Math 535 Homework 4
Due February 29

In this homework, an *entire* function means a function that is analytic on the whole complex plane.

1) Show that there is no analytic function $f(z)$ on a neighborhood of the origin such that $f(\frac{1}{n}) = (-1)^n \frac{1}{n^2}$ for all integers $n > 0$. Is the answer different if we replace n^2 with n^k here, for a different positive integer k?

2) Let $f(z)$ be an entire function such that $f(z) = f(3z + z^2)$ for all z. Show that $f(z)$ is constant.

3) Let A denote the set of analytic functions $f(z)$ on $B(0,1)$ (the unit disk centered at the origin) such that $Re(f(z)) > 0$ for all $z \in B(0,1)$ and such that $f(0) = 2$. Find $\sup_{f \in A} |f'(0)|$, and determine the functions in A that achieve this supremum, if there are any.

4) Suppose $g(z)$ is a nonconstant analytic function on $B(0,1)$ that extends to a continuous function on the closure $cl(B(0,1))$ such that $|g(z)| = 1$ whenever $|z| = 1$. Show that there is some z in $B(0,1)$ for which $g(0) = 0$.

5) Let $p(z)$ be an entire function such that there is a constant $C > 0$ and a positive integer n such that $|p(z)| \leq C(1 + |z|^n)$ for all z. Show that $p(z)$ is a polynomial of degree at most n.

6) Suppose $q(z)$ is an entire function such that $\int_{\mathbf{R}^2} |q(x+iy)|\, dx\, dy$ is finite. Show that $q(z) = 0$ for all z. (Hint: Look up the mean value theorem for analytic functions, which you may use.)