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Planar Quasi-Static Motion of a Lamina with Uncertain Contact Friction

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Abstract

Planning the motion of bodies in contact requires a model of contact mechanics in order to predict sliding, rolling, and jamming. Such a model typically includes estimates of the coefficients of friction. Though, usually assumed to be constant, these coefficients vary and are difficult to measure. In this paper, we treat the coefficients of friction as independent variables and derive regions in the space of friction coefficients that correspond to particular modes of motion; three sliding contacts or one sliding and one rolling contact, all other contacts separating.

1 Introduction

The planning of dexterous manipulation and assembly by robotic systems involves predicting the motions of systems of bodies in contact, where the positions and orientations of some bodies are actively controlled while others move only in response to the motion of the controlled bodies. The goal of planning is to determine a sequence of manipulator motion commands which, if executed, would achieve a prespecified relative arrangement of the bodies, *e.g.*, a new grasp or completed assembly. However, because of the large computational demands of planning geometrically valid trajectories, the physical models of body interactions are usually simplified, so that the computational cost is not greatly increased.

The most commonly made simplifying assumptions are that the bodies are rigid, that "dry" or Coulomb friction acts at the contact points, and that the system's motion is quasi-static. The quasi-static assumption is applicable when dynamic effects are negligible [9] and implies that the system's equilibrium equations are satisfied at all times. In this case, the accuracy of motion prediction is tied to the quality of the estimates of the coefficients of friction, the parameters describing the body's geometry, the contact positions, the contact normals, the external wrench applied to the body, and the motion of the controlled bodies. However, because these parameters can never be measured exactly, many researchers have studied motion planning problems under parameter uncertainty (*e.g.*, Mason[9], Brost[2], Erdmann[5], Caine[4], Buckley[3], Lozano-Perez *et al.* [8]).

It has been shown that the difficulty of one of the most important assembly tasks, peg-in-hole insertions, is greatly increased with increasing friction [14] [12] [16] [4]. It is also known that the effective coefficients of friction depend strongly on microscopic details of the contacting surfaces and on foreign fluids and particles in the contact interfaces. Thus the coefficients can vary quickly through relatively large excursions during manipulation [11]. Despite these facts, the open literature does not contain papers directly addressing the effects of the variations of the coefficients of friction on the quasi-static motion of systems of rigid bodies in contact. The purpose of this paper is to study those effects for the case of a single passive body moving in response to frictional contact with a number of moving point bodies.

1.1 Relation to Previous Work

The most closely related line of research was initiated by Mason[9] in the 1980's, who studied sliding friction during general planar motion of a rigid body. His work was motivated by difficulties in executing manipulation and assembly tasks with robot manipulators. Mason primarily was concerned with the prediction of the instantaneous velocity of a pushed object moving quasi-statically on a supporting plane. His results were based on the fact that the "center of friction" was computable even though the supporting force distribution was unknown. Mason[9], Peshkin[13], and others have used the results with success in parts orienting experiments with single "pushers" with one or more points of contact.

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Also relevant to the present work are previous quasi-static analyses of the "peg-in-hole" problem. The most thorough study of the effect of variations in the coefficient of friction was performed by Whitney[16]. He reformulated Simunovic's work [14], using small angle approximations to derive closed-form solutions for reaction forces and moments occurring during peg insertions. These solutions were presented in [16], as families of plots parameterized by the coefficient of friction, thus showing how the insertion force increased with increasing friction.

In this paper, we study the planar quasi-static motion of an arbitrary rigid lamina in frictional contact with any number of moving rigid point objects. Our goal is to show how the instantaneous velocity of the lamina depends on the coefficients of friction by partitioning the space of friction coefficients into cells such that within each cell the motion of the lamina is governed by a single set of kinematic constraints.

2 Problem Statement

An arbitrary rigid lamina moves quasi-statically in a plane due to frictional contact with n_c moving rigid point bodies. The exact positions, contact normals, and instantaneous velocities of the point bodies and the position and orientation of the lamina are known. Given the external force acting on the lamina, our purpose is to determine the instantaneous velocity of the lamina and the contact forces.

It is convenient to define a "world" frame, chosen arbitrarily, and several "contact" frames. One contact frame is assigned to each contact point and is positioned with its origin at the contact point and with its "n"-axis, $\hat{\mathbf{n}}_i$, aligned with the contact normal (pointing inward with respect to the lamina's surface), its "t"-axis, $\hat{\mathbf{t}}_i$, aligned with the contact tangent such that the cross product of $\hat{\mathbf{n}}_i$ and $\hat{\mathbf{t}}_i$, points out of the plane of motion (see Figure 1).

