilitated the task of checking whether all the contact force magnitudes were positive, because then they could be uniquely determined. When friction is present, we must check the satisfaction of inequalities (25) and (26). This will be facilitated by imposing the condition that the applicable wrench intensity vector, $\mathbf{c}_A$ (and therefore the contact forces) be uniquely determined by the generalized applied force.

Comparing equation (29) to equation (12) implies that the following conditions must be satisfied for the generalized applied forces to determine the contact forces uniquely:

$$Rank(\mathbf{W}_{A\mu}) = 6 \tag{30}$$
$$\exists\ \mathbf{E}_Q \ni \mathbf{Q}_I \text{ is nonsingular} \tag{31}$$

where $\mathbf{E}_Q$ is a selection matrix and $\mathbf{Q}_I = [\mathbf{W}_{A\mu}^T \quad (\mathbf{E}_Q \mathbf{J}_{A\mu}^T)^T]^T$. Here the implication is that the joints corresponding to $\boldsymbol{\tau}_I = \mathbf{E}_Q \boldsymbol{\tau}$ should be effort-controlled. Also, the selection matrices $\mathbf{E}_P$ and $\mathbf{E}_Q$ imply two control mode partitionings of the joints, but since only one partitioning can be applied at a given time, $\mathbf{E}_P$ must equal $\mathbf{E}_Q$.[9]

The satisfaction of the conditions above prevents the loss of contacts and the conversion of contacts from rolling to sliding. This leaves one final requirement: *no sliding contact may convert to rolling.* Since Coulomb's Law is ambiguous in this situation, we must bring other aspects of the model into the picture to determine when conversion is possible. In particular, by assuming that such conversions take place, we derive conditions under which this assumption is contradicted.

Suppose that contact i is sliding. Then the contact force lies on the surface of the friction cone (see Figure 3). Denote the unit wrench associated with that contact force by $\mathbf{w}_{ix}$ and let c_{ix} be the corresponding wrench intensity. Note that $\mathbf{w}_{ix}$ (scaled by $\sqrt{1+\mu_i^2}$) is a column of the matrix $\mathbf{W}_{A\mu}$. Next, let $\mathbf{w}_{iy}$ be the unit wrench, with intensity c_{iy}, corresponding to a force in the contact tangent plane and perpendicular to the sliding contact force. Further, let $\mathbf{w}_{iz}$ be the unit wrench, with intensity c_{iz}, corresponding to a force lying on the friction cone diametrically opposite to the actual sliding contact force direction. Notice that c_{ix} and c_{iz} must be nonnegative and because the contact is sliding, c_{iy} and c_{iz} are initially zero. Consistent with constraint (26), c_{ix} is initially positive.

Assume contact i converts to rolling. This conversion adds two new kinematic constraints to equation (4), typically leading to kinematic inconsistency. Thus the planned joint trajectories (or at least a subset of them) are now impossible to execute accurately. Henceforth, we will refer to such as situation as "jammed" even though the joints may continue to move due to compliance (active or passive). The position-controllers cannot sense the jam instantly, so they continue execution "unaware." As errors accrue, the joint efforts in the position-controlled joints build accordingly. Assuming PID position-controllers, we have:

$$\mathbf{E}_J sgn(\mathbf{e}_{II}) = \mathbf{E}_J sgn(\Delta \boldsymbol{\tau}_{II}) \tag{32}$$

where $\mathbf{E}_J$ is the selection matrix which selects the jammed joints (*i.e.*, the position-controlled joints whose trajectories are altered by the new kinematic constraints), $\mathbf{e}_{II}$ is the vector of joint errors at the jammed joints, and $\Delta \boldsymbol{\tau}_{II}$ is the vector of changes in the control efforts at those same joints. Note that effort-controlled joints do not jam; they just continue to apply the desired effort regardless of the motion of the rest of the system.

[9] A detailed discussion of this issue can be found in [43].

