Dynamic Multi-Rigid-Body Systems with Concurrent Distributed Contacts

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Abstract. Consider a system of bodies with multiple concurrent contacts. The multi-rigid-body contact problem is to predict the accelerations of the bodies and the normal and friction loads acting at the contacts. This paper presents theoretical results for the multi-rigid-body contact problem under the assumptions that one or more contacts occur over locally planar, finite regions and friction forces are consistent with the maximum work inequality. We present an existence and uniqueness result for this problem under some mild assumptions on the system inputs. The application of our results to two examples is discussed.

1 Introduction

Multi-body dynamic systems are ubiquitous in our society: motors, engines, and the automation devices used to build portions of these machines are common examples. Where possible, machine designers use joints that provide bilateral kinematic constraints between the connected bodies (*e.g.*, pin joints). In some situations, however, design constraints dictate the use of "joints" which provide only unilateral kinematic constraint (*e.g.*, a cam and follower). In the domain of automated manufacturing, robots are used to position and orient parts for "presentation" to other robots and automated devices for further processing. Under normal operations, robots can position and orient only those parts that are light enough and small enough to be grasped securely and lifted. However, with a solid understanding of contact mechanics, a robot can use pushing operations to reliably position and orient objects that are too heavy to lift and too large to grasp securely [9].

Analyses of multi-body dynamic systems are typically based on the simplifying assumption that the bodies are rigid. Then the Newton-Euler equations, the kinematic equations and inequalities (arising from the bilateral and unilateral constraints, respectively), and a friction model for the contacts are used to formulate a governing system of differential-algebraic equations [4]. We note however, that formulation as a system of *equations* requires prior knowledge of the impending contact state (*i.e.*, rolling, sliding, or breaking at each contact). Once formulated, the equations will have a unique solution if the system Jacobian matrix has full row rank. If there are "too many" contacts, the contact forces cannot be uniquely determined through a rigid body model. One way to resolve this indeterminacy is to incorporate a model of contact compliance [14]. However, this "remedy" suffers from its own set of problems. For example, the original differential-algebraic system becomes a system of *stiff* differential equations whose solution then depends on the contact stiffnesses, which in turn, depend on the global geometries of the parts in contact.

As alluded to above, the differential algebraic system arising from the rigid body assumption cannot be formulated at times when the impending contact state is not known *a priori*. In simulation, the usual approach to predicting the ensuing contact state is to assume it will be the same as the current one. Then after solving the corresponding differential-algebraic system, the solution is checked against a contact model. If the normal force at a contact has become negative, then the simulation is backed up to the time when the component became zero and the system equations are reformulated under the assumption that the corresponding contact has separated. Analogous logic is applied to determine all other possible contact state transitions.

The above approach clearly relies on force history information, which is often absent for the first time step in a simulation. It also depends implicitly upon the assumption that the contact forces are continuous functions of time. This is violated whenever a collision occurs, since the contacts experience impulsive forces at those times. Clearly one cannot base his assumption of the contact state just after a collision on the contact forces immediately before it. In this situation, one would have to resort to other means to "guess" the impending contact state. For example, one could test all possible new contact states and choose one which is consistent [7, 8]. However, there are two problems with this approach. First, the number of possible contact states grows exponentially with the number of contacts, so enumeration becomes impractical quickly as the number of contacts grows. Second, it is possible that no consistent contact mode exists (with finite contact forces).

The difficulties described above suggest that it is important to understand the multi-rigid-body contact problem formulated *without* assuming prior knowledge of the ensuing contact state. This philosophy leads to the appearance of kinematic inequality constraints in the system model, and then naturally to a complementarity problem combining the Newton-Euler equations, the kinematic constraints, and a contact friction model [6, 11, 12] in the unknown accelerations and contact forces. In this paper, we extend previous results for systems having isolated point contacts to those having one or more contacts distributed over finite, locally planar areas [5]. The limit surface formalism which is based on the maximum work inequality [3] is used to model the load-motion behavior at the contacts. Here we would like to stress that the small contact patches are used to allow the transmission of a friction moment along the contact normal. This is *not* to say that rigid bodies in contact would develop finite patches of contact. Rather, it is an acknowledgement of the fact that bodies are not rigid, and that the friction moments transmitted along the contact normals can have a significant affect on their motions.