Let the vector $\mathbf{c}_i = [c_{in}, c_{it}]^T$ represent the force at the i^{th} contact such that c_{in} and c_{it} are the normal and tangential components, where ^{T} is the matrix transpose operator. The vector $\mathbf{c}_i$ is known as the individual wrench intensity vector of the i^{th} contact. To write the equations of equilibrium, we transform the contact forces into the world frame by premultiplying each wrench intensity vector by its corresponding wrench matrix, $\mathbf{W}_i$, which is defined as follows

$$\mathbf{W}_i = [\mathbf{w}_{in} \;\; \mathbf{w}_{it}] = \begin{bmatrix} \hat{\mathbf{n}}_i & \hat{\mathbf{t}}_i \\ \mathbf{r}_i \otimes \hat{\mathbf{n}}_i & \mathbf{r}_i \otimes \hat{\mathbf{t}}_i \end{bmatrix}_{(3\times2)}, \quad (1)$$

where $\hat{\mathbf{n}}_i$, $\hat{\mathbf{t}}_i$, $\mathbf{r}_i$ are all expressed in the world coordinate frame, $\mathbf{r}_i$ is the position of the i^{th} contact point, the $\otimes$ operator applied to two vectors, $[a_1, a_2] \otimes [b_1, b_2]$ is defined as $a_1b_2 - a_2b_1$, and the subscript $_{(3\times2)}$ indicates the dimension of the matrix. Summing all forces and moments in the world frame yields the equations of static equilibrium

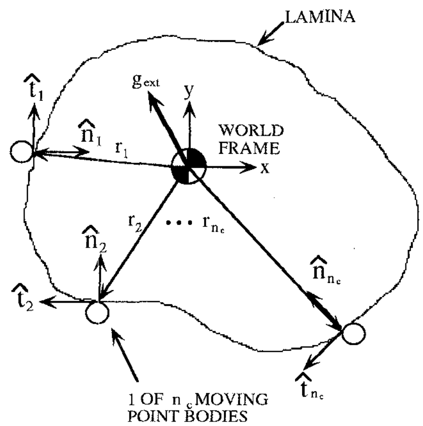

Figure 1: Lamina in Contact with Moving Points.

$$\sum_{i=1}^{n_c} \mathbf{W}_i \mathbf{c}_i + \mathbf{g}_{ext} = \mathbf{0} \quad (2)$$

where $\mathbf{g}_{ext}$ is the external wrench (*i.e.*, force and moment) applied to the lamina. Casting equation (2) into matrix form yields

$$\mathbf{W}\mathbf{c} + \mathbf{g}_{ext} = \mathbf{W}_n\mathbf{c}_n + \mathbf{W}_t\mathbf{c}_t + \mathbf{g}_{ext} = \mathbf{0} \quad (3)$$

where $\mathbf{W}$ and $\mathbf{c}$ are known as the global wrench matrix and the global wrench intensity vector and have dimensions $(3 \times 2n_c)$ and $(2n_c \times 1)$, respectively. The normal and tangential wrench matrices, $\mathbf{W}_n$ and $\mathbf{W}_t$, both of dimension $(3 \times n_c)$, are formed by the horizontal concatenation of all the individual normal and tangential contact wrenches $\mathbf{w}_{in}$ and $\mathbf{w}_{it}$. Correspondingly, the normal and tangential wrench intensity vectors, $\mathbf{c}_n$ and $\mathbf{c}_t$, both have length n_c and are formed by the vertical concatenation of all the normal and tangential wrench intensity components, c_{in} and c_{it}.

The motion of the lamina is also subject to kinematic velocity constraints, the satisfaction of which ensures that the point bodies do not penetrate the lamina's surface. Denoting the relative linear velocity at the i^{th} contact expressed with respect to the i^{th} contact frame as $\mathbf{v}_i = [v_{in}, v_{it}]^T$, then the nonpenetration constraint is given by the following inequality

$$v_{in} \geq 0; \; \forall i. \quad (4)$$

If we let $\dot{\mathbf{q}} = [\dot{q}_x, \dot{q}_y, \dot{q}_\theta]^T$ represent the linear and angular velocity of the point on the lamina coincident with the origin of the world frame, then $\mathbf{W}_i^T\dot{\mathbf{q}}$ is the linear velocity of the i^{th} contact point on the lamina [6]. Assuming that the bodies in contact with the lamina are points on manipulators, then the linear velocity of the i^{th} point body may be written as the product of the Jacobian matrix (excluding the rows corresponding to rotational velocity) associated with the point, $\mathbf{J}_i = \begin{bmatrix} \mathbf{j}_{in} \\ \mathbf{j}_{it} \end{bmatrix}_{(2\times n_\theta)}$, and the joint velocity vector, $\dot{\theta}$ of length n_θ, the number of joints in the manipulators. Thus the relative linear velocity at the i^{th} contact is given by

$$\mathbf{W}_i^T\dot{\mathbf{q}} - \mathbf{J}_i\dot{\theta} = \mathbf{v}_i \ . \tag{5}$$