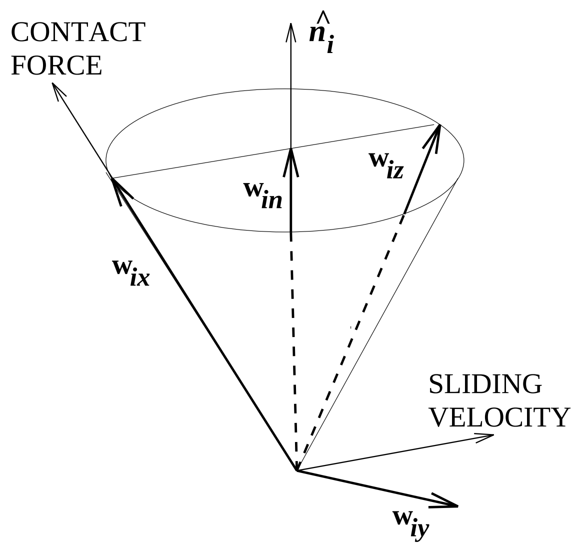

Figure 3: The Wrenches, $\mathbf{w}_{ix}$, $\mathbf{w}_{iy}$, and $\mathbf{w}_{iz}$ in Relation to the Friction Cone

Consider a situation in which only one contact, i, is sliding. The equilibrium equations can be written as follows:

$$\begin{bmatrix} \mathbf{W}_{A\mu} & \mathbf{w}_{iy} & \mathbf{w}_{iz} \\ \mathbf{J}_{A\mu}^T & \mathbf{j}_{iy}^T & \mathbf{j}_{iz}^T \end{bmatrix} \begin{bmatrix} \mathbf{c}_A \\ c_{iy} \\ c_{iz} \end{bmatrix} = \begin{bmatrix} -\mathbf{g}_{obj} \\ \boldsymbol{\tau} - \mathbf{g}_{man} \end{bmatrix} \tag{33}$$

where the vectors, $\mathbf{j}_{iy}^T$ and $\mathbf{j}_{iz}^T$, are the columns of the Jacobian matrix corresponding to the unit wrenches, $\mathbf{w}_{iy}$ and $\mathbf{w}_{iz}$. If contact i converts to rolling, then the system of equations, (32) and (33), must be feasible for some nonnegative c_{iz}. If not, the rolling assumption is contradicted.

Using $\mathbf{E}_Q$ to partition equation (33) yields:

$$\begin{bmatrix} \mathbf{Q}_I & \mathbf{y}_{iI} & \mathbf{z}_{iI} \\ \mathbf{Q}_{II} & \mathbf{y}_{iII} & \mathbf{z}_{iII} \end{bmatrix} \begin{bmatrix} \mathbf{c}_A \\ c_{iy} \\ c_{iz} \end{bmatrix} = \begin{bmatrix} \mathbf{g}_I \\ \boldsymbol{\tau}_{II} - \mathbf{g}_{manII} \end{bmatrix}. \tag{34}$$

The vector of efforts at the position-controlled joints, $\boldsymbol{\tau}_{II}$, can now be written as:

$$\begin{aligned} \boldsymbol{\tau}_{II} &= \mathbf{Q}_{II}\mathbf{Q}_I^{-1} \begin{bmatrix} -\mathbf{g}_{obj} \\ \boldsymbol{\tau}_I - \mathbf{g}_{manI} \end{bmatrix} + \mathbf{g}_{manII} \\ &+ (\mathbf{y}_{iII} - \mathbf{Q}_{II}\mathbf{Q}_I^{-1}\mathbf{y}_{iI})c_{iy} + (\mathbf{z}_{iII} - \mathbf{Q}_{II}\mathbf{Q}_I^{-1}\mathbf{z}_{iI})c_{iz}. \end{aligned} \tag{35}$$

When $c_{iy} = c_{iz} = 0$, $\boldsymbol{\tau}_{II}$ takes on its value just prior to jamming. This implies that the third and fourth terms of the right hand side of equation (35) represent the changes in the efforts of the jammed joints' position controllers, $\Delta\boldsymbol{\tau}_I I$, due to the position errors induced by the jam.