The contribution of this paper is a set of new existence and uniqueness results that provide strict theoretical guidelines for the use of our model in the analysis of multi-rigid-body dynamic systems. This result subsumes most previous results and generalizes them to accommodate any friction law obeying the Maximum Work Principle [3].

2 The Model

The derivation of our mathematical model describing the motion of a system rigid bodies with locally planar, finite areas of contact is analogous to the model with isolated points contacts developed previously[12]. Due to space restrictions, we will only discuss the aspects of the friction limit surface model.

The mathematical model describing the dynamic, three-dimensional, multi-rigid-body contact problem with point and distributed contacts consists of four sets of equations and inequalities. In order to describe the model, let n_c denote the number of contacts at the time instant for which the model is formulated (*i.e.*, the current time). By definition, the normal components of the relative linear velocities at these contacts, $v_{in}, i = 1, \ldots, n_c$, are all zero; the two orthogonal components of the relative linear velocity in the contact tangent plane are denoted by (v_{it}, v_{io}) (see Figure 1). The relative angular velocity component in the direction of the ith contact normal is v_{ir}. We assume that a distributed contact forms as a result of compressive normal loading between two contacting bodies; we are not concerned with the geometry of contact region. The modeling of this regions will be necessary in some situations, but we do not consider that here.

Let us classify a contact i as *rolling* if $v_{it} = v_{io} = v_{ir} = 0$ and *non-rolling* otherwise. Let $\mathcal{R}$ and $\mathcal{N}$ denote, respectively, the sets of rolling and non-rolling contacts; these two index sets partition $\{1, \ldots, n_c\}$. Let $\mathcal{M}$ and $\mathcal{J}$ denote, respectively, the system inertia and constraint Jacobian matrices; we note that the former is a symmetric positive definite matrix whereas the latter is defined by the contact geometry. Define

$$\boldsymbol{A} \equiv \mathcal{J}^T \mathcal{M}^{-1} \mathcal{J}.$$

Letting n_d and n_p be the numbers of distributed and point contacts, respectively, one can show that $\boldsymbol{A}$ is a $(3n_p + n_d) \times (3n_p + n_d)$ symmetric positive semidefinite matrix; its null space coincides with the null space of $\mathcal{J}$. In particular, $\boldsymbol{A}$ is positive definite if and only if $\mathcal{J}$ has linearly independent columns. The unknown vectors of relative accelerations and contact forces and and moments are denoted by (a_n, a_t, a_o, a_r) and (c_n, c_t, c_o, c_r) respectively. Here each vector subscripted by n, t, or o is of order n_c and refers to the normal or tangential components of the relative linear accelerations or forces at the contacts. The vectors subscripted by r are of order n_d and represent the relative angular accelerations or transmitted moments about the normals of the distributed contacts.

The basic model consists of the following:
(i) the combined kinematic/Newton-Euler equations

of motion,

$$\begin{bmatrix} a_n \\ a_t \\ a_o \\ a_r \end{bmatrix} = A \begin{bmatrix} c_n \\ c_t \\ c_o \\ c_r \end{bmatrix} + \begin{bmatrix} b_n \\ b_t \\ b_o \\ b_r \end{bmatrix},$$

where (b_n, b_t, b_o, b_r) is a constant vector that contains the known external forces applied to the system and velocity product forces;

(ii) the nontensile restrictions on the contact forces, the unilateral kinematic constraints, and the complementarity conditions on the normal contact forces and accelerations,

$$(a_n, c_n) \geq 0, \quad (a_n)^T c_n = 0;$$

(iii) the Coulomb friction limit surface condition suggested by Howe and Cutkosky [5] (based upon a series of contact friction experiments),

$$\frac{c_{it}^2}{e_{it}^2} + \frac{c_{io}^2}{e_{io}^2} + \frac{c_{ir}^2}{e_{ir}^2} \leq \mu_i^2 c_{in}^2, \quad i = 1, \ldots, n_c, \quad (1)$$

where e_{it}, e_{io}, and e_{ir} are given positive constants and μ_i is the coefficient of friction (assumed positive); for an arbitrary scalar $\lambda \geq 0$, let

$$\mathcal{F}_i(\lambda) \equiv \left\{ (c_{it}, c_{io}, c_{ir}) \in R^3 : \frac{c_{it}^2}{e_{it}^2} + \frac{c_{io}^2}{e_{io}^2} + \frac{c_{ir}^2}{e_{ir}^2} \leq \lambda^2 \right\};$$