Casting constraints (4) in matrix form yields

$$\mathbf{W}_n^T\dot{\mathbf{q}} - \mathbf{J}_n\dot{\theta} \geq \mathbf{0} \tag{6}$$

where $\mathbf{J}_n$ was formed by vertically concatenating the rows, $\mathbf{j}_{in}$, of the individual contact Jacobian matrices $\mathbf{J}_i$. The remaining constraints enforce the Coulomb friction model, which for the planar case may be written as a system of linear inequalities as follows

$$\mathbf{B}\mathbf{c} \geq \mathbf{0} \tag{7}$$

where

$$\mathbf{B} = \begin{bmatrix} \mathbf{U} & \mathbf{I} \\ \mathbf{U} & -\mathbf{I} \end{bmatrix}_{(2n_c\times 2n_c)}, \quad \mathbf{c} = \begin{bmatrix} \mathbf{c}_n \\ \mathbf{c}_t \end{bmatrix}_{(2n_c\times 1)}, \tag{8}$$

$\mathbf{U} = diag\{\mu_1, \ldots, \mu_{n_c}\}$ is the diagonal matrix of effective coefficients of friction, and $\mathbf{I}$ is the $(n_c \times n_c)$ identity matrix. To complete the contact model, we also require the following complementary constraints which define the three possible types of contact interactions:

Rolling contact: $\forall i \in \mathcal{R}$

$$v_{in} = v_{it} = 0, \quad c_{in} \geq 0, \quad -\mu_i c_{in} \leq c_{it} \leq \mu_i c_{in}, \tag{9}$$

Sliding contact: $\forall i \in \mathcal{S}$

$$v_{in} = 0, \ c_{in} \geq 0, \quad \left\{ \begin{array}{ll} v_{it} < 0, & c_{it} = \mu_i c_{in} \\ v_{it} > 0, & c_{it} = -\mu_i c_{in} \end{array} \right\}, \tag{10}$$

Breaking contact: $\forall i \in \mathcal{B}$

$$v_{in} > 0, \quad c_{in} = 0, \ c_{it} = 0, \tag{11}$$

where the disjoint sets $\mathcal{R}$, $\mathcal{S}$, and $\mathcal{B}$ contain the indices of the contacts assumed to be rolling, sliding, and breaking, respectively.

3 Solution Approach

To find the solution(s), to the quasi-static motion relationships given above, we must consider every possible combination of assumptions of rolling, sliding, and breaking at the contacts. This gives rise to 3^{n_c} possible *modes of motion* and their corresponding systems of equations and inequalities. To facilitate our analysis, we introduce the selection matrices, $\mathbf{E}_R$, $\mathbf{E}_S$, and $\mathbf{E}_B$, which identify the currently considered motion mode. Letting n_R be the number of rolling contacts and $\mathbf{e}_i^T$ be the row vector of length n_c with i^{th} element equal to one; all others zero. Then $\mathbf{E}_R$ is the $n_R \times n_c$ matrix formed by the vertical concatenation of one row vector, $\mathbf{e}_i^T$, for each $i \in \mathcal{R}$. The matrices $\mathbf{E}_S$ and $\mathbf{E}_B$ and the numbers n_S and n_B are analogously defined.

Given an hypothesized set of contact interactions, the corresponding set of kinematic requirements can be written as follows

$$\mathbf{W}_A^T\dot{\mathbf{q}} \ = \mathbf{J}_A\dot{\theta} \tag{12}$$

where the active wrench and Jacobian matrices are defined as follows

$$\mathbf{W}_A^T = \begin{bmatrix} \mathbf{W}_{nR}^T \\ \mathbf{W}_{tR}^T \\ \mathbf{W}_{nS}^T \end{bmatrix}_{((2n_R+n_S)\times 3)}, \quad \mathbf{J}_A = \begin{bmatrix} \mathbf{J}_{nR} \\ \mathbf{J}_{tR} \\ \mathbf{J}_{nS} \end{bmatrix}_{((2n_R+n_S)\times n_\theta)}, \tag{13}$$

$\mathbf{W}_{\alpha\beta}^T = \mathbf{E}_\beta\mathbf{W}_\alpha^T$, and $\mathbf{J}_{\alpha\beta} = \mathbf{E}_\beta\mathbf{J}_\alpha$, with $\alpha \in \{n, t\}$ and $\beta \in \{R, S, B\}$. Additionally, the contacts assumed to be breaking must satisfy the following inequality

$$\mathbf{W}_{nB}^T\dot{\mathbf{q}} > \mathbf{J}_{nB}\dot{\theta}. \tag{14}$$

The wrench intensity vectors of the sliding contacts are known to lie along the edges of their respective friction cones. To write these constraints in matrix form, we define the diagonal *tangential directions matrix*, Ξ, as follows

$$\Xi = diag\{\xi_1, \ldots, \xi_{n_c}\} \tag{15}$$

where $\xi_i = sgn(\mathbf{w}_{it}^T\dot{\mathbf{q}} - \mathbf{j}_{it}^T\dot{\theta})$ and $sgn()$ is the signum function. Given these definitions, the tangential wrench intensity vector for the sliding contacts is given by