Let $\mathbf{X}$ denote the diagonal matrix with i^{th} diagonal entries given by $sgn(e_{iII})$, where e_{iII} is the i^{th} element of $\mathbf{e}_{II}$. Equation (32) implies the following inequality:

$$\mathbf{X}\mathbf{E}_J \Delta \boldsymbol{\tau}_{II} > \mathbf{0}. \tag{36}$$

The physical interpretation of inequality (36) is that the directions of the changes in the efforts of the jammed joints must be consistent with their errors. Note that it is possible that inequality (36) be satisfied by equality only if the conversion of sliding to rolling happens to generate constraints that do not make the kinematic velocity constraints (4) inconsistent. Since this situation is extremely rare, we will not consider the possibility further.

Substituting the third and fourth terms on the right hand side of equation (35) into inequality (36) yields the conditions for *possible* jamming (*i.e.*, the following are necessary conditions for jamming):

$$\mathbf{X}\mathbf{E}_J[(\mathbf{y}_{iII} - \mathbf{Q}_{II}\mathbf{Q}_I^{-1}\mathbf{y}_{iI}) \quad (\mathbf{z}_{iII} - \mathbf{Q}_{II}\mathbf{Q}_I^{-1}\mathbf{z}_{iI})] \begin{bmatrix} c_{iy} \\ c_{iz} \end{bmatrix} > 0 \tag{37}$$

$$c_{iz} \geq 0. \tag{38}$$

We stress that satisfaction of the system (37) and (38) implies that the situation is "ripe" for jamming. If a conversion from sliding to rolling occurs, then c_{iz} will increase driving the contact force inside the friction cone, thus sustaining the jam. However, if the system of inequalities is not feasible, then contact i cannot convert to rolling, because the changes in the joint efforts corresponding to the change in the i^{th} contact force are not consistent with the joint position errors, $\mathbf{e}_{II}$.

Extending the possible jamming conditions, inequalities (37) and (38), to situations with more than one sliding contacts leads to:

$$\mathbf{X}\mathbf{E}_J[(\mathbf{Y}_{II} - \mathbf{Q}_{II}\mathbf{Q}_I^{-1}\mathbf{Y}_I) \quad (\mathbf{Z}_{II} - \mathbf{Q}_{II}\mathbf{Q}_I^{-1}\mathbf{Z}_I)] \begin{bmatrix} \mathbf{c}_y \\ \mathbf{c}_z \end{bmatrix} > \mathbf{0} \tag{39}$$

$$\mathbf{c}_z \geq \mathbf{0} \tag{40}$$

where the matrices $\mathbf{Y}_I$, $\mathbf{Y}_{II}$, $\mathbf{Z}_I$, and $\mathbf{Z}_{II}$ are formed by horizontally concatenating n_S vectors $\mathbf{y}_{iI}$, $\mathbf{y}_{iII}$, $\mathbf{z}_{iI}$, and $\mathbf{z}_{iII}$. The matrix, $\mathbf{X}\mathbf{E}_J[(\mathbf{Y}_{II} - \mathbf{Q}_{II}\mathbf{Q}_I^{-1}\mathbf{Y}_I) \quad (\mathbf{Z}_{II} - \mathbf{Q}_{II}\mathbf{Q}_I^{-1}\mathbf{Z}_I)]$, is referred to as the *jamming matrix*. The numbers of rows of the matrices subscripted by I and II are $3n_R + n_S$ and $6 + n_\theta - 3n_R - n_S$, respectively, so the numbers of rows and columns in the jamming matrix are the number of position-controlled joints, $6 + n_\theta - 3n_R - n_S$, and the number of sliding contacts, n_S, respectively.