(iv) the maximum work inequality: for each $i \in \mathcal{N}$,

$$\begin{aligned}
(c_{it}, c_{io}, c_{ir}) \in \quad & \mathrm{argmax}\{-(v_{it} c_{it}' + v_{io} c_{io}' + v_{ir} c_{ir}') : \\
& (c_{it}', c_{io}', c_{ir}') \in \mathcal{F}_i(\mu_i c_{in})\},
\end{aligned}$$

and for each $i \in \mathcal{R}$,

$$\begin{aligned}
(c_{it}, c_{io}, c_{ir}) \in \quad & \mathrm{argmax}\{-(a_{it} c_{it}' + a_{io} c_{io}' + a_{ir} c_{ir}') : \\
& (c_{it}', c_{io}', c_{ir}') \in \mathcal{F}_i(\mu_i c_{in})\},
\end{aligned}$$

where $\mathrm{argmax}\{f(x) : x \in X\}$ denotes the set of optimal solutions of the maximization problem:

$$\text{maximize} \quad f(x) \; : \; x \in X.$$

By introducing Lagrange multipliers and writing down the Karush-Kuhn-Tucker optimality conditions for the above maximation problems, condition (iv) can be replaced by the following equivalent system of equations: for all $i = 1, \ldots, n_c$,

$$\left.\begin{aligned}
e_{it}^2 \mu_i c_{jn} \sigma_{it} + \sqrt{e_{it}^2 \sigma_{it}^2 + e_{io}^2 \sigma_{io}^2 + e_{ir}^2 \sigma_{ir}^2}\, c_{it} = 0 \\
e_{io}^2 \mu_i c_{in} \sigma_{io} + \sqrt{e_{it}^2 \sigma_{it}^2 + e_{io}^2 \sigma_{io}^2 + e_{ir}^2 \sigma_{ir}^2}\, c_{io} = 0 \\
e_{ir}^2 \mu_i c_{in} \sigma_{ir} + \sqrt{e_{it}^2 \sigma_{it}^2 + e_{io}^2 \sigma_{io}^2 + e_{ir}^2 \sigma_{ir}^2}\, c_{ir} = 0
\end{aligned}\right\}$$

where

$$(\sigma_{it}, \sigma_{io}, \sigma_{ir}) = \begin{cases} (v_{it}, v_{io}, v_{ir}) & \text{if } i \in \mathcal{N}, \\ (a_{it}, a_{io}, a_{ir}) & \text{if } i \in \mathcal{R}. \end{cases}$$

In order to handle other kinds of Coulomb friction laws, we introduce a generalized model in which we replace the quadratic friction cone defined by (1) by an abstract closed convex cone and modify the maximum work inequality accordingly. Specifically, for each $i = 1, \ldots, n_c$, let $\mathcal{F}_i : R_+ \to R^3$ be a set-valued map with the property that for each scalar $\sigma \geq 0$, the image $\mathcal{F}_i(\lambda)$ is a closed convex cone in the 3-dimensional Euclidean space R^3 and that $\mathcal{F}_i(0) = \{0\}$. The latter property of $\mathcal{F}_i$ stipulates that at each contact, if the normal force is zero, then so is the friction force and the transmitted moment. In terms of these abstract friction maps $\mathcal{F}_i$, the generalized dynamic multi-rigid-body problem with concurrent distributed frictional contacts is to find contact forces $(c_{in}, c_{it}, c_{io}, c_{ir})$ and accelerations $(a_{in}, a_{it}, a_{io}, a_{ir})$ satisfying conditions (i), (ii), and (iv). (Note: (iv) implies that (c_{it}, c_{io}, c_{ir}) belongs to $\mathcal{F}_i(\mu_i c_{in})$.)