$$\mathbf{c}_{tS} = -\mathbf{U}_S\Xi_S\mathbf{c}_{nS} \tag{16}$$

where $\mathbf{c}_{\alpha\beta} = \mathbf{E}_\beta\mathbf{c}_\alpha$, $\mathbf{U}_S = \mathbf{E}_\beta\mathbf{U}\mathbf{E}_\beta^T$ and $\Xi_S = \mathbf{E}_\beta\Xi\mathbf{E}_\beta^T$, with $\alpha \in \{n, t\}$ and $\beta \in \{R, S, B\}$. Substituting into the equilibrium equation (3) yields

$$\mathbf{W}_{A\mu}\mathbf{c}_A = [\mathbf{W}_A + \mathbf{W}_\mu]\mathbf{c}_A = -\mathbf{g}_{ext} \tag{17}$$

where $\mathbf{W}_\mu = -\mathbf{W}_{tS}\mathbf{U}_S\Xi_S$. The applicable wrench intensity vector, $\mathbf{c}_A$, is defined as follows

$$\mathbf{c}_A = \begin{bmatrix} \mathbf{c}_{nR} \\ \mathbf{c}_{tR} \\ \mathbf{c}_{nS} \end{bmatrix} \tag{18}$$

where $\mathbf{c}_{nR}$ and $\mathbf{c}_{tR}$ are the rolling normal and tangential wrench intensity vectors, respectively, and $\mathbf{c}_{nS}$ is the sliding normal wrench intensity vector.

To satisfy the Coulomb model and to guarantee that all contact forces are compressive, the wrench intensities must also satisfy the following system of inequalities

$$\mathbf{B}_A \mathbf{c}_A \geq \mathbf{0} \tag{19}$$

where $\mathbf{B}_A$ is defined as follow

$$\mathbf{B}_A = \begin{bmatrix} \mathbf{U}_R & \mathbf{I}_R & \mathbf{0} \\ \mathbf{U}_R & -\mathbf{I}_R & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_S \end{bmatrix}_{((2n_R+n_S)\times(2n_R+n_S))}, \tag{20}$$

where $\mathbf{I}_R$ and $\mathbf{I}_S$ are the n_R- and n_S-dimensional identity matrices, respectively.

The final undiscussed quantity is $\mathbf{g}_{ext}$. It represents all external forces from sources other than contact with the moving point bodies and is assumed to be known.

To predict the quasi-static velocity of the lamina, one must find all vectors, $\dot{\mathbf{q}}$ and $\mathbf{c}$, and coefficients of friction, μ_i; $i \in \{1, \ldots, n_c\}$ which satisfy the kinematic constraints and render the Coulomb and equilibrium constraints feasible. If the coefficients of friction were known, then we could use the following straight forward approach. Given the configuration of the system, $\mathbf{g}_{ext}$, and $\dot{\theta}$: first, hypothesize a mode of motion; second, solve the applicable kinematic equations (12) for $\dot{\mathbf{q}}$; third, substitute $\dot{\mathbf{q}}$ into (14) to ensure that presumed breaking contacts will actually break; fourth, solve the applicable equilibrium equations, (17), for $\mathbf{c}_A$; and fifth, substitute $\mathbf{c}_A$ into inequality (19) to ensure that the Coulomb friction constraints are satisfied. If all constraints are satisfied, then the hypothesized mode of motion is feasible, so the sliding tangential wrench intensity vector, $\mathbf{c}_{tS}$, can be computed using equation (16).

Since we assume that the coefficients of friction at the contact points are unknown, the above approach must be slightly modified. Since, the kinematic constraints do not depend on the coefficients of friction, the first three steps above need not be altered. However, in the fourth step, we must solve for $\mathbf{c}_A$ analytically and then substitute the result into the applicable Coulomb friction constraints (19) to yield a set of inequalities in the unknown coefficients of friction.

In this paper, we consider only the situations for which $\mathbf{W}_A$ is nonsingular, placing no constraints on $\dot{\theta}$. We further assume that the presumed breaking contacts satisfy the required constraints (14). Cases for which $\mathbf{W}_A$ is singular are discussed in [17].

4 Exact Decomposition of μ-space

Given that the dimension of $\mathbf{W}_A$ is $(3 \times (2n_R + n_S))$ and our restriction that $\mathbf{W}_A^{-1}$ must exist, the modes of motion considered here must have either three sliding contacts with all other contacts breaking or one sliding and one rolling contact, all others breaking. We denote these modes of motion by $3S$ and RS. Note that there are $\binom{n_c}{3}$ and $n_c(n_c - 1)$ distinct modes of types $3S$ and RS, respectively.

