Math 535 Homework 1
Due January 19

1) Show that if $A > 0$ and $A \neq 1$, and if z_1 and z_2 are distinct complex numbers, then $\{z \in \mathbf{C} : |z - z_1| = A|z - z_2|\}$ is a circle in the complex plane. Find the center and radius of this circle.

2) Show that complex numbers z_1, z_2, z_3 are vertices of an equilateral triangle if and only if $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_1 z_3$. (Hint: If you can show that this equation is invariant under scalings, translations, and rotations, then the problem is reduced to a much less general question.)

3) Find all complex numbers z satisfying

a) $z^4 = -1$
b) $z^4 + z^3 + z^2 + z^2 + 1 = 0$
c) $z^6 + 2z^3 + 2 = 0$

4) Show that if z_1 and z_2 are complex numbers with $|z_1|, |z_2| < 1$, then $\left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| < 1$

5) Given three complex numbers z_1, z_2, z_3 not all on a line, find the center and radius of the circle which contains all three points. (Hint: Look at the perpendicular bisectors of the segments $[z_1, z_2]$ and $[z_2, z_3]$).

6) Note that we saw in class that $\prod_{j=0}^{n-1}(z - e^{\frac{2\pi i j}{n}}) = z^n - 1$.

a) Show that $1 + z + z^2 + ... + z^{n-1} = \prod_{j=1}^{n-1}(z - e^{\frac{2\pi i j}{n}})$.
b) Suppose $z_0, ..., z_{n-1}$ are vertices of a regular n-gon such that the vertices are all on a circle of radius 1. Show that $\prod_{j=1}^{n-1} |z_j - z_0| = n$.