Examples of $\mathcal{F}_i(\lambda)$ include (a) the quadratic cone defined by (1); (b) approximations of such a cone by a convex polyhedron:

$$\begin{aligned}
\mathcal{F}_i(\lambda) \equiv \{ (c_{it}, c_{io}, c_{ir}) \in R^3 : \\
\alpha_{ij} c_{it} + \beta_{ij} c_{io} + \gamma_{ij} c_{ir} \leq \lambda, \; j = 1, \ldots, m_i \},
\end{aligned}$$

where α_{ij}, β_{ij} and γ_{ij} are some given scalars and m_i is a positive integer; and (c) mixtures of elliptic and polyhedral friction constraints: *e.g.*, $\mathcal{F}_i(\lambda)$ is given by

$$\left\{ (c_{it}, c_{io}, c_{ir}) \in R^3 : \frac{c_{it}^2}{e_{it}^2} + \frac{c_{io}^2}{e_{io}^2} \leq \lambda^2, |c_{ir}| \leq \lambda \right\}.$$

For planar problems, we can let

$$\mathcal{F}_i(\lambda) \equiv \{ (c_{it}, 0, 0) \in R^3 : |c_{it}| \leq \lambda \}.$$

Examples (a) and (c) pertain to axi-symmetric friction laws; whereas (b) do not necessary correspond to such laws. Other axi-asymmetric friction laws can also be modeled by using the friction map $\mathcal{F}_i$.

3 Existence/Uniqueness of Solutions

Employing a unified approach, we provide sufficient conditions for the existence and uniqueness of solutions to the basic model presented in the last section. Similar results can be established for variations of this model, such as those based on the abstract friction maps $\mathcal{F}_i$. Due to space limitation, we will focus our

discussion on the basic model under the Coulomb friction limit surfaces.

Let $\mathcal{F}$ consist of all force tuples (c_n, c_t, c_o, c_r) such that $c_n \geq 0$ and for all $i \in \mathcal{N}$,

$$\left.\begin{array}{l} e_{it}^2 \mu_i c_{in} v_{it} + \sqrt{e_{it}^2 v_{it}^2 + e_{io}^2 v_{io}^2 + e_{ir}^2 v_{ir}^2}\; c_{it} = 0 \\[2mm] e_{io}^2 \mu_i c_{in} v_{io} + \sqrt{e_{it}^2 v_{it}^2 + e_{io}^2 v_{io}^2 + e_{ir}^2 v_{ir}^2}\; c_{io} = 0 \\[2mm] e_{ir}^2 \mu_i c_{in} v_{ir} + \sqrt{e_{it}^2 v_{it}^2 + e_{io}^2 v_{io}^2 + e_{ir}^2 v_{ir}^2}\; c_{ir} = 0 \end{array}\right\},$$

and

$$\frac{c_{it}^2}{e_{it}^2} + \frac{c_{io}^2}{e_{io}^2} + \frac{c_{ir}^2}{e_{ir}^2} \leq \mu_i^2 c_{in}^2, \quad \forall i \in \mathcal{R}.$$

Let

$$\mathcal{F}_{\mathcal{J}} \equiv \mathcal{F} \cap \text{ null space of } \mathcal{J}.$$

The main result of this paper is summarized in the following theorem.

Theorem 1 *Let $A \equiv \mathcal{J}^T \mathcal{M}^{-1} \mathcal{J}$ with $\mathcal{M}$ being symmetric positive definite.*

(A) If A is positive definite, then there exists a scalar friction bound $\bar{\mu} > 0$ such that whenever $\mu_i \in [0, \bar{\mu}]$ for all $i \in \mathcal{N}$, the rigid-body contact model defined by conditions (i)–(iv) has a solution. If in addition $\mu_i \in [0, \bar{\mu}]$ for all $i \in \mathcal{R}$, then the solution is unique.