4.1 Regions in μ-Space for $3S$

Since for the $3S$ modes of motion, all maintained contacts are sliding, the equilibrium equation, (17), and Coulomb constraints, (19), simplify to yield

$$(\mathbf{W}_{nS} - \mathbf{W}_{tS}\Xi_S\mathbf{U}_S)\mathbf{c}_{nS} = -\mathbf{g}_{ext} \tag{21}$$

$$\mathbf{c}_{nS} \geq \mathbf{0}\ . \tag{22}$$

Analytically determining the inverse of $\mathbf{W}_{nS} - \mathbf{W}_{tS}\Xi_S\mathbf{U}_S$ yields the following inequalities which must be satisfied for the $3S$ solution to be valid

$$c_{nSi} = \frac{Y_i}{X} \quad \forall i \in \{1, 2, 3\} \tag{23}$$

where

$$\begin{aligned} X &= A\mu_1\mu_2\mu_3 + B_1\mu_2\mu_3 + B_2\mu_1\mu_3 + B_3\mu_1\mu_2 \\ &\quad + C_1\mu_1 + C_2\mu_2 + C_3\mu_3 + D \qquad (24) \\ Y_1 &= E_1\mu_2\mu_3 + F_{12}\mu_2 + F_{13}\mu_3 + G_1 \qquad (25) \\ Y_2 &= E_2\mu_1\mu_3 + F_{21}\mu_1 + F_{23}\mu_3 + G_2 \qquad (26) \\ Y_3 &= E_3\mu_1\mu_2 + F_{31}\mu_1 + F_{32}\mu_2 + G_3 \qquad (27) \end{aligned}$$

where the coefficients, A_i, B_i, C_i, D, E_i, F_{ij}, and G_i can be written as three-by-three determinants with columns consisting of individual contact wrenches and the external wrench multiplied by up to three elements of the tangential directions matrix (see [17] for definitions). Thus the analytical expressions for the regions of a valid $3S$ motion in μ-space are as follows,

$$\begin{aligned} X &> 0 \\ Y_i &\geq 0; \quad \forall i \end{aligned} \tag{28}$$

OR

$$\begin{aligned} X &< 0 \\ Y_i &\leq 0; \quad \forall i \end{aligned} \tag{29}$$

If all μ_i are different, Collins' decomposition could be used to generate descriptions of regions of μ-space in which $3S$ motion can exist. However, if one can justify that the friction coefficients are nominally the same and don't vary greatly, then the above inequalities reduce to cubic and quadratic inequalities in μ. The roots of these inequalities can be used to determine the range(s) of μ for which $3S$ motion is consistent with the quasi-static and Coulomb friction assumptions. It is important to note, however, that the coefficients of the inequalities, and therefore the $3S$ regions in μ-space, are dependent on $\dot{\theta}$. Thus, $3S$ motion of the lamina in one direction, $\dot{\theta}$, may be possible for "large" coefficients of friction, while $3S$ motion in the opposite direction may only be possible for "small" friction coefficients.

Figure 2 below shows a randomly chosen grasp for which the $3S$ regions in mu-space were computed. Figure 3 shows two slices of the $3S$ regions taken perpendicular to the μ_3-axis. The areas in the μ_1-μ_2 planes containing the 1's represent slices of the $3S$ region. The areas containing the 2's belong to the RS region corresponding to contact 1 rolling, contact 2 sliding, and contact 3 breaking. The dotted lines are the exact $3S$ boundaries defined by the most restrictive inequalities in the sets (28) and (29). The thin, solid, horizontal line, and the dotted line with positive slope bound the RS region (the portion of the solid line to the left of the dotted lines is not constraining). The blank areas denote that the proposed mode of motion is infeasible, *i.e.*, jamming or instability will occur. As one would expect, for this grasp, the points in the infeasible region correspond to large coefficients of friction. If the coefficients of friction are constrained to be equal, then the $3S$ regions is given by $0 \leq \mu \leq 0.8$, where 0.8 is the approximate value of μ for which the friction cones of contacts 1 and 2, in the words of Nguyen, "see each other [10]."

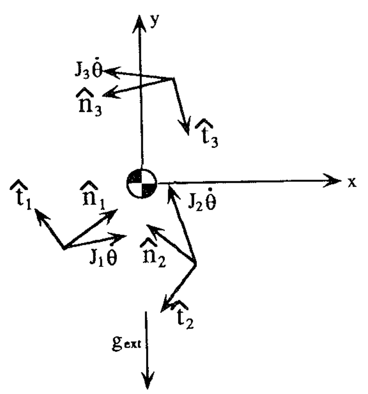

Figure 2: Randomly Chosen Grasp.

Inequalities (28) and (29) embody the feasibility of the equilibrium and Coulomb constraints given the presumed mode of motion. As pointed out by Zeng [17], they also have a simple graphical interpretation which is summarized in the following corollary.