The above development is summarized by the following theorems and corollaries.

Theorem 2 *An active, three-dimensional, multi-rigid-body system with Coulomb friction acting at the contacts cannot jam if the system of inequalities (39) and (40) is infeasible.*

Proof: The preceding derivation serves as a proof. .... *q.e.d.*

Corollary 2 *An active, two-dimensional, multi-rigid-body system with Coulomb friction acting at the contacts cannot jam if any row of the matrix,* $\mathbf{X}\mathbf{E}_J(\mathbf{Z}_{II} - \mathbf{Q}_{II}\mathbf{Q}_I^{-1}\mathbf{Z}_I)$*, has all nonpositive elements.*

Proof: In the planar case, $\mathbf{c}_y$ is zero by definition. Also, jamming is impossible only if the product $(\mathbf{Z}_{II} - \mathbf{Q}_{II}\mathbf{Q}_I^{-1}\mathbf{Z}_I)\mathbf{c}_z$ has all positive elements. Since $\mathbf{c}_z$ is nonnegative, it is clear that the system of inequalities (39) and (40) is infeasible if it has at least one row with all elements nonnegative. .... *q.e.d.*

Theorem 3 *An active, three-dimensional, multi-rigid-body system with Coulomb friction acting at the contacts has first-order stability if equations (23-26), (30) and (31) are satisfied with* $\mathbf{E}_P = \mathbf{E}_Q$*, the joints corresponding to* $\boldsymbol{\tau}_I$ *are effort-controlled, the other joints are position-controlled, and the system of inequalities (39) and (40) is infeasible.*

Proof: The preceding derivation serves as a proof. .... *q.e.d.*

Corollary 3 *A first-order stability cell of an active, three-dimensional, multi-rigid-body system with Coulomb friction acting at the contacts must have between 6 and* $n_\theta + 6$ *contact constraints,* i.e., $6 \leq 3n_R + n_S \leq 6 + n_\theta$.

Proof: This Corollary follows immediately from the rank condition on $\mathbf{W}_A$ and the existence of $\mathbf{P}_I$ which is nonsingular. .... *q.e.d.*

Jamming Example

Figure 4 shows a simple planar system in which the stick finger begins at an angle just less than $\pi/2$ and rotates clockwise under position control, pushing the block to the right. The contact mode of interest is the one maintaining the three contacts shown (the edge-edge contact is modeled as two point contacts between the palm and two corners of the block). For the commanded clockwise finger motion, this system exhibits jamming in some configurations and first-order stability in others. For simplicity, assume that the coefficients of friction, μ_i, at the three contacts are equal. Then, given that the world frame has its origin on the axis of the revolute joint of the stick finger and the external force (in this example, the gravitational force) acts in the $-y$-direction, the relevant wrench and Jacobian matrices

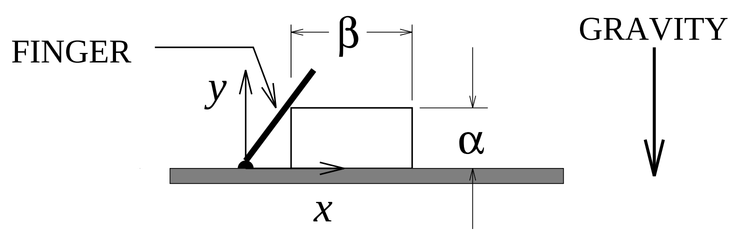

Figure 4: A Planar Manipulation System

are defined as follows:

$$
\begin{array}{ll}
\mathbf{W}_n = \begin{bmatrix} 0 & 0 & sin(\theta) \\ 1 & 1 & -cos(\theta) \\ x & x+\beta & \gamma \end{bmatrix} & \mathbf{J}_n = \begin{bmatrix} 0 \\ 0 \\ -\gamma \end{bmatrix} \\
\mathbf{W}_t = \begin{bmatrix} -1 & -1 & cos(\theta) \\ 0 & 0 & sin(\theta) \\ 0 & 0 & 0 \end{bmatrix} & \mathbf{J}_t = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\
\mathbf{W}_{A\mu} = \mathbf{Q}_I = \begin{bmatrix} -\mu & -\mu & sin(\theta) - \mu cos(\theta) \\ 1 & 1 & -cos(\theta) - \mu sin(\theta) \\ x & x+\beta & -\gamma \end{bmatrix} & \mathbf{J}_{A\mu} = \mathbf{Q}_{II}^T \begin{bmatrix} 0 \\ 0 \\ -\gamma \end{bmatrix} \\
\mathbf{W}_z = \mathbf{Z}_I = \begin{bmatrix} \mu & \mu & sin(\theta) + \mu cos(\theta) \\ 1 & 1 & -cos(\theta) + \mu sin(\theta) \\ x & x+\beta & -\gamma \end{bmatrix} & \mathbf{J}_z = \mathbf{J}_{A\mu} = \mathbf{Z}_{II}^T = \begin{bmatrix} 0 \\ 0 \\ -\gamma \end{bmatrix}
\end{array}
\tag{41}
$$

where $\gamma = \sqrt{x^2 + \alpha^2}$. Assuming $\mathbf{g}_{man} = 0$, the generalized applied force is given as:

$$
\begin{bmatrix} -\mathbf{g}_{obj} \\ \boldsymbol{\tau} - \mathbf{g}_{man} \end{bmatrix} = \begin{bmatrix} 0 \\ mg \\ (x + \beta/2)mg \\ \tau \end{bmatrix} \tag{42}
$$

where m is the mass of the block and g is the magnitude of the gravitational acceleration.

If a contact converts to rolling while the other contacts are maintained, it is clear that motion will cease. Then, since the planned finger motion was clockwise rotation, the error $\mathbf{e}_{II}$ will be negative, leading to $\mathbf{X} = -1$, and because the joint was position-controlled and its ability to accurately execute its planned trajectory has been affected, we have $\mathbf{E}_J = 1$.

To further simply our discussion and the algebra, let $\mu = 0.5$, $\alpha = 1$ and $\beta = 2$. Now, the matrices given above satisfy equations (23), (24), (30) and (31) as long as the left edge of the block is not positioned at $x = 3/4$ or $x = \infty$ (in those configurations, the matrices $\mathbf{W}_{A\mu}$ and $\mathbf{W}_A$ are nonsingular, respectively). Since all contacts are sliding, inequality (25) is irrelevant, but inequality (26) must be satisfied. Also, the system of inequalities (39) and (40) must be infeasible to be sure that jamming is impossible.

Maintaining the chosen contact mode allows the elimination of the finger joint angle from the equations. Then, the applicable wrench intensity vector, $\mathbf{c}_A$, is given as:

$$\mathbf{c}_A = \begin{bmatrix} -\frac{mg(3+x)/4}{2x-3/2} \\ -\frac{5mg(1-x)/4}{2x-3/2} \\ -\frac{mg\sqrt{x^2+1}}{2x-3/2} \end{bmatrix} \tag{43}$$

and the (1×3) jamming matrix $\mathbf{XE}_J(\mathbf{Z}_{II} - \mathbf{Q}_{II}\mathbf{Q}_I^{-1}\mathbf{Z}_I)$ is:

$$\begin{pmatrix} \frac{2(x^2+1)}{2x-3/2} & \frac{2(x^2+1)}{2x-3/2} & \sqrt{x^2+1} - \frac{(2x-5/2)\sqrt{x^2+1}}{2x-3/2} \end{pmatrix}. \tag{44}$$

Note that all three elements of $\mathbf{c}_A$ are positive and all the elements of the jamming matrix are negative when $0 < x < 3/4$. Thus the system is first-order stable over that open interval. When $x > 3/4$, all the elements of the jamming matrix are positive, so jamming is possible and first-order stability is lost. As mentioned earlier, a system's FS-cell can be represented by an n_θ-dimensional set; in this case, $n_\theta = 1$ and the FS-cell projected onto the x-axis is the open interval, $(0, 3/4)$.