(B) If $\mathcal{N} = \emptyset$, and

$$\begin{bmatrix} b_n \\ b_t \\ b_o \\ b_r \end{bmatrix}^T \begin{bmatrix} c_n \\ c_t \\ c_o \\ c_r \end{bmatrix} \geq 0 \quad \text{for all} \quad \begin{bmatrix} c_n \\ c_t \\ c_o \\ c_r \end{bmatrix} \in \mathcal{F}_{\mathcal{J}},$$

then for any positive $\{\mu_i : i = 1, \ldots, n_c\}$, the first conclusion of (A) holds.

The conditions in the two statements (A) and (B) of the theorem are different. The conditions in (A) require the entire matrix A be positive definite and the friction coefficients at the non-rolling contacts be small; in this case if the friction coefficients at the rolling contacts are also sufficiently small, then the solution must be unique. A theoretical estimate for the friction bound $\bar{\mu}$ can be computed as discussed in [12]. Such an estimate tends to be very conservative and can be expected to be much smaller than one would expect to encounter in real systems.

Part (B) pertains to the all-rolling case; in this case, there is no condition imposed on the friction coefficients; also A is not required to be positive definite. The proofs of Theorem 1 are straight forward extensions of those in the papers [10, 12]; background results needed in the proofs are in [1, 2].

4   Examples

Two example problems were formulated and solved for various parameter values. These problems are based on the system depicted in Figure 2. There are two moveable bodies initially at rest: a uniform cube of side 2 with six (passive) degrees of freedom and a rod with two (active) degrees of freedom and a second rod fixed in space. The active rod can translate along and rotate about its axis, which is parallel to the x-axis of the inertial frame. We also assume that the rod's actuators can apply a force τ_1 and moment τ_2 along the rod. The cube contacts the environment at three locally planar, distributed contacts: first, the front face of the cube contacts the active rod, second, the bottom face contacts a fixed, slightly curved, convex surface, and third, the right vertical face contacts the fixed rod. These examples can be viewed as a robot manipulating a box in contact with the floor.

4.1   Example 1

In the first example, the third contact (with the fixed rod), is assumed to be absent; initially, there are two contacts. The contact between the cube and the active rod is at the position (η, ζ) on the front face, and the contact between the hump and the bottom face of the cube is at the face's geometric center (directly below the center of gravity). The simple geometry and initial conditions were chosen so that the solutions obtained could be easily checked against our intuition. For example, if the rod does not push against the cube (i.e., $\tau_1 \leq 0$), then the cube will not accelerate.

Since the system has eight degrees of freedom and there are four unknown contact force components at each contact, the system Jacobian matrix[1] $\mathcal{J}$ is 8×8 and given by:

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -\zeta+1 & 1 & -\eta+1 & 0 & -1 & 0 \\ -\zeta+1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ \eta-1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

where the columns (in order from left to right) correspond to contact forces in the $\hat{n}_1$, $\hat{n}_2$, $\hat{t}_1$, $\hat{t}_2$, $\hat{o}_1$, and $\hat{o}_1$ directions, and contact moments in the $\hat{n}_1$, and $\hat{n}_2$ directions. The last two rows of the matrix are the manipulator Jacobian (of the movable rod).

It can be shown that the determinant of $\mathcal{J}$ is given by $-\zeta$. Geometrically, this means that as long as the

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[1] This Jacobian is expressed in the coordinate frame with the same orientation as the inertial frame, but with its origin at the center of mass of the cube.

contact between the rod and the cube is not along the cube's bottom edge (of the front face), the system Jacobian will be nonsingular and the matrix $\boldsymbol{A}$ will be positive definite.

The nonlinear complementarity problem corresponding to the rod-cube system was formulated following the approach laid out in [12] and this paper. We assumed that the system mass matrix was $\mathrm{diag}(1,1,1,3,3,3,4,2)$, $e_{it} = e_{io} = 1 \quad \forall i$, $e_{1r} = 1$, $e_{2r} = 0.25$, $\tau_1 = 5$, $\tau_2 = 3$, and $\eta = 1.5$ The value of ζ and the coefficients of friction were varied to generate data satisfying the two parts of Theorem 1. Solutions were determined using an interior point algorithm developed by Wang, Monteiro, and Pang [13], and implemented as a script file (an M-file) in Matlab installed on a 486/40 IBM compatible pc. On average, the algorithm found a solution for a given data set in about 2 seconds. Since script files are "interpreted" by Matlab, it would not be difficult to reduce the solution time by a factor of 100 with careful coding.