Corollary 1 *If the kinematic constraints are assumed to be satisfied already, then the sufficient and necessary conditions for the existence of a* $3S$ *solution are as follows.*

- *If the lines of action of the three contact wrenches do not intersect at one point, then for every pair of contacts, the lines of actions of the external wrench and the unpaired contact wrench must produce moments of opposite sense about the intersection point of the lines of action of the paired contacts.*
- *If the lines of action of the three contact wrenches do intersect at one point, then the line of action of the external wrench must pass through the intersection point and its direction must lie in the cone defined by the negative span of the directions of the lines of action of the contact wrenches.*

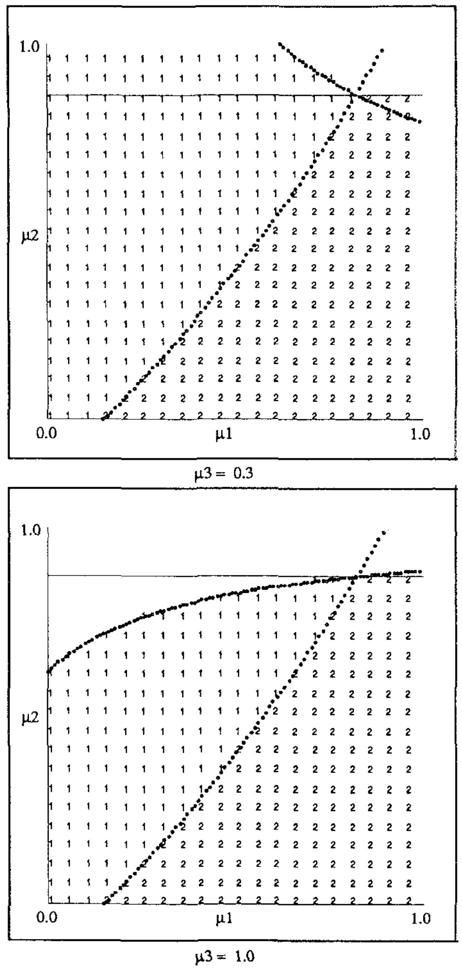

Figure 3: μ-Space Decomposition for Random Grasp

This Corollary is graphically illustrated in Figure 4. Given that the kinematic constraints are feasible, then $\dot{\mathbf{q}}$ is known. This implies that the lines of action of the three sliding contacts are also known. In Figure 4, these lines of action are labeled $\mathbf{W}_1$, $\mathbf{W}_2$, and $\mathbf{W}_3$. Figures 4a and 4b illustrate two possible sets of contact wrenches which do not intersect as a common point. In Figure 4a, the corollary requires the line of action of the possible external wrenchs (the unlabeled arrows) to pass through the segments $\overline{P_{12}P_{23}}$ and $\overline{P_{31}P_{12}}$ such that in points into the cone formed by the negative span of the directions of the lines of action of the contact wrenches, $\mathbf{W}_1$, $\mathbf{W}_2$, and $\mathbf{W}_3$, (shown to the left of the grasp). In Figure 4b, the lines of action of the contact wrenches, $\mathbf{W}_1$, $\mathbf{W}_2$, and $\mathbf{W}_3$, negatively span the plane of the paper, so the direction of the line of action of the external wrench is unconstrained. However, its line of action must cause a negative moment with respect to all points in the triangle, $\triangle P_{12}P_{23}P_{31}$. Several external wrenches consistent with feasible, quasi-static, $3S$ motion are shown. Note that the wrenches acting at the sliding contacts and the external wrench form two cones which "see each other."

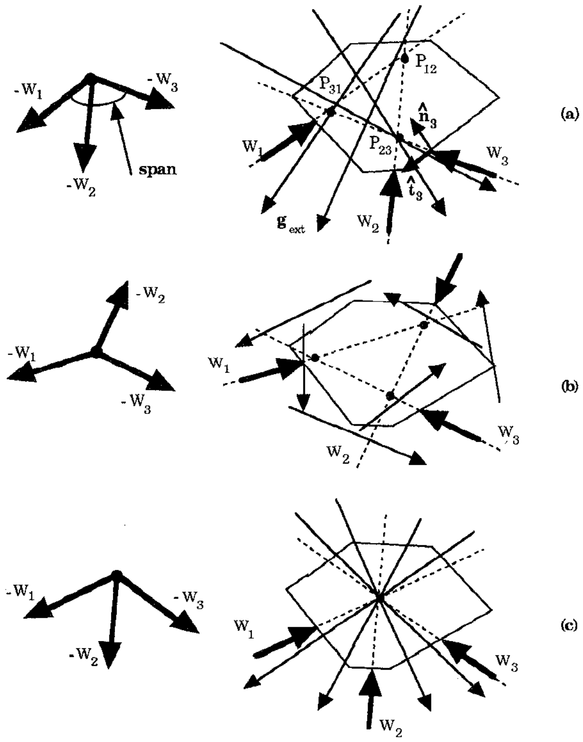

Figure 4: Conditions for Existence of $3S$ Mode.