Notice that at the point $x = 3/4$ the active edges of the friction cones are all parallel, but that no pair of the cones (or negative cones) see each other as is required for frictional form closure [4, 42].[10] In fact, only the cone on the finger and the cone on the left side of the block can ever see each other, and this first happens as x increases beyond 2.0. However, once x increases beyond 3/4, the system of inequalities, (37) and (38) is infeasible, and one can show that no contact mode other than three rolling contacts is feasible. Therefore the system must jam even though the grasp does not have frictional form closure. According to Omata, the two friction cones on the palm could replaced by a single "equivalent" friction cone [28]). This cone has its apex 2 distance units below the center of the bottom of the block and its edges are colinear with the outer edges of the two individual friction cones. This cone and the cone on the finger do not "see each other" until $x > 1.0$. So jamming still will occur before frictional form closure is achieved by Omata's model. Our explanation for this counter-intuitive result is that previous form closure results do not take the kinematic structure of the grasping mechanism into account, but we do.

The reason for this apparent inconsistency is that Nguyen's results do not take the kinematic structure of the mechanism into account and they are only valid for two point contacts (in planar problems).

6 Conclusion and Future Research

We have introduced the concept of first-order stability cells (FS-cells) for spatial, quasistatic, multi-rigid-body systems with Coulomb friction acting at the contact points. These cells are

[10] Recall that Nguyen refers to frictional form closure [4] as "force closure" [27]. However, we reserve the use of the term "force closure" for situations originally identified by Reuleaux [32].

n_θ-dimensional subsets of C-space in which paths corresponding to whole-arm manipulation tasks which can be executed without jamming. Due to the generality of the systems considered here, our jamming results are more general that previous ones for quasistatic systems. In specific, our results take into account the kinematic structure of the whole-arm manipulator and the joint control modes.

FS-cells have been partitioned into two classes: passive FS-cells and active FS-cells. They are characteristically different in that the system can maintain passive first-order stability simply position controlling the joints, while the maintenance of active first-order stability requires joint compliance, which would typically be achieved through active compliance control. While active compliance control is usually more difficult to implement than position control, its implementation in a whole-arm manipulation system can significantly increase the array of tasks that can be successfully executed. This is partly because some tasks can simply not be executed without compliant control (*e.g.*, manipulation tasks that require the use of highly stable, form-closed grasps).

Assuming that the external forces acting on the system are functions of the system configuration (*e.g.*, gravitational forces), the relationships defining frictionless FS-cells are functions of the C-space variables; $\mathbf{q}$ and $\boldsymbol{\theta}$ for passive FS-cells and $\mathbf{q}$, $\boldsymbol{\theta}$, and $\boldsymbol{\tau}$, for active FS-cells. When Coulomb friction is present, the defining relationships still are functions of $\mathbf{q}$, $\boldsymbol{\theta}$, and $\boldsymbol{\tau}$, but some of them (those ensuring system equilibrium and preventing jamming) are also functions of a subset of the velocity variables (those corresponding to the position-controlled joints). The velocity variables cannot be eliminated, because they reflect the fundamental dependence of the friction forces on the relative velocities at the contacts. Thus, in general, a planner based on the results of this paper must be able to handle nonholonomic constraints.

We have made some progress toward implementing an FS-cell-based planner for planar systems in a parallel/distributed computing environment using the software tools, HeNCE and PVM. Once fully implemented, plans generated will be downloaded to our prototype, planar, whole-arm manipulator. After gaining sufficient experience with this system, we intend to implement our planner for spatial systems.

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