Four hundred data sets were generated using the numbers given above, setting $\zeta = 2$, while μ_1 and μ_2 was varied (each in 20 steps) over the range $[0.1, 1.0]$. Thus, for all of these data sets, the system Jacobian was full rank. Since the contacts were initially rolling, part (A) of Theorem 1 implies that a solution exists for any values of the coefficients of friction. The algorithm converged to a valid solution for every pair of friction coefficients, and the solution was used to compute the magnitude of the tangential acceleration of contact 2 (under the cube). This acceleration is plotted against the coefficients of friction in Figure 3. Notice that the solutions appeal to our intuition; when the coefficient of friction at contact 2 is small, sliding initiates.

Figure 3 shows the magnitude of the tangential acceleration of contact 2 when contact 1 (between the rod and cube) was moved down to the bottom edge of the front face, $i.e.$, $\zeta = 0$. All other data values were retained. With the rod contact in this position, the system Jacobian matrix was singular. However, because the contacts were initially rolling and the bodies began at rest, one can show that part (B) of Theorem 1 would be satisfied for any values of the friction coefficients. Again, a solution was found for every pair of friction coefficients, and those solutions were intuitively appealing.

4.2 Example 2

Because the geometry of the first example was so simple, the contact with the hump was moved to the extreme back, left corner of the bottom face, and a third contact was made with the fixed rod on the right face of the cube. This led to the (8×12) system Jacobian matrix $\mathcal{J}$ in display (2): the first 3 columns of

$\mathcal{J}$ correspond to forces in the $\hat{\boldsymbol{n}}$ directions, the second three to the $\hat{\boldsymbol{t}}$ directions and so on; and the last 2 rows are again the Jacobian of the movable rod.

In studying this problem, we searched for multiple qualitatively distinct solutions for a single data set. In other words, given the initial conditions, input torque and force, $etc.$, we searched for problem solutions with different final contact states. To find such solutions, each of several data sets was solved several times using different controlling parameters and initial iterates in the Wang-Monteiro-Pang algorithm. We also systematically varied the coefficient of friction at the contacts, because complementarity theory suggests that multiple solutions are more likely as the coefficient of friction increases. Multiple solutions were found for several data sets with large coefficients of friction (near 1.0). The first data set we found with multiple solutions is as follows: $\mathcal{M} = \mathrm{diag}(1,1,1,3,3,3,4,2)$, $\mu_i = 0.9 \quad \forall i$, $e_{it} = e_{io} = 1 \quad \forall i$, $e_{ir} = 2.0 \quad \forall i$, $\tau_1 = 6$, $\tau_2 = 3$, $\eta = \zeta = \gamma = 1.5$, and $\delta = 0.5$. The bodies were initially at rest, and thus all contacts were considered to be initially rolling. The two solutions found are shown in Table 1.

$$
\mathcal{J} = \left[
\begin{array}{cccccc}
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 \\
0 & -1 & \gamma-1 & -\zeta+1 & 1 & -1 \\
-\zeta+1 & 1 & 0 & 0 & 0 & 1-\delta \\
\eta-1 & 0 & \delta-1 & 1 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{array}
\right.
$$

$$
\left.
\begin{array}{cccccc}
0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
-\eta+1 & 0 & 0 & -1 & 0 & 0 \\
1 & 1 & \gamma-1 & 0 & 0 & -1 \\
0 & -1 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{array}
\right] \tag{2}
$$

Table 1 is organized as follows. The left-hand column contains the names of the unknowns. Above the triple horizontal lines dividing the Table are the values of the variables which have one element per contact point. Below are the values of the accelerations of the bodies. The acceleration vector $\ddot{\boldsymbol{q}}$ is partitioned into linear and angular acceleration subvectors, $\ddot{\boldsymbol{q}}_{\mathrm{linear}}$ and $\ddot{\boldsymbol{q}}_{\mathrm{angular}}$, respectively.