When the area of the triangle, $\triangle P_{12}P_{23}P_{31}$, approaches zero, the denominator, X, of c_{nSi} also approaches zero. If the line of action of the external wrench does not pass through the intersection point (the degenerate triangle), then the numerator has a finite value. If this is the case, the elements of the normal wrench intensity vector go to infinity, which implies that no feasible solution exists.

When the friction coefficients change, the lines of action of the contact wrenches change. If such change does not result in the change of the moments of the external wrench with respect to the vertices of the triangle, then the same $3S$ motion is valid. Otherwise, the external wrench will not be balanced by the three modified contact wrenches.

4.2 Regions in μ-Space for RS

For the RS motion, $\mathbf{W}_{A\mu}$, $\mathbf{B}_A$, and $\mathbf{c}_A$ are defined as follows

$$\mathbf{W}_{A\mu} = [\mathbf{w}_{rn} \;\; \mathbf{w}_{rt} \;\; (\mathbf{w}_{sn} - \xi_s \mu_s \mathbf{w}_{st})] \quad (30)$$

$$\mathbf{c}_A = \begin{bmatrix} c_{rn} \\ c_{rt} \\ c_{sn} \end{bmatrix} \qquad \mathbf{B}_A = \begin{bmatrix} \mu_r & 1 & 0 \\ \mu_r & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (31)$$

where the subscripts r and s are the indices of the rolling and sliding contacts, respectively, and ξ_s is the direction of relative tangential velocity at the sliding contact. Solving for the wrench intensities yields

$$\begin{aligned} c_{rn} &= \frac{C_1\mu_S + D_1}{A\mu_S + B} \\ c_{rt} &= \frac{C_2\mu_S + D_2}{A\mu_S + B} \\ c_{sn} &= \frac{D_3}{A\mu_S + B} \end{aligned}$$

where the coefficients A, B, C_i, and D_i, can be written as three-by-three determinants with columns consisting of individual contact wrenches and the external wrench multiplied by up to three elements of the tangential directions matrix (see [17] for definitions).

Substituting the wrench intensities into the Coulomb friction constraint (19) yields the analytical expressions for the region of μ-space in which the chosen RS motion mode exists

$$\left\{ \begin{array}{l} A\mu_s + B > 0 \\ C_1\mu_r\mu_s + D_1\mu_r - C_2\mu_s - D_2 \geq 0 \\ C_1\mu_r\mu_s + D_1\mu_r + C_2\mu_s + D_2 \geq 0 \end{array} \right\} \; if D_3 \geq 0 \quad (32)$$

OR

$$\left\{ \begin{array}{l} A\mu_s + B < 0 \\ C_1\mu_r\mu_s + D_1\mu_r - C_2\mu_s - D_2 \leq 0 \\ C_1\mu_r\mu_s + D_1\mu_r + C_2\mu_s + D_2 \leq 0 \end{array} \right\} \; if D_3 \leq 0 \quad (33)$$

The graphical interpretation of the RS mode inequalities (32) and (33) can be shown to identical to that of the $3S$ mode. One need merely replace two wrenches generated by sliding contacts with the wrenches corresponding to the two edges of the friction cone of the rolling contact.

Figure 5 below shows a thin rigid rod supported by two points of contact which move slowly together. The coefficient of friction at contact 1 takes on a value μ_{1R} during rolling and a lower value, μ_{1S}, when sliding. Similarly the effective coefficient of friction at the second contact is either μ_{2R} or μ_{2S}. Of the nine possible modes of motion, five involve one or more breaking contacts. These are quasi-statically infeasible, because equilibrium cannot be maintained. The 2R mode is kinematically infeasible since the bodies are rigid. Thus we need only consider three modes; $2S$, R_1S_2, and R_2S_1.

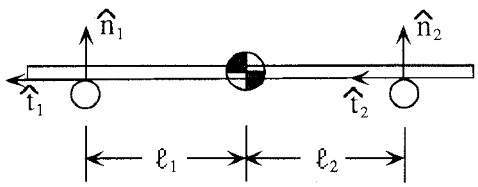

Figure 5: Rod on Two Fingers

Inequalities (32) and (33) for the R_2S_1 mode can be shown to exist if the coefficients of friction define a point in the portion of μ-space above or on the line, $l_1\mu_2 = l_2\mu_1$, which we call the partitioning line (see Figure 6). Similarly, points in μ-space below or on the partitioning line correspond to the R_1S_2 mode.

Suppose l_1 and l_2 are such that the rectangle formed by the coefficients of friction lies entirely within the R_2S_1 region as shown. As the supports move, l_1 will reduce, l_2 will be unchanged, and the coefficients of friction will assume the values μ_{1S} and μ_{2R}, defining the upper left vertex of the rectangle.