From the two solutions displayed in Table 1, one may infer that the block and rod system may jam (Solution 1) or move with all contacts sliding (Solution 2). Note that jamming (in the first solution) is indicted by the fact that $\ddot{\boldsymbol{q}} = \ddot{\boldsymbol{\theta}} = \boldsymbol{0}$. The jamming

interpretation is further corroborated by all components of the contact accelerations being zero and the positive values of the variable "slack." This variable is simply the difference between the left- and right-hand sides of equation (1). It is not a variable of the model *per se*, rather it is a useful indicator of whether or not a given contact force is on the boundary of its friction cone. For the jammed case, the slacks are all positive, indicating that the contact forces are inside their friction cones, and hence the contacts are sticking.

The second solution implies sliding at all contacts. Notice that $a_n = 0$, which means that the contacts are maintained. The difference in the character of this solution is evident in the elements of "slack" and the acceleration values. Since the values of "slack" are all zero, every contact could begin to slide (Coulomb's Law is ambiguous on this point). However, sliding is clearly indicated by the nonzero values of the contact acceleration vectors, a_t, a_o, and a_r.

5 Conclusion

We have formulated the dynamic equations of a general, spatial, multi-rigid-body system with multiple distributed contacts as a complementarity problem, and provided two sufficient conditions for solution existence and uniqueness. The first condition guaranteeing solution existence requires linear independence of the columns of the system Jacobian and constrains the maximum coefficient of friction at the non-rolling contacts. If the coefficients of friction at the rolling contacts are also small, then the solution is unique. The second condition guaranteeing existence pertains to problems in which all contacts are initially rolling (without twisting). It is important to note that this condition does *not* restrict the coefficients of friction.

The ultimate goal of this work is to develop efficient algorithms for solving the multi-rigid-body contact problem. Enumerative algorithms exist for solving for general complementarity problems that include our formulation as a special case, but we anticipate that these algorithms will not be able to compete with algorithms specialized to our problem. In animation applications, a specialized algorithm is likely to perform even better, if we exploit solution coherence over time (to allow "hot starts").

unknowns	Solution 1: Jamming		
	contact 1	contact 2	contact 3
c_n	6	4.2039	3.9117
c_t	1.5767	2.3350	-1.6069
c_o	-4.1892	-2.9716	3.0284
c_r	3	-0.2292	0.6971
slack	6.8	0.019	0.52
a_n	0	0	0
a_t	0	0	0
a_o	0	0	0
a_r	0	0	0
$\ddot{q}_\text{linear}$	0	0	0
$\ddot{q}_\text{angular}$	0	0	0
θ	0	0	
unknowns	Solution 2: Sliding		
	contact 1	contact 2	contact 3
c_n	5.6399	4.1562	1.7016
c_t	-0.8763	2.2585	-0.6783
c_o	-4.8224	-2.9756	1.3717
c_r	2.6394	-0.3840	-0.1196
slack	0	0	0
a_n	0	0	0
a_t	0.0815	-0.9582	0.7526
a_o	0.4483	1.2624	-1.5219
a_r	-0.2454	0.1629	0.1327
$\ddot{q}_\text{linear}$	-1.2926	-0.3194	-0.3431
$\ddot{q}_\text{angular}$	-0.4759	-0.1327	0.1629
θ	1.4404	0.7212	

Table 1: *For a single data set, the multi-rigid-body contact problem can have multiple solutions. For the problem described in Example 2, at least two qualitatively different solutions exist when the coefficient of friction is taken to be 0.9 at all contacts: jamming and sliding.*

There remain a number of open questions. We have not yet determined an efficient procedure for determining the friction bound, $\bar{\mu}$. More generally, an algorithm to determine solution uniqueness is desirable (in some situations) as a means for delineating the domain of applicability of the multi-rigid-body model. Last, there is a need to develop existence and uniqueness conditions for general situations characterized by a system Jacobian without full column rank.

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