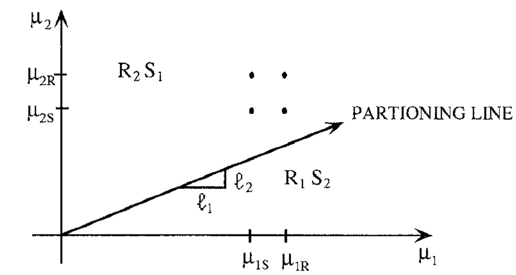

Figure 6: μ-Space Decomposition for Rod

As motion progresses, the partitioning line rotates counterclockwise about the origin. Assuming the coefficients of friction remain fixed, then only the R_2S_1 mode is feasible until the instant the partitioning line passes through the upper left corner of the rectangle. At this point, both RS modes are feasible, but the mode must switch to R_1S_2, because if it did not, then in the next instant, the R_2S_1 mode would become infeasible with only R_1S_2 feasible. At the instant the mode switches, the coefficients of friction switch to their other values. As the supports continue to approach each other, the partitioning line "bounces" back and forth between the upper left and lower right corners of the rectangle, switching between RS modes, seemingly chasing the current point in μ-space. The $2S$ mode is never active, because it is only feasible when the partitioning line reaches the lower left corner of the rectangle and if the coefficients of friction are equal to their sliding values. However, one of the coefficients of friction is always equal to its rolling value. Note that if the coefficients of friction were random variables, with means at the corners of the rectangle, the same "bouncing" behavior would be observed, but the corners of the rectangle would not be fixed.

In the above example, two modes were feasible simultaneously at the instants that the partitioning line reached the upper left and lower right corners of the rectangle. This is not surprising, since dynamic models of rigid body interaction have been shown to yield multiple feasible solutions [7]. One might think that multiple solutions are feasible only for an instant of time, but this is not true.

Figure 7 shows a grasp with four distinct feasible modes of motion ($3S$, R_1S_3, R_2S_1, and R_3S_1), three of which are simultaneously feasible for suitably small perturbations in the coefficients of friction and the positions and orientations of the contacts. Figure 8 shows that multiple modes of motion are simultaneously feasible over large regions of friction space. Note that in Figure 8, 1's, 3's, 4's, and 6's correspond to the modes, $3S$, R_1S_3, R_2S_1, and R_3S_1, respectively.

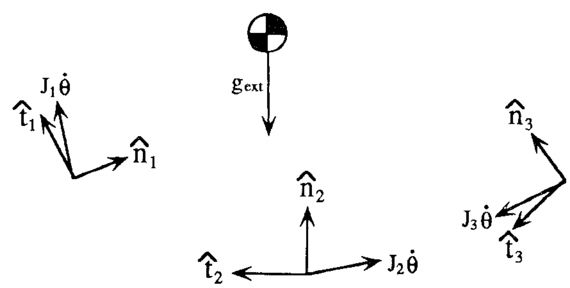

Figure 7: Three-Point Grasp with Multiple Modes

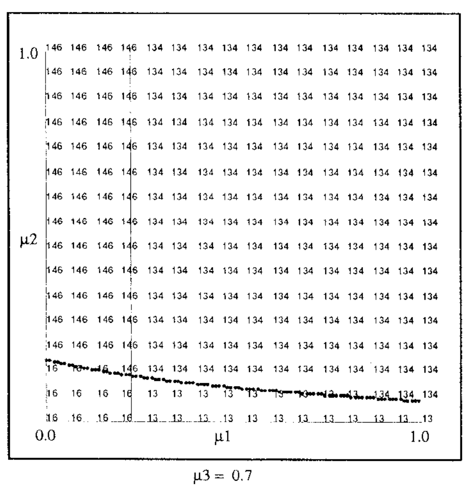

Figure 8: Partial μ-Space Decomposition for Figure 7.

5 Conclusion

The quasi-static planar motion of a rigid lamina in frictional contact with any number of moving point bodies depends on the coefficients of friction at the contacts. Using the equations of equilibrium, the kinematic velocity constraints, and a Coulomb model of friction, we have derived polynomial inequalities defining regions in μ-space in which particular modes of motion are feasible. The examples discussed here have pointed out that the relationships defining quasi-static motion may have no solution or multiple solutions; a fact which is not surprising given that dynamic equations of motion of rigid bodies in contact also exhibit nonunique solution[7].

The analysis presented can be applied to dexterous manipulation and assembly planning for systems consisting of parts of constant cross section. One particular application that we are currently pursuing relates to dexterous manipulation planning. We have succeeded in planning under the frictionless assumption[15], but would like to execute "frictionless" plans in real environments with friction. Since frictionless plans have all sliding contacts, it is possible to apply our results for the valid regions of $3S$ solutions in a point-wise manner along a given path. The results for all the points can be combined to determine a friction bound, below which, the frictionless plan can be successfully executed. Such analyses can yield useful constraints on materials selection during the design of assembly systems, by simulating nominal trajectories or planning nominal tasks prior to manufacture.

6 Acknowledgements

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